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INVESTIGATION OF A NEW COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS IN FRAME OF HILFER-HADAMARD

Ali Abd Alaziz Najem Al-Sudani[1] and Ibrahem Abdulrasool hammood Al-Nuh[2]

[1]Department of Materials Management, Technical Institute for Administration,
Middle Technical University, Baghdad, Iraq
e-mail: `ali _ abdalaziz@mtu.edu.iq`

[2]Department of Accounting Techniques, Technical Institute for Administration,
Middle Technical University, Baghdad, Iraq
e-mail: `ibrahem _ abdalrasool@mtu.edu.iq`

Abstract. The primary focus of this paper is to thoroughly examine and analyze a coupled system by a Hilfer-Hadamard-type fractional differential equation with coupled boundary conditions. To achieve this, we introduce an operator that possesses fixed points corresponding to the solutions of the problem, effectively transforming the given system into an equivalent fixed-point problem. The necessary conditions for the existence and uniqueness of solutions for the system are established using Banach's fixed point theorem and Schaefer's fixed point theorem. An illustrate example is presented to demonstrate the effectiveness of the developed controllability results.

1. Introduction

The fundamental concept of fractional calculus (FC) involves replacing natural numbers with rational numbers in the order of derivation operators. Although this concept may seem simple, it has far-reaching consequences and results that describe phenomena in various fields such as bioengineering, dynamics, modeling, control theory, and medicine For more details, we refer to

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[0]Corresponding author: Ali Abd Alaziz Najim Al-Sudani(`abdalaziz@mtu.edu.iq`).

[11, 13, 15, 22]. Specifically, many authors have shown great interest in the topic of boundary value problems (BVPs) for fractional differential equations (FDEs), and a variety of results have been obtained for BVPs involving different types of boundary conditions, for example, see [6, 10, 12, 18, 20, 24, 25], and the references mentioned therein. Existence and uniqueness theory for classical fractional differential problems involving Hadamard-type operator have been studied by several researchers, see [1, 3, 4, 5, 7, 8] and references therein, and other problems with Hilfer-Hadamard fractional derivative [14, 16, 17, 23].

Coupled systems of FDEs are an extremely significant and important field, due to their applications in many real-world problems. For some theoretical works on the coupled systems of FDEs, we refer the reader to some studied works [2, 5, 19, 21]. In this article, we consider a nonlinear BVP for a coupled system of Hilfer-Hadamard (HH) type FDEs

$$\begin{cases} {}^{H}\mathfrak{D}_{1}^{\rho,\upsilon}\mathcal{H}(\kappa) = \mathcal{M}_1(\kappa, \mathcal{H}(\kappa), \overline{\mathcal{H}}(\kappa)), & \kappa \in I = [1,T], \quad 0 < \rho \leq 1, \\ {}^{H}\mathfrak{D}_{1}^{\rho,\upsilon}\overline{\mathcal{H}}(\kappa) = \mathcal{M}_2(\kappa, \mathcal{H}(\kappa), \overline{\mathcal{H}}(\kappa)), & \kappa \in I, \qquad 0 < \upsilon \leq 1, \end{cases} \tag{1.1}$$

with the following coupled integral boundary conditions

$$\begin{cases} {}^{H}\mathcal{I}_{1}^{\rho,\upsilon}\mathcal{H}(1) = \ell_1\overline{\mathcal{H}}(T), \\ {}^{H}\mathcal{I}_{1}^{\rho,\upsilon}\overline{\mathcal{H}}(1) = \ell_2\mathcal{H}(T), \end{cases} \tag{1.2}$$

where $\rho \in (0,1)$, $\upsilon \in [0,1]$, $w = \rho + \upsilon - \rho\upsilon$, $\mathcal{M}_1$, $\mathcal{M}_2 : I \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions and ${}^{H}\mathfrak{D}_{1}^{\rho,\upsilon}$ is the Hilfer–Hadamard fractional derivative of order ρ and type υ.

This paper mainly examines the uniqueness and existence results of fractional BVPs for a coupled system of HH-type with coupled integral boundary conditions. To the best of our knowledge, coupled systems of HH-type FDEs (1.1) with coupled integral boundary conditions (1.2) have were not widely investigated. In consequence, the coupled system of FDEs with coupled integral boundary conditions will be studied by the HH fractional derivative. Moreover, the main results are obtained by fixed-point theorems of Banach and Schaefer.

The remainder of this paper is organized as follows: In Section 2, we provide a review of fractional calculus notations, definitions, and relevant lemmas that are essential to our research. Additionally, we present an important lemma that allows us to convert the coupled system of HH type problem (1.1)-(1.2) into an equivalent integral equation. Section 3 presents the main findings regarding the existence and uniqueness of solutions for the coupled system of HH type problem (1.1)-(1.2). To illustrate these results, we present a numerical example in Section 4. Finally, we conclude the paper with a summary of our findings in the last section.

2. Preliminary results and essential concepts

Definition 2.1. ([11]) For function $\mathcal{H} : [1, +\infty) \to \mathbb{R}$. The Hadamard fractional integral of order $\rho > 0$ is defined by

$$\mathcal{I}_1^\rho \mathcal{H}(\kappa) = \frac{1}{\Gamma(\rho)} \int_1^\kappa (\log \frac{\kappa}{s})^{\rho-1} \mathcal{H}(s) \frac{ds}{s},$$

provided the right-hand side is point-wise defined on $[1, +\infty)$.

Definition 2.2. ([11]) The Hadamard fractional derivative of order $\rho > 0$, for a continuous function $\mathcal{H} : [1, +\infty) \to \mathbb{R}$ is defined by

$$\left({}^H\mathfrak{D}_1^\rho \mathcal{H}\right)(\kappa) = \xi^n \left(\mathcal{I}_1^{n-\rho} \mathcal{H}\right)(\kappa), \ n = [\rho] + 1,$$

where $\xi^n = \kappa^n \frac{d^n}{d\kappa^n}$, $[\rho]$ and denotes the integer part of the real number ρ.

Definition 2.3. ([16]) Let $\rho \in (0, 1)$, $\upsilon \in [0, 1]$, $w = \rho + \upsilon - \rho\upsilon$, $\mathcal{H} \in L^1(I)$ and ${}^H\mathcal{I}_1^{(1-\rho)(1-\upsilon)}\mathcal{H} \in AC^1(I)$. The Hilfer–Hadamard fractional derivative of order ρ and type υ applied to a function $\mathcal{H}$ is defined as

$$\begin{aligned} \left({}^H\mathfrak{D}_1^{\rho,\upsilon} \mathcal{H}\right)(\kappa) &= {}^H\mathcal{I}_1^{\upsilon(1-\rho)}({}^H\mathfrak{D}_1^w \mathcal{H})(\kappa) \\ &= {}^H\mathcal{I}_1^{\upsilon(1-\rho)}\delta({}^H\mathcal{I}_1^{1-w}\mathcal{H})(\kappa), \quad \text{for a.e } \kappa \in I. \end{aligned} \tag{2.1}$$

This new fractional derivative (2.1) can be thought of as an interpolation of the fractional derivatives of Hadamard and Caputo-Hadamard. As a matter of fact, for $\upsilon = 0$, this derivative reduces to the Hadamard fractional derivative, and for $\upsilon = 1$, we recover the Caputo-Hadamard fractional derivative

$${}^H\mathfrak{D}_1^{\rho,0} = {}^H\mathfrak{D}_1^\rho, \quad {}^H\mathfrak{D}_1^{\rho,1} = {}^{Hc}\mathfrak{D}_1^\rho.$$

Lemma 2.4. ([11]) *For all $\rho > 0$ and $\upsilon > -1$, we have*

$$\frac{1}{\Gamma(\rho)} \int_1^\kappa (\log \frac{\kappa}{s})^{\rho-1} (\log s)^{\upsilon-1} \frac{ds}{s} = \frac{\Gamma(\upsilon)}{\Gamma(\rho + \upsilon)} (\log \kappa)^{\rho+\upsilon-1}.$$

Lemma 2.5. *Let $\phi, h : I \to \mathbb{R}$ be continuous functions. Then the linear coupled HH type FDEs*

$$\begin{cases} {}^H\mathfrak{D}_1^{\rho,\upsilon}\mathcal{H}(\kappa) = \phi(\kappa), & \kappa \in I, \quad 0 < \rho \leq 1, \\ {}^H\mathfrak{D}_1^{\rho,\upsilon}\overline{\mathcal{H}}(\kappa) = h(\kappa), & \kappa \in I, \quad 0 < \upsilon \leq 1, \\ {}^H\mathcal{I}_1^{\rho,\upsilon}\mathcal{H}(1) = \mathcal{H}_1, \\ {}^H\mathcal{I}_1^{\rho,\upsilon}\overline{\mathcal{H}}(1) = \overline{\mathcal{H}}_1, \end{cases} \tag{2.2}$$

is equivalent to the system of integral equations

$$\mathcal{H}(\kappa) = \frac{1}{\Gamma(w)} {}^H\mathcal{I}_1^{1-w}\mathcal{H}(1)(\log \kappa)^{w-1} + {}^H\mathcal{I}_1^\rho \phi(\kappa), \ \kappa \in I \tag{2.3}$$

and

$$\overline{\mathcal{H}}(\kappa) = \frac{1}{\Gamma(w)}\, {}^{H}\mathcal{I}_1^{1-w}\overline{\mathcal{H}}(1)(\log\kappa)^{w-1} + {}^{H}\mathcal{I}_1^{\rho}h(\kappa),\ \kappa \in I. \tag{2.4}$$

Lemma 2.6. *Let $\mathcal{M}_i : I \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ be such that $\mathcal{M}_i(\cdot, \mathcal{H}, \overline{\mathcal{H}}) \in C_{\tau,\log}(I)$, for any $\mathcal{H}, \overline{\mathcal{H}} \in C_{\tau,\log}(I)$. Then system (1.1)–(1.2) is equivalent to the solution of the coupled system*

$$\begin{cases} \mathcal{H}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\varphi_1\ {}^{H}\mathcal{I}_1^{\rho}\mathcal{M}_1(T, \mathcal{H}(T), \overline{\mathcal{H}}(T)) + {}^{H}\mathcal{I}_1^{\rho}\mathcal{M}_1(\kappa, \mathcal{H}(\kappa), \overline{\mathcal{H}}(\kappa)), \\ \overline{\mathcal{H}}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\varphi_2\ {}^{H}\mathcal{I}_1^{\rho}\mathcal{M}_2(T, \mathcal{H}(T), \overline{\mathcal{H}}(T)) + {}^{H}\mathcal{I}_1^{\rho}\mathcal{M}_2(\kappa, \mathcal{H}(\kappa), \overline{\mathcal{H}}(\kappa)), \end{cases} \tag{2.5}$$

where

$$\varphi_1 = \frac{\ell_1}{1 - \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ell_1},\ \varphi_2 = \frac{\ell_2}{1 - \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ell_2}.$$

Proof. Applying of HH type FDEs in (2.2) and using the above lemma, we get

$$\mathcal{H}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ {}^{H}\mathcal{I}_1^{1-w}\mathcal{H}(1) + \frac{1}{\Gamma(\rho)}\int_1^{\kappa}(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_1(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s},$$

$$\overline{\mathcal{H}}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ {}^{H}\mathcal{I}_1^{1-w}\overline{\mathcal{H}}(1) + \frac{1}{\Gamma(\rho)}\int_1^{\kappa}(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_2(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s},$$

by using coupled integral boundary condition, we obtain

$$\mathcal{H}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ell_1\mathcal{H}(T) + \frac{1}{\Gamma(\rho)}\int_1^{\kappa}(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_1(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s}, \tag{2.6}$$

$$\overline{\mathcal{H}}(\kappa) = \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\ell_2\overline{\mathcal{H}}(T) + \frac{1}{\Gamma(\rho)}\int_1^{\kappa}(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_2(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s}. \tag{2.7}$$

Take the limit on both sides of (2.6) and (2.7) as $\kappa \to T$, we get

$$\mathcal{H}(T)\left(1 - \frac{(\log T)^{w-1}}{\Gamma(w)}\ell_1\right) = \frac{1}{\Gamma(\rho)}\int_1^{T}(\log\frac{T}{s})^{\rho-1}\mathcal{M}_1(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s}$$

and

$$\overline{\mathcal{H}}(T)\left(1 - \frac{(\log T)^{w-1}}{\Gamma(w)}\ell_2\right) = \frac{1}{\Gamma(\rho)}\int_1^{T}(\log\frac{T}{s})^{\rho-1}\mathcal{M}_2(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s},$$

which implies

$$\mathcal{H}(T) = \frac{1}{\left(1 - \frac{(\log T)^{w-1}}{\Gamma(w)}\ell_1\right)}\frac{1}{\Gamma(\rho)}\int_1^{T}(\log\frac{T}{s})^{\rho-1}\mathcal{M}_1(s, \mathcal{H}(s), \overline{\mathcal{H}}(s))\frac{ds}{s} \tag{2.8}$$

and

$$\overline{\mathcal{H}}(T)=\frac{1}{\Big(1-\frac{(\log T)^{w-1}}{\Gamma(w)}\ell_2\Big)}\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\mathcal{M}_2(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\frac{ds}{s}. \quad (2.9)$$

Substitute (2.8) and (2.9) in (2.6) and (2.7) respectively, we get that the coupled system (2.5) is satisfied. By some simple direct computation, we get the converse. This completes the proof. □

3. Main results:

For a Banach space $(U,\|\cdot\|)$, with the norm $\|\mathcal{H}\|=\sup\{|\mathcal{H}(\kappa)|,\ \kappa\in I\}$, the product space $(U\times V,\|(\mathcal{H},\overline{\mathcal{H}})\|)$ with the norm defined by $\|(\mathcal{H},\overline{\mathcal{H}})\|=\|\mathcal{H}\|+\|\overline{\mathcal{H}}\|$, is also a Banach space.

In view of Lemma 2.6, introduce an operator $\mathcal{N}:U\times V\to U\times V$ defined by

$$\mathcal{N}(\mathcal{H},\overline{\mathcal{H}})(\kappa)=\begin{pmatrix}\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa)\\ \mathcal{N}_2(\mathcal{H},\overline{\mathcal{H}})(\kappa)\end{pmatrix},$$

where

$$\begin{aligned}\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa) &= \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\varphi_1\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\frac{ds}{s}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_2(\mathcal{H},\overline{\mathcal{H}})(\kappa) &= \frac{(\log\kappa)^{w-1}}{\Gamma(w)}\varphi_2\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\mathcal{M}_2(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\mathcal{M}_2(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\frac{ds}{s}.\end{aligned}$$

Now we are in a position to present our main results. Our first result is concerned with the uniqueness and the existence of solutions for the problem Hilfer- Hadamard-type coupled system (1.1)-(1.2) and relies on Banach's FPT [9] and Schaefer's FPT [9], we need the following hypotheses

(H$_1$): $\mathcal{M}_i:I\times\ \mathbb{R}^2\to\mathbb{R}$ $(i=1,2)$ are continuous and there exist $l_i>0$ $(i=1,2)$ such that for all $\kappa\in I$ and $\mathcal{H}_i,\overline{\mathcal{H}}_i\in\mathbb{R}$ $i=1,2$, we have

$$\left|\mathcal{M}_1(\kappa,\mathcal{H}_1,\mathcal{H}_2)-\mathcal{M}_1(\kappa,\overline{\mathcal{H}}_1,\overline{\mathcal{H}}_2)\right|\le l_1\sum_{i=1}^2\left|\mathcal{H}_i-\overline{\mathcal{H}}_i\right|,$$

$$\left|\mathcal{M}_2(\kappa,\mathcal{H}_1,\mathcal{H}_2)-\mathcal{M}_2(\kappa,\overline{\mathcal{H}}_1,\overline{\mathcal{H}}_2)\right|\le l_2\sum_{i=1}^{2}\left|\mathcal{H}_i-\overline{\mathcal{H}}_i\right|.$$

(H_2): $\mathcal{M}_l : I\times \mathbb{R}^2\to\mathbb{R}$ $(l=1,2)$ are continuous and there exist real constants $k_i,\ w_i\ge 0,\ i=0,1,2$ and $k_0>0,\ w_0>0$ such that $\forall\mathcal{H}_j\in\mathbb{R}(j=1,2)$, we have

$$\begin{aligned}|\mathcal{M}_1(\kappa,\mathcal{H}_1,\mathcal{H}_2)| &\le k_0+k_1|\mathcal{H}_1|+k_2|\mathcal{H}_2|,\\ |\mathcal{M}_2(\kappa,\mathcal{H}_1,\mathcal{H}_2)| &\le w_0+w_1|\mathcal{H}_1|+w_2|\mathcal{H}_2|.\end{aligned}$$

First, we prove the uniqueness of solution for the HH-type coupled system (1.1)-(1.2) relying on the Banach's FPT [9].

Theorem 3.1. *Assume that (H_1) holds. If*

$$\begin{aligned}\theta_1 := &\left[\frac{\varphi_1}{\Gamma(\rho+w)}+\frac{1}{\Gamma(\rho)}(\log T)^{1-w}\right](\log T)^{\rho}l_1\\ &+\left[\frac{\varphi_2}{\Gamma(\rho+w)}+\frac{1}{\Gamma(\rho)}(\log T)^{1-w}\right](\log T)^{\rho}l_2\\ =&\ (\Lambda_1+\Lambda_2)<1. \qquad (3.1)\end{aligned}$$

Then the HH-type coupled system (1.1)-(1.2) *has a unique solution.*

Proof. Define $\sup_{\kappa\in I}\mathcal{M}_i(\kappa,0,0)=P_i<\infty$, for all $(i=1,2)$, and $r>0$, such that $r\ge\frac{\theta_2}{1-\theta_1}$, where $\theta_1<1$ and

$$\begin{aligned}\theta_2 := &\left[\frac{\varphi_1}{\Gamma(w)}+(\log T)^{1-w}\right]\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}P_1\\ &+\left[\frac{\varphi_2}{\Gamma(w)}+(\log T)^{1-w}\right]\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}P_2\\ =&\ \sigma_1P_1+\sigma_2P_2.\end{aligned}$$

At first, we show that $\mathcal{N}\mathcal{B}_r\subset\mathcal{B}_r$, where

$$\mathcal{B}_r=\{(\mathcal{H},\overline{\mathcal{H}})\in C_{w,\log}(I,\mathbb{R}):\left\|(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}<r\}.$$

Therefore,

$$
\begin{aligned}
&\left|(\log\kappa)^{1-w}\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right| \\
&\quad\le \frac{\varphi_1}{\Gamma(w)}\ \frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s} \\
&\qquad+\ ((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s} \\
&\quad\le \frac{\varphi_1}{\Gamma(w)}\ \frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1} \\
&\qquad\times\Big[\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))-\mathcal{M}_1(s,0,0)\right|+|\mathcal{M}_1(s,0,0)|\Big]\frac{ds}{s} \\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1} \\
&\qquad\times\left[\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))-\mathcal{M}_1(s,0,0)\right|+|\mathcal{M}_1(s,0,0)|\right]\frac{ds}{s} \\
&\quad\le \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\left[l_1\left(|\mathcal{H}(T)|+\left|\overline{\mathcal{H}}(T)\right|\right)+P_1\right]\frac{ds}{s} \\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\left[l_1\left(|\mathcal{H}(\kappa)|+\left|\overline{\mathcal{H}}(\kappa)\right|\right)+P_1\right]\frac{ds}{s} \\
&\quad= \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1} \\
&\qquad\times\left[l_1\left((\log\kappa)^{w-1}(\log\kappa)^{1-w}\{|\mathcal{H}(T)|+\left|\overline{\mathcal{H}}(T)\right|\}\right)+P_1\right]\frac{ds}{s} \\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1} \\
&\qquad\times\left[l_1\left((\log\kappa)^{w-1}(\log\kappa)^{1-w}\{|\mathcal{H}(\kappa)|+\left|\overline{\mathcal{H}}(\kappa)\right|\}\right)+P_1\right]\frac{ds}{s} \\
&\quad= \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1} \\
&\qquad\times\Big[l_1\Big((\log\kappa)^{w-1}\{\|\mathcal{H}\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}\right\|_{C_{w,\log}}\}\Big)+P_1\Big]\frac{ds}{s} \\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1} \\
&\qquad\times\Big[l_1\Big((\log\kappa)^{w-1}\{\|\mathcal{H}\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}\right\|_{C_{w,\log}}\}\Big)+P_1\Big]\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T (\log\frac{T}{s})^{\rho-1}(\log\kappa)^{w-1}\left[l_1\left\|(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}\right]\frac{ds}{s} \\
&\quad + \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T (\log\frac{T}{s})^{\rho-1}P_1\frac{ds}{s} \\
&\quad + ((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa (\log\frac{\kappa}{s})^{\rho-1}(\log\kappa)^{w-1}\left[l_1\left\|(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}\right]\frac{ds}{s} \\
&\quad + ((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa (\log\frac{\kappa}{s})^{\rho-1}P_1\frac{ds}{s} \\
&= \frac{\varphi_1}{\Gamma(\rho+w)}(\log T)^{\rho+w-1}l_1\left\|(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}} + \frac{\varphi_1}{\Gamma(w)}P_1\frac{(\log T)^{\rho-w+1}}{\Gamma(\rho+1)} \\
&\quad + ((\log\kappa)^{1-w})\frac{\Gamma(w)}{\Gamma(\rho+w)}(\log\kappa)^{\rho+w-1}l_1\left\|(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}} \\
&\quad + P_1(\log\kappa)^{1-w}\frac{(\log\kappa)^{\rho}}{\Gamma(\rho+1)} \\
&\leq \left[\frac{\varphi_1}{\Gamma(\rho+w)} + \frac{1}{\Gamma(\rho)}(\log T)^{1-w}\right](\log T)^{\rho}l_1 r \\
&\quad + \left[\frac{\varphi_1}{\Gamma(w)} + (\log T)^{1-w}\right](\log T)^{\rho}\frac{P_1}{\Gamma(\rho+1)} \\
&= \Lambda_1 r + \sigma_1 P_1,
\end{aligned}
$$

where

$$\Lambda_1 = \left[\frac{\varphi_1}{\Gamma(\rho+w)} + \frac{1}{\Gamma(\rho)}(\log T)^{1-w}\right](\log T)^{\rho}l_1$$

and

$$\sigma_1 = \left[\frac{\varphi_1}{\Gamma(w)} + (\log T)^{1-w}\right]\frac{(\log T)^{\rho}}{\Gamma(\rho+1)},$$

which lead to

$$\left\|\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right\|_{Cw,\log} \leq \Lambda_1 r + \sigma_1 P_1.$$

Likewise, we can get that

$$\left\|\mathcal{N}_2(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right\|_{C_{w,\log}} \leq \Lambda_2 r + \sigma_2 P_2.$$

Accordingly,

$$
\begin{aligned}
\left\|\mathcal{N}(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right\|_{C_{w,\log}} &\leq \left\|\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right\|_{C_{w,\log}} + \left\|\mathcal{N}_2(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right\|_{C_{w,\log}} \\
&\leq [\Lambda_1 + \Lambda_2]\, r + \sigma_1 P_1 + \sigma_2 P_2 \\
&\leq \theta_1 r + \theta_2 = r.
\end{aligned}
$$

This proves $\mathcal{N}\mathcal{B}_r \subset \mathcal{B}_r$.

We will prove that $\mathcal{N}$ is contraction mapping.

Indeed, for $(\mathcal{H}_1,\overline{\mathcal{H}}_1),(\mathcal{H}_2,\overline{\mathcal{H}}_2)\in C_{w,\log}\times C_{w,\log}$ and for any $\kappa\in I$, we get

$$
\begin{aligned}
&\left|(\log(\kappa))^{1-w}\mathcal{N}_1(\mathcal{H}_1,\overline{\mathcal{H}}_1)(\kappa)-(\log(\kappa))^{1-w}\mathcal{N}_1(\mathcal{H}_2,\overline{\mathcal{H}}_2)(\kappa)\right|\\
&\quad\le \frac{\varphi_1}{\Gamma(w)}\ \frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\\
&\qquad\times\left|\mathcal{M}_1(s,\mathcal{H}_1(s),\overline{\mathcal{H}}_1(s))-\mathcal{M}_1(s,\mathcal{H}_2(s),\overline{\mathcal{H}}_2(s))\right|\frac{ds}{s}\\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\\
&\qquad\times\left|\mathcal{M}_1(s,\mathcal{H}_1(s),\overline{\mathcal{H}}_1(s))-\mathcal{M}_1(s,\mathcal{H}_2(s),\overline{\mathcal{H}}_2(s))\right|\frac{ds}{s}\\
&\quad\le \frac{\varphi_1}{\Gamma(w)}\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}l_1\\
&\qquad\times\left[|\mathcal{H}_1(s)-\mathcal{H}_2(s)|+\left|\overline{\mathcal{H}}_1(s)-\overline{\mathcal{H}}_2(s)\right|\right]\frac{ds}{s}\\
&\qquad+((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}l_1\\
&\qquad\times\left[|\mathcal{H}_1(s)-\mathcal{H}_2(s)|+\left|\overline{\mathcal{H}}_1(s)-\overline{\mathcal{H}}_2(s)\right|\right]\frac{ds}{s}\\
&\quad=\frac{\varphi_1}{\Gamma(w)}l_1\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}(\log T)^{w-1}\\
&\qquad\times\left[\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right]\frac{ds}{s}\\
&\qquad+((\log\kappa)^{1-w})l_1\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}(\log\kappa)^{w-1}\\
&\qquad\times\left[\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right]\frac{ds}{s}\\
&\quad=l_1\left[\frac{\varphi_1}{\Gamma(\rho+w)}(\log T)^{\rho+w-1}+\frac{\Gamma(w)}{\Gamma(\rho+w)}(\log\kappa)^{\rho}\right]\\
&\qquad\times\left(\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right)\\
&\quad\le l_1\left[\frac{\varphi_1}{\Gamma(\rho+w)}(\log T)^{\rho}+\frac{1}{\Gamma(\rho)}(\log T)^{\rho}\right]\\
&\qquad\times\left(\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right)\\
&\quad=l_1\left[\frac{\varphi_1}{\Gamma(\rho+w)}+\frac{1}{\Gamma(\rho)}\right](\log T)^{\rho}\left(\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right)\\
&\quad=k_1\left(\|\mathcal{H}_1-\mathcal{H}_2\|_{C_{w,\log}}+\left\|\overline{\mathcal{H}}_1-\overline{\mathcal{H}}_2\right\|_{C_{w,\log}}\right),
\end{aligned}
$$

where

$$k_1 = l_1 \left[\frac{\varphi_1}{\Gamma(\rho + w)} + \frac{1}{\Gamma(\rho)} \right] (\log T)^{\rho}.$$

Hence, we get

$$\begin{aligned} &\left\| \mathcal{N}_1(\mathcal{H}_1, \overline{\mathcal{H}}_1) - \mathcal{N}_1(\mathcal{H}_2, \overline{\mathcal{H}}_2) \right\|_{C_{w,\log}} \\ &\quad \leq k_1 \left(\|\mathcal{H}_1 - \mathcal{H}_2\|_{C_{w,\log}} + \left\| \overline{\mathcal{H}}_1 - \overline{\mathcal{H}}_2 \right\|_{C_{w,\log}} \right). \end{aligned} \tag{3.2}$$

In similar fashion,

$$\begin{aligned} &\left\| \mathcal{N}_2(\mathcal{H}_1, \overline{\mathcal{H}}_1) - \mathcal{N}_2(\mathcal{H}_2, \overline{\mathcal{H}}_2) \right\|_{C_{w,\log}} \\ &\quad \leq k_2 \left(\|\mathcal{H}_1 - \mathcal{H}_2\|_{C_{w,\log}} + \left\| \overline{\mathcal{H}}_1 - \overline{\mathcal{H}}_2 \right\|_{C_{w,\log}} \right). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\begin{aligned} &\left\| \mathcal{N}(\mathcal{H}_1, \overline{\mathcal{H}}_1) - \mathcal{N}(\mathcal{H}_2, \overline{\mathcal{H}}_2) \right\|_{C_{w,\log}} \\ &\quad \leq (k_1 + k_2) \left(\|\mathcal{H}_1 - \mathcal{H}_2\|_{C_{w,\log}} + \left\| \overline{\mathcal{H}}_1 - \overline{\mathcal{H}}_2 \right\|_{C_{w,\log}} \right). \end{aligned}$$

Since $\theta_1 < 1$, the operator $\mathcal{N}$ is a contraction mapping, we conclude that the coupled system (1.1)-(1.2) has a unique solution due to Banach's FPT [9]. The proof is completed. □

Next, in order to prove the existence of solution for the HH-type coupled system (1.1)-(1.2) we apply Schaefer's FPT [9].

Theorem 3.2. *Assume that* (H_2) *holds. If*

$$|\sigma_1 k_1 + \sigma_2 w_1| < 1 \quad \textit{and} \quad |\sigma_1 k_2 + \sigma_2 w_2| < 1,$$

then the HH-type coupled system (1.1)-(1.2) *has at least one solution.*

Proof. Initially, we show that $\mathcal{N} : C_{w,\log} \times C_{w,\log} \to C_{w,\log} \times C_{w,\log}$ is a completely continuous mapping. Since $\mathcal{M}_i$ $(i = 1, 2)$ are continuous functions, it is obvious that the operator $\mathcal{N}$ is continuous.

Let $\Upsilon \subset C_{w,\log} \times C_{w,\log}$ be bounded. Then there exist $L_i > 0$ $(i = 1, 2)$ such that for all $(\kappa, \mathcal{H}, \overline{\mathcal{H}}) \in I \times \Upsilon$

$$\left| \mathcal{M}_i(\kappa, \mathcal{H}(\kappa), \overline{\mathcal{H}}(\kappa)) \right| < L_i \quad \text{for}(i = 1, 2).$$

Then, for any $(\mathcal{H},\overline{\mathcal{H}})\in\Upsilon$, we have

$$\begin{aligned}
&\left|(\log(\kappa))^{1-w}\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})(\kappa)\right|\\
&\leq \frac{\varphi_1}{\Gamma(w)}\ \frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s}\\
&\quad+\ ((\log\kappa)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s}\\
&\leq \frac{\varphi_1}{\Gamma(w)}\ L_1\frac{1}{\Gamma(\rho)}\int_1^T(\log\frac{T}{s})^{\rho-1}\frac{ds}{s}\\
&\quad+\ ((\log\kappa)^{1-w})L_1\frac{1}{\Gamma(\rho)}\int_1^\kappa(\log\frac{\kappa}{s})^{\rho-1}\frac{ds}{s}\\
&=\left[\frac{\varphi_1}{\Gamma(w)}(\log T)^{\rho}+\frac{1}{\Gamma(\rho+1)}(\log T)^{\rho+1-w}\right]L_1\\
&=\left[\frac{\varphi_1}{\Gamma(w)}+(\log T)^{1-w}\right]\frac{(\log T)^{\rho}}{\Gamma(\rho+1)}L_1\\
&=\sigma_1L_1
\end{aligned}$$

implying that

$$\left\|\mathcal{N}_1(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}\leq\sigma_1L_1.$$

Analogously, we get

$$\left\|\mathcal{N}_2(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}\leq\sigma_2L_2.$$

It follows from the above inequalities that

$$\left\|\mathcal{N}(\mathcal{H},\overline{\mathcal{H}})\right\|_{C_{w,\log}}\leq\sigma_1L_1+\sigma_2L_2:=\zeta.$$

Which shows that $\mathcal{N}$ is uniformly bounded.

Next, in order to establish $\mathcal{N}$ is an equicontinuous. We take $\kappa_1,\kappa_2\in I$ such that $\kappa_1<\kappa_2$. Then for any $(\mathcal{H},\overline{\mathcal{H}})\in\Upsilon$, we have

$$\begin{aligned}
&\left|(\log(\kappa))^{1-w}\mathcal{N}_1(\mathcal{H}(\kappa_2),\overline{\mathcal{H}}(\kappa_2))-(\log(\kappa))^{1-w}\mathcal{N}_1(\mathcal{H}(\kappa_1),\overline{\mathcal{H}}(\kappa_1)\right|\\
&\leq((\log\kappa_2)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^{\kappa_2}(\log\frac{\kappa_2}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s}\\
&\quad+((\log\kappa_1)^{1-w})\frac{1}{\Gamma(\rho)}\int_1^{\kappa_1}(\log\frac{\kappa_1}{s})^{\rho-1}\left|\mathcal{M}_1(s,\mathcal{H}(s),\overline{\mathcal{H}}(s))\right|\frac{ds}{s}
\end{aligned}$$

$$\leq L_1 \left\{ ((\log \kappa_1)^{1-w}) \frac{1}{\Gamma(\rho)} \int_1^{\kappa_1} \left[(\log \frac{\kappa_2}{s})^{\rho-1} - (\log \frac{\kappa_1}{s})^{\rho-1} \right] \frac{ds}{s} \right.$$
$$\left. + ((\log \kappa_2)^{1-w}) \frac{1}{\Gamma(\rho)} \int_{\kappa_1}^{\kappa_2} (\log \frac{\kappa_2}{s})^{\rho-1} \frac{ds}{s} \right\}$$
$$= (\log \kappa_1)^{1-w} \frac{2L_1}{\Gamma(\rho+1)} (\log \kappa_2 - \log \kappa_1)^{\rho}$$
$$+ ((\log \kappa_2)^{1-w}) \frac{L_1}{\Gamma(\rho+1)} (\log \kappa_2)^{\rho} - (\log \kappa_1)^{\rho}. \tag{3.4}$$

In a similar fashion, we can easily get

$$\left| (\log \kappa)^{1-w} \mathcal{N}_2(\mathcal{H}(\kappa_2), \overline{\mathcal{H}}(\kappa_2)) - (\log \kappa)^{1-w} \mathcal{N}_2(\mathcal{H}(\kappa_1), \overline{\mathcal{H}}(\kappa_1) \right|$$
$$\leq (\log \kappa_1)^{1-w} \frac{2L_2}{\Gamma(\rho+1)} (\log \kappa_2 - \log \kappa_1)^{\rho}$$
$$+ ((\log \kappa_2)^{1-w}) \frac{L_2}{\Gamma(\rho+1)} (\log \kappa_2)^{\rho} - (\log \kappa_1)^{\rho}. \tag{3.5}$$

Since $\log(\kappa)$ is uniformly continuous on I, the right-hand sides of the inequalities (3.4) and (3.5) tend to zero as $\kappa_2 \to \kappa_1$. Thus, the equicontinuity of $\mathcal{N}_1$ and $\mathcal{N}_2$ implies that the operator $\mathcal{N}$ is equicontinuous. Hence, the operator $\mathcal{N}$ is compact by Arzela-Ascoli theorem.

Finally, we establish the boundedness of the set

$$\Omega = \{ (\mathcal{H}, \overline{\mathcal{H}}) \in C_{w,\log} \times C_{w,\log} : (\mathcal{H}, \overline{\mathcal{H}}) = \lambda \mathcal{N}(\mathcal{H}, \overline{\mathcal{H}}), 0 \leq \lambda \leq 1 \}.$$

Let $(\mathcal{H}, \overline{\mathcal{H}}) \in \Omega$, such that $(\mathcal{H}, \overline{\mathcal{H}}) = \lambda \mathcal{N}(\mathcal{H}, \overline{\mathcal{H}})$. Then for any $\kappa \in I$, we have

$$\mathcal{H}(\kappa) = \lambda \mathcal{N}_1(\mathcal{H}, \overline{\mathcal{H}})(\kappa), \quad \overline{\mathcal{H}}(\kappa) = \lambda \mathcal{N}_2(\mathcal{H}, \overline{\mathcal{H}})(\kappa).$$

Assumption (H_2), gives

$$\begin{aligned}
\left| (\log \kappa)^{1-w} \mathcal{H}(\kappa) \right| &= \lambda \left| (\log \kappa)^{1-w} \mathcal{N}_1(\mathcal{H}, \overline{\mathcal{H}})(\kappa) \right| \\
&\leq \left| (\log \kappa)^{1-w} \mathcal{N}_1(\mathcal{H}, \overline{\mathcal{H}})(\kappa) \right| \\
&\leq \left[\frac{\varphi_1}{\Gamma(w)} (\log T)^{\rho} + \frac{1}{\Gamma(\rho+1)} (\log T)^{\rho+1-w} \right] \\
&\quad \times (k_0 + k_1 |\mathcal{H}_1| + k_2 |\mathcal{H}_2|) \\
&= \left[\frac{\varphi_1}{\Gamma(w)} + (\log T)^{1-w} \right] \frac{(\log T)^{\rho}}{\Gamma(\rho+1)} (k_0 + k_1 |\mathcal{H}_1| + k_2 |\mathcal{H}_2|) \\
&= \sigma_1 k_0 + \sigma_1 k_1 |\mathcal{H}_1| + \sigma_1 k_2 |\mathcal{H}_2|,
\end{aligned}$$

implying that

$$\|\mathcal{H}\|_{C_{w,\log}} \leq \sigma_1 k_0 + \sigma_1 k_1 \|\mathcal{H}_1\| + \sigma_1 k_2 \|\mathcal{H}_2\|.$$

In a similar method, we can get

$$\left\|\overline{\mathcal{H}}\right\|_{C_{w,\log}} \leq \sigma_2 w_0 + \sigma_2 w_1 \left\|\mathcal{H}_1\right\| + \sigma_2 w_2 \left\|\mathcal{H}_2\right\|.$$

Collecting the above norms, we find that

$$\|\mathcal{H}\| + \left\|\overline{\mathcal{H}}\right\| \leq \sigma_1 k_0 + \sigma_2 w_0 + (\sigma_1 k_1 + \sigma_2 w_1) \|\mathcal{H}_1\| + (\sigma_1 k_2 + \sigma_2 w_2) \|\mathcal{H}_2\|,$$

which implies

$$\left\|\mathcal{H} + \overline{\mathcal{H}}\right\| \leq \|\mathcal{H}\| + \left\|\overline{\mathcal{H}}\right\| \leq \frac{\sigma_1 k_0 + \sigma_2 w_0}{\chi_0},$$

where $\chi_0 = \min\left\{1 - (\sigma_1 k_1 + \sigma_2 w_1), 1 - (\sigma_1 k_2 + \sigma_2 w_2)\right\}$.

In consequence Ω is bounded, Thus, the conclusion of Schaefer's FPT [9] applies and hence the operator $\mathcal{N}$ has at least one fixed point, which is indeed a solution of coupled system (1.1)-(1.2). □

Example 3.3. Let $\rho = \upsilon = \frac{1}{2}$, $\ell_1 = \frac{4}{5}$, $\ell_2 = \frac{1}{2}$,

$$\begin{aligned} \mathcal{M}_1(\kappa, \mathcal{H}, \overline{\mathcal{H}}) &= \frac{1}{2(\kappa+2)^2} \frac{|\mathcal{H}(\kappa)|}{1 + |\mathcal{H}(\kappa)|} + 1 \\ &\quad + \frac{1}{16} \sin^2(\overline{\mathcal{H}}(\kappa)) + \frac{1}{\sqrt{\kappa^2+1}} \end{aligned}$$

and

$$\mathcal{M}_2(\kappa, \mathcal{H}, \overline{\mathcal{H}}) = \frac{1}{16\pi} \sin(2\pi \mathcal{H}(\kappa)) + \frac{\left|\overline{\mathcal{H}}(\kappa)\right|}{8\left(1 + \left|\overline{\mathcal{H}}(\kappa)\right|\right)} + \frac{1}{2}.$$

We consider the coupled Hadamard type FDEs

$$\begin{cases} \mathfrak{D}_1^{\frac{1}{2}, \frac{1}{2}} \mathcal{H}(\kappa) = \mathcal{M}_1(\kappa, \mathcal{H}, \overline{\mathcal{H}}), & \kappa \in [1, e], \\ \mathfrak{D}_1^{\frac{1}{2}, \frac{1}{2}} \overline{\mathcal{H}}(\kappa) = \mathcal{M}_2(\kappa, \mathcal{H}, \overline{\mathcal{H}}), & \kappa \in [1, e], \end{cases} \tag{3.6}$$

with the coupled integral boundary conditions

$$\begin{cases} \mathcal{I}_1^{\frac{1}{2}, \frac{1}{2}} \mathcal{H}(1) = \frac{4}{5} \overline{\mathcal{H}}(e), \\ \mathcal{I}_1^{\frac{1}{2}, \frac{1}{2}} \overline{\mathcal{H}}(1) = \frac{1}{2} \mathcal{H}(e). \end{cases} \tag{3.7}$$

For $\kappa \in [1, e]$ and $\mathcal{H}, \mathcal{H}^*, \overline{\mathcal{H}}, \overline{\mathcal{H}}^* \in \mathbb{R}^+$,

$$\left|\mathcal{M}_1(\kappa, \mathcal{H}, \mathcal{H}^*) - \mathcal{M}_1(\kappa, \overline{\mathcal{H}}, \overline{\mathcal{H}}^*)\right| \leq \frac{1}{8} \left|\mathcal{H} - \overline{\mathcal{H}}\right| + \frac{1}{8} \left|\mathcal{H}^* - \overline{\mathcal{H}}^*\right|$$

and

$$\left|\mathcal{M}_2(\kappa, \mathcal{H}, \mathcal{H}^*) - \mathcal{M}_2(\kappa, \overline{\mathcal{H}}, \overline{\mathcal{H}}^*)\right| \leq \frac{1}{8} \left|\mathcal{H} - \overline{\mathcal{H}}\right| + \frac{1}{8} \left|\mathcal{H}^* - \overline{\mathcal{H}}^*\right|.$$

Hence the condition (H$_1$) holds with $l_1 = l_2 = \frac{1}{8}$. $\sup_{\kappa \in [1,e]} \mathcal{M}_1(\kappa, 0, 0) = 1 + \frac{1}{\sqrt{2}} = P_1 < \infty$ and $\sup_{\kappa \in [1,e]} \mathcal{M}_2(\kappa, 0, 0) = \frac{1}{2} = P_2 < \infty$. We shall check that condition (3.1) holds, indeed, by some simple calculations we find that

$\varphi_2 = 0.844\,63, \varphi_1 = 2.\,304\,4$. With the given data, we see that $\theta_1 = 0.575\,32 < 1$. Therefore, by Theorem 3.1, we conclude that problem (3.6)-(3.7) has a unique solution.

4. Conclusion

We have discussed the essential requirements for boundary value problems for a coupled system of HH-type FDEs with coupled integral boundary conditions. Through the application of tools from fixed-point theory, we have obtained novel findings that contribute to and generalize the existing literature on this topic. Our results encompass various new results as special cases, making a significant contribution to the field of boundary value problems associated with HH-type FDEs. Future work in this area could involve exploring the stability properties of the solutions obtained, investigating numerical methods for solving the coupled system, and extending the analysis to more complex scenarios or higher dimensions. Additionally, it would be valuable to explore applications of the obtained results in diverse fields, such as physics, engineering, and biology, to further demonstrate the practical significance of the findings.

References

[1] S. Abbas, M. Benchohra, N. Hamidi and Y. Zhou, *Implicit coupled Hilfer-Hadamard fractional differential systems under weak topologies*, Adv. Diff. Equ., **2018**(1) (2018), 1–17.

[2] B.N. Abood, S.S. Redhwan, O. Bazighifan and K. Nonlaopon, *Investigating a generalized fractional quadratic integral equation*. Fractal and Fractional, **6**(5) (2022), 251.

[3] Y. Adjabi, F. Jarad, D. Baleanu and T. Abdeljawad, *On Cauchy problems with Caputo Hadamard fractional derivatives*, J. Comput. Anal. Appl., **21**(1) (2016), 661–681.

[4] B. Ahmad, SK. Ntouyas and A. Alsaedi, *On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions*, Chaos Solitons Fractals, **83** (2016), 234–241.

[5] H.H. Alsulami, S.K. Ntouyas, R.P. Agarwal, B. Ahmad and A. Alsaedi, *A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions*, Boundary Value Prob., **2017**(1) (2017), 68.

[6] N. Anakira, Z. Chebana, T-E. Oussaeif, I. M. Batiha and A. Ouannas, *A study of a weak solution of a diffusion problem for a temporal fractional differential equation* , Nonlinear Funct. Anal. Appl., **27** (3) (2022), 679-689.

[7] D.B. Dhaigude and S.P. Bhairat, *Local Existence and Uniqueness of Solution for Hilfer-Hadamard fractional differential problem*, Nonlinear Dyn. Syst. Theory, **18**(2) (2018), 144–153.

[8] Y. Gambo, F. Jarad, D. Baleanu and F. Jarad, *On Caputo modification of the Hadamard fractional derivatives*, Adv. Diff. Equ., **10** (2014), 1–12.

[9] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York 2005.

[10] M. M. A. Hasan and S. A. Alkhatibd, *Analytic Solution of High Order Fractional Boundary Value Problems* , Nonlinear Funct. Anal. Appl., **28** (3) (2023), 601-612.

[11] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier, Amsterdam 2006.

[12] S. Liang and J. Zhang, *Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval*, Math. Comput. Model., **54** (2011), 1334–1346.

[13] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[14] L. Palve, M.S. Abdo and S.K. Panchal, *Some existence and stability results of Hilfer-Hadamard fractional implicit differential fractional equation in a weighted space*, preprint: arXiv, arXiv:1910.08369v1, math.GM, 2019.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[16] M.D. Qassim, K.M. Furati and N.E. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative*, Abstr. Appl. Anal., **2012**, Article ID 391062 (2012).

[17] M.D. Qassim and N.E. Tatar, *Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative*. Abstr. Appl. Anal., **2013**, Article ID 605029 (2013).

[18] S. Redhwan and S. Shaikh, *Implicit fractional differential equation with nonlocal integral-multipoint boundary conditions in the frame of Hilfer fractional derivative*, J. Math. Anal. Model., **2**(1) (2021), 62–71.

[19] S. Redhwan and S. Shaikh, *Analysis of implicit type of a generalized fractional differential equations with nonlinear integral boundary conditions*, J. Math. Anal. Model., **1**(1) (2020), 64–76.

[20] S. Redhwan, S. Shaikh and A.B.D.O. Mohammed, *Caputo-Katugampola-type implicit fractional differential equation with anti-periodic boundary conditions*, Results Nonlinear Anal., **5**(1), (2022), 12–28.

[21] S. Redhwan, A.M. Suad, S. Shaikh and A.B.D.O. Mohammed, *A coupled non-separated system of Hadamard-type fractional differential equations*, Adv. Theory Nonlinear Anal. Appl., **6**(1) (2021), 33–44.

[22] J. Sabatier, O.P. Agrawal and JAT (eds.) Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht 2007.

[23] W. Saengthong, E. Thailert and S.K. Ntouyas, *Existence and uniqueness of solutions for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions*, Adv. Diff. Equ., **2019**(1) (2019), 525.

[24] H. Wang, *Existence of solutions for fractional anti-periodic BVP*, Results Math., **68** (2015), 227–245.

[25] J.R. Wang, Y. Zhou and M. Feckan, *On recent developments in the theory of boundary value problems for impulsive fractional differential equations*, Comput. Math. Appl., **64** (2012), 3008–3020.