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GENERAL MIXED HARMONIC VARIATIONAL INEQUALITIES

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Abstract. In this paper, some iterative methods are used to discuss the behavior of general mixed-harmonic variational inequalities. We employ the auxiliary principle technique and g-strongly harmonic monotonicity of the operator to obtain results on the existence of solutions to a generalized class of mixed harmonic variational inequality.

1. Introduction

For the last six decades, it has been found that the theory of variational inequality has a very rich and varied source. Stampacchia [31], who first introduced and studied this theory in 1964, used it as a very effective unifying model to address equilibrium issues. Later on, it has explosive and dynamical growth in both pure and applied mathematics. This field has a wide range of applications in industry, physical, regional, engineering, pure and applied sciences.

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Variational inequality theory has been extended and generalized in different directions by applying novel and innovative techniques. One of the most convenient and major generalizations of variational inequalities is the mixed variational inequality, introduced and studied by Lions and Stampacchia [7] and further extended by Noor [12]. The most important and significant generalization of a convex function is a harmonic convex function. Iscan [6] and Noor et. al. [28, 29] have obtained several Hermite-Hadamard, Simpson, and Cholowski type integral inequalities for the harmonic convex function and their variant forms. For recent advancements in the work related to harmonic variational inequalities and mixed variational inequalities, see [2, 3, 8–10, 27, 30] and the references therein.

Glowinski and Tremolieres [5] used the auxiliary principal technique to study the existence of solutions to mixed variational inequalities. In recent years, this technique has been used to suggest and analyze many iterative methods for solving various classes of variational inequalities [22]. For other applications, formulations, numerical methods and other aspects of variational inequalities, see, for example, [1, 4, 11, 13–23] and the references therein. Motivated and inspired by the work of Noor et al. [24–26], we use the auxiliary principle technique, which is mainly due to Lions and Stampacchia [31] for the existence of solutions to the general mixed harmonic variational inequalities.

The rest of the paper is organized as follows: In Section 2, we formulate the general mixed harmonic variational inequality with some special cases as remarks are discussed and recall some important definitions from the literature for their subsequent usage. In Section 3, we employ the auxiliary principle technique for obtaining an approximate solution to the general mixed harmonic variational inequality.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and the symbols $\|.\|$ and $\langle .,. \rangle$ represent the norm and inner product on $\mathcal{H}$, respectively. Let K be a nonempty nonzero closed harmonic convex set in $\mathcal{H}$.

Let $T, g : \mathcal{H} \to \mathcal{H}$ be continuous nonlinear mappings and $\psi : \mathbb{R} \to \mathbb{R}^+ \cup \{\infty\}$ be a semi-continuous linear functional. Then the problem of finding $u \in \mathcal{H}$ such that

$$\left\langle Tu, \frac{g(u)g(v)}{g(u) - g(v)} \right\rangle + \psi(g(v)) - \psi(g(u)) \geq 0, \ \forall \ g(v) \in \mathcal{H} \tag{2.1}$$

known as *general mixed harmonic variational inequality* which appears to be new.

Remark 2.1. We see, if we take $g(u) = -\frac{(g(u)-g(v))^2}{g(v)}$ and $g(v) = -\frac{(g(u)-g(v))^2}{g(u)}$ in the second component of $\langle .,.\rangle$ in (2.1), then

$$\langle Tu, g(v) - g(u)\rangle + \psi(g(v)) - \psi(g(u)) \geq 0,\ \forall\ g(u) \in \mathcal{H}, \tag{2.2}$$

known as general mixed variational inequality or general variational inequality of 2nd kind, introduced and studied by Noor [12].

Remark 2.2. If we take $g \equiv I$, identity operator, then the above inequalities (2.1) and (2.2) are analogous to finding u in H such that

$$\left\langle Tu, \frac{uv}{u-v}\right\rangle + \psi(v) - \psi(u) \geq 0,\ \ \forall\ v \in \mathcal{H}, \tag{2.3}$$

known as *mixed harmonic variational inequality* which appears to be new one and the inequality (2.2) implies

$$\langle Tu, v - u\rangle + \psi(v) - \psi(u) \geq 0,\ \ \forall\ u \in \mathcal{H}, \tag{2.4}$$

known as mixed variational inequality introduced by Lions and Stampachhia[7].

Note that the function ψ is an indicator function for the closed convex harmonic set K in $\mathcal{H}$, defined as

$$\psi(u) = I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

The problem (2.1) is equivalent to find $u \in \mathcal{H},\ g(u) \in K$ such that

$$\left\langle Tu, \frac{g(u)g(v)}{g(u)-g(v)}\right\rangle \geq 0,\ \ \forall g(v) \in K \tag{2.6}$$

known as *general harmonic variational inequality* and the problem of the type (2.2) is equivalent to

$$\langle Tu, g(v) - g(u)\rangle \geq 0,\ \ \forall g(v) \in K, \tag{2.7}$$

is known as *general variational inequality.*

Again, if we consider $g \equiv I$, the identity operator, then the above inequalities (2.6) and (2.7) reduce to the original harmonic variational inequality introduced and studied by Noor [25] and classical variational inequality by Lions and Stampacchia [31], respectively. Also, we can conclude that inequalities (2.3) and (2.4) are special cases of (2.1) and (2.2), respectively.

Definition 2.3. Let $\mathcal{H}$ be a Hilbert space and $K \subset \mathcal{H} \setminus \{0\}$ be a nonempty set. Then the set K is said to be harmonic convex, if

$$\frac{uv}{v + \lambda(u-v)} \in K,\ \forall u, v \in K,\ \lambda \in [0, 1].$$

Definition 2.4. Let $f : K \subset \mathcal{H} \setminus \{0\} \to \mathbb{R}$ be a map. Then f is said to be harmonic convex, if

$$f\left(\frac{uv}{v+\lambda(u-v)}\right) \leq (1-\lambda)f(u)+\lambda f(v),$$

$\forall u, v \in K,\ \lambda \in [0,1]$

Note. If -f is harmonic convex, then f is called harmonic concave, and vice versa.

Definition 2.5. Let $f : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function. Then f is said to be harmonic quasi-convex, if

$$f\left(\frac{uv}{v+\lambda(u-v)}\right) \leq \max\{f(u), f(v)\},\ \forall u, v \in K,\ \lambda \in [0,1].$$

Note: If $-f$ is harmonic quasi-convex then f is called harmonic quasi-concave, and vice versa. Also, whenever $f(v) \geq f(u)$, f is called harmonic quasi-convex and this implies $f(v)$ is greater than or equal to the value of f at the point of the path $\frac{uv}{v+\lambda(u-v)}$. If the strict inequality holds for $f(u) \neq f(v)$, then f is called strictly harmonic quasi-convex.

Definition 2.6. Let a function f be defined as $f : K \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$. Then f is said to be logarithmic harmonic convex if

$$f\left(\frac{uv}{v+\lambda(u-v)}\right) \leq (f(u))^{1-\lambda}(f(v))^{\lambda},\ \forall u, v \in K,\ \lambda \in [0,1],$$

where $f(.) > 0$.

From the above definitions, we have

$$\begin{aligned} f\left(\frac{uv}{v+\lambda(u-v)}\right) &\leq (f(u))^{1-\lambda}(f(v))^{\lambda} \\ &\leq (1-\lambda)f(u)+\lambda f(v) \\ &\leq \max\{f(u), f(v)\}. \end{aligned}$$

From this, we conclude that harmonic logarithmic convex functions are harmonic convex functions and harmonic convex functions are quasi-harmonic convex functions. But the converse need not be true.

Again, from the above definition, we have

$$\log f\left(\frac{uv}{v+\lambda(u-v)}\right) \leq (1-\lambda)\log f(u)+\lambda \log f(v), \forall u, v \in K,\ \lambda \in [0,1].$$

Lemma 2.7. *Let $\mathcal{H}$ be a real Hilbert space and for all u, v in $\mathcal{H}$, we have*

$$\langle u, v\rangle = \frac{1}{2}\{\|u+v\|^2 - \|u\|^2 - \|v\|^2\}. \tag{2.8}$$

Proof.

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2.\end{aligned}$$

Hence $\langle u, v\rangle = \frac{1}{2}\{\|u+v\|^2 - \|u\|^2 - \|v\|^2\}$. □

3. Main results

In this section, we use the auxiliary principle technique, which is mainly due to Lions and Stampacchia [31] and developed by Noor [22], in order to obtain the existence results.

Definition 3.1. Let $\mathcal{H}$ be a real Hilbert space and $T, g : \mathcal{H} \to \mathcal{H}$ be two operators. Then T is said to be *g- partially relaxed harmonic strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\left\langle Tu - Tv, \frac{g(v)g(z)}{g(v)-g(z)}\right\rangle \geq -\alpha\|g(u) - g(z)\|^2, \ \ \forall u, v, z \in \mathcal{H}.$$

Note: If we take $z = u$, then g-partially relaxed harmonic strongly monotonicity implies g-harmonic monotonicity.

$$\text{That is, } \left\langle Tu - Tv, \frac{g(u)g(v)}{g(v)-g(u)}\right\rangle \geq 0,$$

which is called the g-harmonic monotonicity of the operator T.

Algorithm 3.2. For a given $u_0 \in \mathcal{H}$ and $\rho > 0$, we compute the approximate solution u_{n+1} by the iterative scheme.

$$\begin{aligned}\left\langle \rho Tu_n, \frac{g(u_{n+1})g(v)}{g(u_{n+1})-g(v)}\right\rangle &+ \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &+ \rho\psi(g(v)) - \rho\psi(g(u_{n+1})) \geq 0.\end{aligned} \tag{3.1}$$

Theorem 3.3. *Let $\mathcal{H}$ be a real Hilbert space, $\bar{u} \in \mathcal{H}$ be the solution of general mixed harmonic variational inequality* (2.1) *and u_{n+1} be the approximate solution acquired from Algorithm* 3.2. *If $T : \mathcal{H} \to \mathcal{H}$ is a g-partially relaxed harmonic strongly monotone operator with constant $\alpha > 0$, then*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - (1 - 2\rho\alpha)\|g(u_{n+1}) - g(u_n)\|^2.$$

Proof. Let $\bar{u} \in \mathcal{H}$ be the solution of (2.1), we have

$$\left\langle \rho T(\bar{u}), \frac{g(\bar{u})g(v)}{g(\bar{u}) - g(v)} \right\rangle + \rho\psi(g(v)) - \rho\psi(g(\bar{u})) \geq 0, \ \ \forall v \in \mathcal{H}, \tag{3.2}$$

where $\rho > 0$ is a constant. Take $v = u_{n+1}$ in (3.2) and $v = \bar{u}$ in (3.1), we have

$$\left\langle \rho T\bar{u}, \frac{g(\bar{u})g(u_{n+1})}{g(\bar{u}) - g(u_{n+1})} \right\rangle + \rho\psi(g(u_{n+1})) - \rho\psi(g(\bar{u})) \geq 0 \tag{3.3}$$

and

$$\left\langle \rho Tu_n, \frac{g(u_{n+1})g(\bar{u})}{g(u_{n+1}) - g(\bar{u})} \right\rangle + \langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1}) \rangle \\ + \rho\psi(g(\bar{u})) - \rho\psi(g(u_{n+1})) \geq 0. \tag{3.4}$$

On adding (3.3) and (3.4), and in view of g- partially relaxed harmonic strongly monotonicity of the operator T, it follows that

$$\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1}) \rangle \geq \rho \left\langle Tu_n - T\bar{u}, \frac{g(\bar{u})g(u_{n+1})}{g(\bar{u}) - g(u_{n+1})} \right\rangle \\ \geq -\alpha\rho\|g(u_n) - g(u_{n+1})\|^2. \tag{3.5}$$

Using Lemma 2.7, we have

$$\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1}) \rangle = \frac{1}{2}\{\|g(\bar{u}) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 \\ - \|g(\bar{u}) - g(u_{n+1})\|^2\}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\frac{1}{2}\{\|g(\bar{u}) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 - \|g(\bar{u}) - g(u_{n+1})\|^2\} \\ \geq -\alpha\rho\|g(u_n) - g(u_{n+1})\|^2,$$

it implies that

$$\|g(\bar{u}) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2 - \|g(\bar{u}) - g(u_{n+1})\|^2 \\ + 2\alpha\rho\|g(u_{n+1}) - g(u_n)\|^2 \geq 0.$$

Hence we have

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - (1 - 2\rho\alpha)\|g(u_{n+1}) - g(u_n)\|^2.$$

□

Definition 3.4. Let $\mathcal{H}$ be a Hilbert space and $T, g : \mathcal{H} \to \mathcal{H}$ be two operators. Then T is said to be:

(i) strongly g-monotone, if there exists a constant $\gamma > 0$, such that

$$\langle Tu - Tv, g(u) - g(v) \rangle \geq \gamma\|g(u) - g(v)\|^2, \ \ \forall u, v \in \mathcal{H}.$$

(ii) Lipschitz g-continuous, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|g(u) - g(v)\|, \quad \forall u, v \in \mathcal{H}.$$

Note: If we take $g \equiv I$, the identity operator, then strongly g-monotonicity and Lipschitz g-continuity reduce to strongly monotonicity and Lipschitz continuity of the operator T, respectively.

Algorithm 3.5. Consider the problem of finding $w \in \mathcal{H}$ such that $g(w) \in \mathcal{H}$, satisfying the general auxiliary harmonic variational inequality.

$$\left\langle \rho Tu + g(w) - g(u), \frac{g(w)g(v)}{g(w) - g(v)} \right\rangle + \rho\psi(g(v)) - \rho\psi(g(u)) \geq 0, \tag{3.7}$$

where $\rho > 0$ is a constant.

Note: In particular, if we take $w = u$, then we see that w is a solution of general mixed harmonic variational inequality (2.1).

Theorem 3.6. *Let K be a nonzero nonempty closed harmonic convex subset of a real Hilbert space $\mathcal{H}$ and $T, g : \mathcal{H} \to \mathcal{H}$ be two operators such that T is Lipschitz g-continuous and strongly g- monotone. Then for*

$$0 < \rho < \frac{2\gamma}{\beta^2}, \tag{3.8}$$

there exists a solution to the general mixed harmonic variational inequality (2.1).

Proof. For $u \in \mathcal{H}$ such that $g(u) \in K$, we see that inequality (3.7) defines a mapping $g(u)$ *to* $g(w)$ *in* K. To prove the existence of a solution, we need to show that the mapping $g(u) \to g(w)$ has a fixed point. For this, it is enough to show that $g(w)$ is a contraction mapping.

Let $g(w_1) \neq g(w_2)$ be two solutions of (3.7) corresponding to $g(u_1) \neq g(u_2)$ in K, respectively. Then for a constant $\rho > 0$, we have

$$\left\langle \rho Tu_1 + g(w_1) - g(u_1), \frac{g(w_1)g(v)}{g(w_1) - g(v)} \right\rangle + \rho\psi(g(v)) - \rho\psi(g(u_1)) \geq 0 \tag{3.9}$$

and

$$\left\langle \rho Tu_2 + g(w_2) - g(u_2), \frac{g(w_2)g(v)}{g(w_2) - g(v)} \right\rangle + \rho\psi(g(v)) - \rho\psi(g(u_2)) \geq 0. \tag{3.10}$$

Taking $g(v) = g(w_2)$ in (3.9) and $g(v) = g(w_1)$ in (3.10) and then adding, we have

$$\left\langle \rho Tu_2 - \rho Tu_1 + g(u_1) - g(u_2), \frac{g(w_1)g(w_2)}{g(w_2) - g(w_1)} \right\rangle + \rho(\psi(g(w_1)) + \psi(g(w_2))) - \rho(\psi(g(u_1)) + \psi(g(u_2))) \geq \left\langle g(w_1) - g(w_2), \frac{g(w_1)g(w_2)}{g(w_2) - g(w_1)} \right\rangle. \quad (3.11)$$

Since ψ is an indicator function, then from (3.11), we have

$$\left\langle \rho Tu_2 - \rho Tu_1 + g(u_1) - g(u_2), \frac{g(w_1)g(w_2)}{g(w_2) - g(w_1)} \right\rangle \geq \left\langle g(w_1) - g(w_2), \frac{g(w_1)g(w_2)}{g(w_2) - g(w_1)} \right\rangle.$$

Therefore

$$\|\rho(Tu_2 - Tu_1) + g(u_1) - g(u_2)\| \geq \|g(w_1) - g_(w_2)\|. \quad (3.12)$$

Now consider

$$\begin{aligned}
&\|\rho(Tu_2 - Tu_1) + g(u_1) - g(u_2)\|^2 \\
&= \langle \rho(Tu_2 - Tu_1) + g(u_1) - g(u_2), \rho(Tu_2 - Tu_1) + g(u_1) - g(u_2)\rangle \\
&= \rho^2\|(Tu_2 - Tu_1)\|^2 + \|g(u_1) - g(u_2)\|^2 + 2\rho\langle Tu_2 - Tu_1, g(u_1) - g(u_2)\rangle \\
&= \rho^2\|(Tu_1 - Tu_2)\|^2 + \|g(u_1) - g(u_2)\|^2 - 2\rho\langle Tu_1 - Tu_2, g(u_1) - g(u_2)\rangle \\
&\leq \rho^2\beta^2\|g(u_1) - g(u_2)\|^2 - 2\rho\gamma\|g(u_1) - g(u_2)\|^2 + \|g(u_1) - g(u_2)\|^2 \\
&\leq (\rho^2\beta^2 - 2\rho\gamma + 1)\|g(u_1) - g(u_2)\|^2. \quad (3.13)
\end{aligned}$$

From (3.12) and (3.13), we have

$$\|g(w_1) - g_(w_2)\|^2 \leq (\rho^2\beta^2 - 2\rho\gamma + 1)\|g(u_1) - g(u_2)\|^2$$

and hence

$$\|g(w_1) - g_(w_2)\| \leq \delta\|g(u_1) - g(u_2)\|,$$

where $\delta = \sqrt{\rho^2\beta^2 - 2\rho\gamma + 1} < 1$, follows from (3.8).

This shows that the mapping $g(w)$ is a contraction mapping and hence has a fixed point $g(w) = g(u) \in K$, satisfying the general mixed harmonic variational inequality (2.1). □

4. Conclusion

In this paper, we introduced and studied general mixed harmonic variational inequality and established the existence of a solution to this inequality by using auxiliary principle techniques and some iterative methods. The developments of such algorithms applied for obtaining solutions are interesting to find most suitable to establish their approximate solution and some of their characterizations are also discussed. Further efforts are required to explore several applications of this new class of variational inequality in pure as well as applied sciences. The concept of general mixed harmonic variational inequality is a new class of general mixed variational inequality, and it is highly expected that the ideas and methods used in this paper may stimulate further research in this area.

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