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FIXED POINT THEOREMS OF EXTENSION AND MODIFIED EXTENSION α-F-CONTRACTION ON COMPLETE METRIC SPACE

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Abstract. The concept of an extension α-F-contraction and it's modified counterpart represents an advancement in the theory of metric space contractions. Through our study of the contraction principles and it's relationship to extension and modified extension, we found different conditions somewhat lengthy. In our paper, we create a development of the conditions for the extension of α-F-contraction and a modified α-F-contraction by reducing the conditions and make them easier. Our propose conditions are notably simple and effective. They serve as the foundation for proving theorems and solving examples that belong to our study. Moreover, they have remarkable significance in the condition of mathematical analysis and problem-solving. Thus, we find that these new conditions that we mention in the definitions achieve what is require and through them, we choose $\lambda = 1$ and we choose $\lambda \in (0, 1)$ to clarify our ideas.

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1. Introduction

The fixed point theorems, the metric space and the metric contractions have been studied by many researchers. Many concepts and theories have been developed in this aspect. Moreover, several mathematicians have studied how these concepts can be used in different practical situations. Some important principles of Banach contractions as fixed point functional analysis on metric space were presented by researchers, see the results ([2, 3, 9, 11, 12, 13, 17, 23]) and examples ([4, 5, 6, 8, 10, 15, 19, 20, 21, 22, 24]).

In this study we discuss the extension and modified α-F-contraction by presenting some concepts and examples with the help of some previous studies of researchers such as Lakshmi Kanta day, Boom Kumam and Tanusri Senapati in their research article that was published in 2018, as it's shown in the definitions 2.5, 2.6.

In this research paper we reach the ideas with a brief base also we choose a suitable title for this extension and it is apply to the examples addressed by the aforementioned researchers and we get the correct results see definitions, theorems and examples in ([7]).

For more information about the F-contraction and F-weak contraction in the examples we mention in main results, see ([1]).

2. Preliminaries

In this paper different groups of numbers will be symbolized as follows:

- $\mathbb{R}^+$=$(0, +\infty)$ be the set of positive real numbers;
- $\mathbb{R}$=$(-\infty, +\infty)$ be the set of real numbers;
- $\mathbb{N}^+ \cup \{0\}$=$[0, +\infty)$ be the set of non-negative integer numbers;
- $\mathbb{N}$=$[1, +\infty)$ be the set of natural numbers.

At the beginning of our article, we show some definitions which related to our study.

Definition 2.1. ([18]) A self-map $\Omega : X \to X$ is called an α-admissible, if there exists $\alpha : X \times X \to \mathbb{R}^+$ such that

$$\alpha(\varrho, \varsigma) \geq 1 \quad \Rightarrow \quad \alpha(\Omega(\varrho), \Omega(\varsigma)) \geq 1, \ \forall\, \varrho, \varsigma \in X. \tag{2.1}$$

Definition 2.2. ([16]) A self-map $\Omega : X \to X$ be called a triangular α-admissible, if there exists $\alpha : X \times X \to \mathbb{R}^+$ such that

(1) $\alpha(\varrho, \varsigma) \geq 1 \quad \Rightarrow \quad \alpha(\Omega(\varrho), \Omega(\varsigma)) \geq 1, \ \forall\, \varrho, \varsigma \in X$;

(2) $\alpha(\varrho, \varsigma) \geq 1, \alpha(\varsigma, \zeta) \geq 1 \quad \Rightarrow \quad \alpha(\varrho, \zeta) \geq 1, \forall\, \varrho, \varsigma, \zeta \in X$.

Note: ([16]) Suppose that Ω is a triangular α-admissible map, if $\{\varrho_n\}$ is any sequence defined by $\Omega(\varrho_n) = \varrho_{n+1}$ and $\alpha(\varrho_n, \varrho_{n+1}) \geq 1$, then for all $n, m \in \mathbb{N}$, we have $\alpha(\varrho_n, \varrho_m) \geq 1$.

Definition 2.3. ([14]) Let a self-map $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to \{-\infty\} \cup \mathbb{R}^+$, then Ω is called α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(d(\varrho, \varsigma)). \tag{2.2}$$

Definition 2.4. ([14]) Let a self-map $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to \{-\infty\} \cup \mathbb{R}^+$, then Ω is called α-F-weak contraction, if there exists $\tau > 0$ such that for all $\varrho, \varsigma \in X$,

$$\begin{aligned} \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\max\{& d(\varrho, \varsigma), d(\varrho, \Omega(\varrho)), d(\varsigma, \Omega(\varsigma)), \\ & \frac{d(\varrho, \Omega(\varsigma)) + d(\varsigma, \Omega(\varrho))}{2}\}). \end{aligned} \tag{2.3}$$

Note: F in the above definition belongs to the family of all contraction mappings $\mathcal{F} = \{F / F : \mathbb{R}^+ \to \mathbb{R}\}$ is achieved the following conditions:

- (F1) F is strictly increasing function, that is, $\alpha, \beta \in (0, +\infty)$ with $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
- (F2) every sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F3) there exists $r \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^r F(\alpha) = 0$.

Definition 2.5. ([13]) Let a self-map $\Omega : X \to X$ be defined on a set of a metric space (X, d), then Ω is called a modified generalized α-F-contraction of type (A), if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{N}_\Omega(\varrho, \varsigma)), \tag{2.4}$$

where

$$\begin{aligned} \mathcal{N}_\Omega(\varrho, \varsigma) = \max\{ & d(\varrho, \varsigma), \frac{d(\varrho, \Omega(\varsigma)) + d(\varsigma, \Omega(\varrho))}{2}, \frac{d(\Omega^2(\varrho), \varrho) + d(\Omega^2(\varrho), \Omega(\varsigma))}{2}, \\ & d(\Omega^2(\varrho), \Omega(\varrho)), d(\Omega^2(\varrho), \varsigma), d(\Omega(\varrho), \varsigma) + d(\varsigma, \Omega(\varsigma)), \\ & d(\Omega^2(\varrho), \Omega(\varsigma)) + d(\varrho, \Omega(\varrho))\} \end{aligned} \tag{2.5}$$

and F is satisfies the following conditions:

- (a) F is continuous;
- (b) F is strictly increasing.

Definition 2.6. ([7]) Let $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is called a generalized α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{M}_\Omega(\varrho, \varsigma)), \tag{2.6}$$

where

$$\begin{aligned}\mathcal{M}_\Omega(\varrho, \varsigma) = \max\{&d(\varrho, \varsigma), d(\varrho, \Omega(\varrho)), d(\varsigma, \Omega(\varsigma)), \frac{d(\varrho, \Omega(\varsigma)) + d(\varsigma, \Omega(\varrho))}{2},\\ &\frac{d(\Omega^2(\varrho), \varrho) + d(\Omega^2(\varrho), \Omega(\varsigma))}{2}, d(\Omega^2(\varrho), \Omega(\varrho)),\\ &d(\Omega^2(\varrho), \varsigma), d(\Omega^2(\varrho), \Omega(\varsigma))\}.\end{aligned} \tag{2.7}$$

Note: In this article, we use the concept of extension instead of generalized because generalized α-F-contractions and modified generalized α-F-contractions provide valuable extensions to traditional contraction mapping principles, enabling a more flexible and adaptable framework for studying fixed points in metric spaces.

3. Main results

In the realm of fixed point theory an extension α-F-contractions and a modified extension α-F-contractions are powerful extensions of traditional contraction mapping principles. These concepts have proved to be essential tools in the analysis of fixed points in metric spaces, offering a more flexible and adaptable framework.

In this article, we delve into the definitions, key properties of an extension α-F-contractions, a modified extension α-F-contractions and explore their implications in fixed point theory. Let us begin by examining the concept of extension α-F-contraction as follows:

Definition 3.1. Let a self-map $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is called an extension α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{M}_\Omega(\varrho, \varsigma)), \tag{3.1}$$

where

$$\begin{aligned}\mathcal{M}_\Omega(\varrho, \varsigma) = \max\{&d(\varrho, \varsigma), d(\varrho, \Omega(\varsigma)), d(\varsigma, \Omega(\varrho)), \frac{d(\varrho, \Omega(\varrho)) + d(\varsigma, \Omega(\varsigma))}{\lambda}\},\\ &\lambda \in (0, 1].\end{aligned} \tag{3.2}$$

Note: Intuitively an extension α-F-contraction can be understood as a mapping that gradually shrinks the distances between points in the metric space, with the rate of shrinking determined by the function α. This extension allows for greater adaptability in addressing various types of contractions as different choices of α can correspond to different contraction rates. Building upon the concept of extension α-F-contractions a modified extension α-F-contractions introduce into the framework.

Definition 3.2. Let a self-map $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is called a modified extension α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{N}_\Omega(\varrho, \varsigma)), \tag{3.3}$$

where

$$\mathcal{N}_\Omega(\varrho, \varsigma) = \max\{d(\varrho, \varsigma), d(\varrho, \Omega(\varrho)), d(\varsigma, \Omega(\varsigma)), \frac{d(\varrho, \Omega(\varrho)) + d(\varsigma, \Omega(\varrho))}{\lambda}\}, \quad \lambda \in (0, 1]. \tag{3.4}$$

Note: In particular, we take $\lambda = 1$ in equations (3.2), (3.4), we obtain the following definitions:

Definition 3.3. Let $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is called extension α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{M}_\Omega(\varrho, \varsigma)), \tag{3.5}$$

where

$$\mathcal{M}_\Omega(\varrho, \varsigma) = \max\{d(\varrho, \varsigma), d(\varrho, \Omega(\varsigma)), d(\varsigma, \Omega(\varrho)), d(\varrho, \Omega(\varrho)) + d(\varsigma, \Omega(\varsigma))\}. \tag{3.6}$$

Definition 3.4. Let a self map $\Omega : X \to X$ be defined on a set of a metric space (X, d) and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is called a modified extension α-F-contraction, if there exists $\tau > 0$ such that

$$\forall\ \varrho, \varsigma \in X,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{N}_\Omega(\varrho, \varsigma)), \tag{3.7}$$

where

$$\mathcal{N}_\Omega(\varrho, \varsigma) = \max\{d(\varrho, \varsigma), d(\varrho, \Omega(\varrho)), d(\varsigma, \Omega(\varsigma)), d(\varrho, \Omega(\varrho)) + d(\varsigma, \Omega(\varrho))\}. \tag{3.8}$$

Theorem 3.5. *Suppose that $\Omega : X \to X$ is a self-map and (X, d) is α-complete metric space and let a function $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is a modified extension α-F-contraction, if the function Ω follows these conditions*:

(a) Ω *is α-admissible and α-continuous map*;

(b) *there exists* $\varrho_0 \in X$ *and* $\alpha(\varrho_0, \Omega(\varrho_0)) \geq 1$*, so that* Ω *has a unique fixed point.*

Proof. Since, there exists $\varrho_0 \in X$ and $\alpha(\varrho_0, \Omega(\varrho_0)) \geq 1$, we define the sequence $\{\varrho_j\}$ by $\varrho_{j+1} = \Omega(\varrho_j)$, for all $j \in \mathbb{N}_0$, if there exists $j \in \mathbb{N}$, $\varrho_j = \Omega(\varrho_j)$, then ϱ_j is a fixed point of Ω, so that the proof is completed.

Now, let there exists j for $\varrho_j = \Omega(\varrho_j)$, $\alpha(\varrho_0, \Omega(\varrho_0)) \geq 1 \Rightarrow \alpha(\varrho_0, \varrho_1) \geq 1$, when Ω is α-admissible map for all $j \in \mathbb{N}_0$, we have $\alpha(\varrho_j, \varrho_{j+1}) \geq 1$ as $d(\Omega(\varrho_{j-1}), \Omega(\varrho_j)) > 0$ and Ω is a modified extension α-F-contraction, there is $\tau > 0$ and we get

$$\begin{aligned} F(d(\varrho_j, \varrho_{j+1})) &= F(d(\Omega(\varrho_{j-1}), \Omega(\varrho_j))) \\ &\leq \tau + \alpha(\varrho_{j-1}, \varrho_j) F(d(\Omega(\varrho_{j-1}), \Omega(\varrho_j))) \\ &\leq F(\mathcal{N}_\Omega(\varrho_{j-1}, \varrho_j)). \end{aligned} \tag{3.9}$$

Now, we have

$$\begin{aligned} \mathcal{N}_\Omega(\varrho_{j-1}, \varrho_j) &= \max\{d(\varrho_{j-1}, \varrho_j), d(\varrho_{j-1}, \Omega(\varrho_{j-1})), d(\varrho_j, \Omega(\varrho_j)), \\ &\qquad d(\varrho_{j-1}, \Omega(\varrho_{j-1}) + d(\varrho_j, \Omega(\varrho_{j-1}))\} \\ &= \max\{d(\varrho_{j-1}, \varrho_j), d(\varrho_{j-1}, \varrho_j), d(\varrho_j, \varrho_{j+1}), d(\varrho_{j-1}, \varrho_j) + d(\varrho_j, \varrho_j)\} \\ &= \max\{d(\varrho_{j-1}, \varrho_j), d(\varrho_j, \varrho_{j+1})\}. \end{aligned}$$

If $\max\{d(\varrho_{j-1}, \varrho_j), d(\varrho_j, \varrho_{j+1})\} = d(\varrho_j, \varrho_{j+1})$, then (2.7) proves that

$$\tau + \alpha(\varrho_{j-1}, \varrho_j) F(d(\varrho_j, \varrho_{j+1})) \leq F(d(\varrho_j, \varrho_{j+1})),$$

which is a contradiction, so that

$$\max\{d(\varrho_{j-1}, \varrho_j), d(\varrho_j, \varrho_{j+1})\} = d(\varrho_{j-1}, \varrho_j).$$

Therefore, by (3.7), (3.8) we get

$$\begin{aligned} F(d(\varrho_j, \varrho_{j+1})) &\leq \alpha(\varrho_{j-1}, \varrho_j) F(d(\varrho_j, \varrho_{j+1})) \\ &\leq F(d(\varrho_{j-1}, \varrho_j)) - \tau, \end{aligned}$$

as $\tau > 0$

$$d(\varrho_j, \varrho_{j+1}) < d(\varrho_{j-1}, \varrho_j). \tag{3.10}$$

This shows that $\{\varrho_j\}$ is a decreasing sequence of non-negative real number.

We claim that

$$\lim_{j \to \infty} d(\varrho_{j+1}, \varrho_j) = 0.$$

If possible for some $\delta > 0$

$$\lim_{j \to \infty} d(\varrho_{j+1}, \varrho_j) = \delta.$$

Now, for all $j \in \mathbb{N}$, we obtain $d(\varrho_j, \varrho_{j+1}) \geq \delta$, by (F1) and (3.8) we get

$$\begin{aligned} F(\delta) \leq F(d(\varrho_j, \varrho_{j+1})) &\leq \alpha(\varrho_{j+1}, \varrho_j) F(d(\varrho_j, \varrho_{j+1})) \\ &< F(d(\varrho_{j-1}, \varrho_j)) - \tau \\ &< F(d(\varrho_{j-2}, \varrho_{j-1})) - 2\tau \\ &\vdots \\ &< F(d(\varrho_0, \varrho_1)) - j\tau, \end{aligned} \tag{3.11}$$

as $\lim_{j\to\infty} (F(d(\varrho_0, \varrho_1)) - j\tau) = -\infty$ so that we get some $i \in \mathbb{N}$ and

$$F(d(\varrho_0, \varrho_1)) - j\tau < F(\delta), \ \forall\ j > i,$$

which is a contradiction too. So, $\lim_{j\to\infty} d(\varrho_j, \varrho_{j+1}) = 0$.

Next, we claim $\{\varrho_j\}$ is a Cauchy sequence by (F3), there exists $r \in (0,1)$ and

$$\lim_{j\to\infty} (\alpha_j)^r F(\alpha_j) = 0, \tag{3.12}$$

where $\lim_{j\to\infty} \alpha_j = \lim_{j\to\infty} (\varrho_j, \varrho_{j+1}) = 0$.

Again, from (3.9), (3.10) we can get

$$\lim_{j\to\infty} (\alpha_j)^r (F(\alpha_j) - F(\alpha_0)) \leq \lim_{j\to\infty} (\alpha_j)^r \cdot j\tau \leq 0, \tag{3.13}$$

it implies that

$$\lim_{j\to\infty} j(\alpha_j)^r = 0, \ as\ \tau > 0. \tag{3.14}$$

Also, we can get some $j_0 \in \mathbb{N}$ and $j(\alpha_j^r) \leq 1$, for all $j \geq j_0$ such that

$$\alpha_j \leq \frac{1}{j^{\frac{1}{r}}}, \ \forall\ j \geq j_0 \tag{3.15}$$

by (3.13), for all $(i > j > j_0)$ we get

$$\begin{aligned} d(\varrho_j, \varrho_i) \leq d(\varrho_j, \varrho_{j+1}) + d(\varrho_{j+1}, \varrho_{j+2}) + \ldots + d(\varrho_{i-1}, \varrho_i) < \sum_{s=1}^{\infty} \alpha_s \\ \leq \sum_{s=1}^{\infty} \frac{1}{s^{\frac{1}{r}}}, \ as\ \frac{1}{r} > 0 \end{aligned}$$

so that the above series is convergent. Therefore, $\lim_{j,i\to\infty} d(\varrho_j, \varrho_i) = 0$ such that (X, d) is α-complete metric space also $\{\varrho_j\}$ be a Cauchy sequence with $\alpha(\varrho_j, \varrho_{j+1}) \geq 1$, for all $j \geq \mathbb{N}$ we get $\varrho_j \in X$ and $\varrho_j \to \varrho$ since $j \to \infty$.

We claim that ϱ is a unique fixed point of Ω.

Since, $\varrho_j \to \varrho$ as $j \to \infty$ and $\alpha(\varrho_j, \varrho_{j+1}) \geq 1$, for all $j \geq \mathbb{N}_0$, the α-continuity property of Ω implies that $\Omega(\varrho_j) \to \Omega(\varrho)$ as $j \to \infty$.

Finally, we have $\varrho_{j+1} = \Omega(\varrho_j)$, $\lim_{j\to\infty} \varrho_{j+1} = \lim_{j\to\infty} \Omega(\varrho_j)$ this means $\varrho = \Omega(\varrho)$, so that ϱ be a unique fixed point of Ω. □

Theorem 3.6. *Suppose that $\Omega : X \to X$ is a self-map and let (X, d) be α-complete metric space, let $\alpha : X \times X \to [0, +\infty)$, $F \in \mathcal{F}$, then Ω is a modified extension α-F-contraction, if satisfies the following conditions*:

(a) *Ω is α-admissible*;
(b) *there exists $\varrho_0 \in X$ with $\alpha(\varrho_0, \Omega(\varrho_0)) \geq 1$;*
(c) *$\{\varrho_j\} \subset X$ is a sequence with $\alpha(\varrho_j, \varrho_{j+1}) \geq 1$, for all $j \in \mathbb{N}_0$ and $\varrho_j \to \varrho$ as $j \to \infty$ we get $\alpha(\varrho_j, \varrho) \geq 1$, for all $j \in \mathbb{N}_0$, so that Ω has a unique fixed point.*

Proof. Following the theorem's proof (3.9). Thus, $\{\varrho_j\}$ is a Cauchy sequence with $\alpha(\varrho_j, \varrho_{j+1}) \geq 1$, for all $j \in \mathbb{N}_0$ and it convergent to the some point $\varrho \in X$ by (c), we have $\alpha(\varrho_j, \varrho) \geq 1$, for all $j \in \mathbb{N}_0$.

We claim that ϱ is a fixed point of Ω. On the other hand, assume that $\Omega(\varrho) \neq \varrho$ and $d(\varrho, \Omega(\varrho)) > 0$ and we can get $j \in \mathbb{N}$ such that $d(\varrho_i, \Omega(\varrho)) > 0$, for all $(i \geq n)$ this implies that $d(\varrho_{i-1}, \Omega(\varrho)) > 0$. So, based on the theorem's condition and F's property, we can get some $\tau > 0$ and

$$\tau + \alpha(\varrho_{i-1}, \varrho) F(d(\Omega(\varrho_{i-1}), \Omega(\varrho))) < F(\mathcal{N}_\Omega(\varrho_{i-1}, \varrho)),$$

as $\alpha(\varrho_{i-1}, \varrho) \geq 1$, $\tau > 0$,

$$d(\Omega(\varrho_{i-1}), \Omega(\varrho)) < \mathcal{N}_\Omega(\varrho_{i-1}, \varrho) \Rightarrow \lim_{i\to\infty} d(\varrho_i, \Omega(\varrho)) < \lim_{i\to\infty} \mathcal{N}_\Omega(\varrho_{i-1}, \varrho). \tag{3.16}$$

Now, we get

$$\begin{aligned}\mathcal{N}_\Omega(\varrho_{i-1}, \varrho) &= \max\{d(\varrho_{i-1}, \varrho), d(\varrho_{i-1}, \Omega(\varrho_{i-1})), d(\varrho, \Omega(\varrho)),\\ &\qquad d(\varrho_{i-1}, \Omega(\varrho_{i-1}) + d(\varrho, \Omega(\varrho_{i-1}))\}\\ &= \max\{d(\varrho_{i-1}, \varrho), d(\varrho_{i-1}, \varrho_i), d(\varrho, \Omega(\varrho)), d(\varrho_{i-1}, \varrho_i) + d(\varrho, \varrho_i)\}.\end{aligned}$$

When we compensate the above equality, we get

$$\lim_{i\to\infty} d(\varrho_i, \Omega(\varrho)) < \max\{d(\varrho, \varrho), d(\varrho, \Omega(\varrho))\},$$

which is a contradiction. So, we must have $d(\varrho, \Omega(\varrho)) = 0$ such that ϱ is a unique fixed point of Ω. □

Theorem 3.7. *Let Ω be a modified extension α-F-contraction. If Ω has two fixed points $\varrho, \varsigma \in X$, $\alpha(\varrho, \varsigma) \geq 1$, then $\varrho = \varsigma$.*

Proof. For given $\varrho, \varsigma \in Fix(\Omega)$ if $\varrho \neq \varsigma$ then $\Omega(\varrho) \neq \Omega(\varsigma)$ and $d(\Omega(\varrho), \Omega(\varsigma)) > 0$, for all $j \in \mathbb{N}$, if $\Omega^j(\varrho) = \varrho$, $\Omega^j(\varsigma) = \varsigma$, then Ω is α-F-contraction.
So,

$$\exists\, \tau > 0,\ d(\Omega(\varrho), \Omega(\varsigma)) > 0 \Rightarrow \tau + \alpha(\varrho, \varsigma) F(d(\Omega(\varrho), \Omega(\varsigma))) \leq F(\mathcal{N}_\Omega(\varrho, \varsigma)),$$

or

$$\tau+\alpha(\varrho,\varsigma)F(d(\Omega(\varrho),\Omega(\varsigma)))<F(d(\varrho,\varsigma)),$$

therefore

$$F(d(\Omega(\varrho),\Omega(\varsigma)))<F(d(\varrho,\varsigma)),$$

as $\alpha(\varrho,\varsigma)\geq 1$, $\tau>0$ we obtain $F(d(\varrho,\varsigma))<F(d(\varrho,\varsigma))$ that is a contradiction, so $\varrho=\varsigma$. □

Example 3.8. Let $X=\{-2,-1,0,1,2\}$,

$$d(\varrho,\varsigma)=\begin{cases}0, & \text{iff } \varrho=\varsigma,\\ \frac{2}{3}, & (\varrho,\varsigma)\in\{(-2,1),(1,-2)\},\\ \frac{1}{4}, & \text{otherwise.}\end{cases}$$

Let $\Omega:X\to X$ is define as the following:

$$\Omega(-2)=\Omega(1)=-1,\ \ \Omega(-1)=\Omega(2)=1,\ \ \Omega(0)=0,$$

$d(\Omega(\varrho),\Omega(\varsigma))>0$, therefore

$$[\varrho\in\{-2,1\}\wedge\varsigma\in\{-1,2\};\ \varrho\in\{-2,1\}\wedge\varsigma=0;\ \varrho\in\{-1,2\}\wedge\varsigma=0].$$

Now, we find $\mathcal{N}_\Omega(\varrho,\varsigma)$, there are many cases:

Case (1):

Let $\varrho\in\{-2,1\}$; $\varsigma\in\{-1,2\}$. So that $(\varrho,\varsigma)\in\{(-2,-1),(-2,2),(1,-1),(1,2)\}$. Thus, we conclude that $d(\Omega(\varrho),\Omega(\varsigma))=d(-1,1)=\dfrac{1}{4}$.

Now, if $(\varrho,\varsigma)=(-2,-1)$, then

$$\begin{aligned}\mathcal{N}_\Omega(-2,-1)&=\max\{d(-2,-1),d(-2,\Omega(-2)),d(-1,\Omega(-1)),\\&\qquad d(-2,\Omega(-2))+d(-1,\Omega(-2))\}\\&=\max\{d(-2,-1),d(-2,-1),d(-1,1),d(-2,-1)+d(-1,-1)\}\\&=\max\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\}=\frac{1}{4}.\end{aligned}$$

But

$$\begin{aligned}\mathcal{M}_\Omega(-2,-1)&=\max\{d(-2,-1),d(-2,\Omega(-1)),d(-1,\Omega(-2)),\\&\qquad d(-2,\Omega(-2))+d(-1,\Omega(-1))\}\\&=\max\{d(-2,-1),d(-2,1),d(-1,-1),d(-2,-1)+d(-1,1)\}\\&=\max\{\frac{1}{4},\frac{2}{3},0,\frac{1}{2}\}=\frac{2}{3}.\end{aligned}$$

If $(\varrho, \varsigma) \in (-2, 2)$, then

$$\begin{aligned}\mathcal{N}_\Omega(-2,2) &= \max\{d(-2,2), d(-2,\Omega(-2)), d(2,\Omega(2)),\\ &\qquad d(-2,\Omega(-2)) + d(2,\Omega(-2))\}\\ &= \max\{d(-2,2), d(-2,-1), d(2,1), d(-2,-1)+d(2,-1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

If $(\varrho, \varsigma) = (1, -1)$, then

$$\begin{aligned}\mathcal{N}_\Omega(1,-1) &= \max\{d(1,-1), d(1,\Omega(1)), d(-1,\Omega(-1)),\\ &\qquad d(1,\Omega(1)) + d(-1,\Omega(1))\}\\ &= \max\{d(1,-1), d(1,-1), d(-1,1), d(1,-1)+d(-1,-1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\} = \frac{1}{4}.\end{aligned}$$

But

$$\begin{aligned}\mathcal{M}_\Omega(1,-1) &= \max\{d(1,-1), d(1,\Omega(-1)), d(-1,\Omega(1)),\\ &\qquad d(1,\Omega(1)) + d(-1,\Omega(-1))\}\\ &= \max\{d(1,-1), d(1,1), d(-1,-1), d(1,-1)+d(-1,1)\}\\ &= \max\{\frac{1}{4}, 0, 0, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

If $(\varrho, \varsigma) = (1, 2)$, then

$$\begin{aligned}\mathcal{N}_\Omega(1,2) &= \max\{d(1,2), d(1,\Omega(1)), d(2,\Omega(2)), d(1,\Omega(1))+d(2,\Omega(1))\}\\ &= \max\{d(1,2), d(1,-1), d(2,1), d(1,-1)+d(2,-1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

Case (2):

Let $\varrho \in \{-2, 1\}$; $\varsigma = 0$. So that for all $(\varrho, \varsigma) \in \{(-2, 0), (1, 0)\}$ and we get

$$d(\Omega(\varrho), \Omega(\varsigma)) = d(-1, 0) = \frac{1}{4}.$$

If $(\varrho, \varsigma) = (-2, 0)$, then

$$\begin{aligned}\mathcal{N}_\Omega(-2,0) &= \max\{d(-2,0), d(-2,\Omega(-2)), d(0,\Omega(0)),\\ &\qquad d(-2,\Omega(-2)) + d(0,\Omega(-2))\}\\ &= \max\{d(-2,0), d(-2,-1), d(0,0), d(-2,-1)+d(0,-1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

If $(\varrho, \varsigma) = (1, 0)$, then

$$\begin{aligned}\mathcal{N}_\Omega(1,0) &= \max\{d(1,0), d(1,\Omega(1)), d(0,\Omega(0)), d(1,\Omega(1)) + d(0,\Omega(1))\}\\ &= \max\{d(1,0), d(1,-1), d(0,0), d(1,-1) + d(0,-1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

Case (3):

Let $\varrho \in \{-1, 2\}$; $\varsigma = 0$. So that for all $(\varrho, \varsigma) \in \{(-1, 0), (2, 0)\}$ and we get

$$d(\Omega(\varrho), \Omega(\varsigma)) = d(1, 0) = \frac{1}{4}.$$

If $(\varrho, \varsigma) = (-1, 0)$, then

$$\begin{aligned}\mathcal{N}_\Omega(-1,0) &= \max\{d(-1,0), d(-1,\Omega(-1)), d(0,\Omega(0)),\\ &\qquad d(-1,\Omega(-1)) + d(0,\Omega(-1))\}\\ &= \max\{d(-1,0), d(-1,1), d(0,0), d(-1,1) + d(0,1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

If $(\varrho, \varsigma) = (2, 0)$, then

$$\begin{aligned}\mathcal{N}_\Omega(2,0) &= \max\{d(2,0), d(2,\Omega(2)), d(0,\Omega(0)), d(2,\Omega(2)) + d(0,\Omega(2))\}\\ &= \max\{d(2,0), d(2,1), d(0,0), d(2,1) + d(0,1)\}\\ &= \max\{\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

From the above cases, we conclude that

$$(\varrho, \varsigma) \in \{(-2, -1), (1, -1)\}, d(\Omega(\varrho), \Omega(\varsigma)) = \mathcal{N}_\Omega(\varrho, \varsigma) = \frac{1}{4}.$$

Since, the function F is increasing, we can not get any $\tau > 0$ such that

$$\tau + F(d(\Omega(\varrho), \Omega(\varsigma))) \le F(\mathcal{N}_\Omega(\varrho, \varsigma)),$$

that proves Ω is a modified extension and Ω is not F-weak contraction and not F-contraction, but Ω is an extension F-contraction.

Example 3.9. In the last example, we have

$$d(\Omega(\varrho), \Omega(\varsigma)) = \mathcal{N}_\Omega(\varrho, \varsigma) = \frac{1}{4},$$

when $(\varrho, \varsigma) \in \{(-2, -1), (1, -1)\}$. Let $F(\varrho) = Ln(\varrho)$, for all $\varrho > 0$ and $F \in \mathcal{F}$, we define $\alpha : X \times X \to [0, +\infty)$ by:

$$\alpha(\varrho, \varsigma) = \begin{cases} \frac{1}{3}, & (\varrho, \varsigma) \in \{(-2, -1), (1, -1)\},\\ \\ 1, & \text{otherwise}.\end{cases}$$

We get $\tau > 0$ such that

$$\tau+\alpha(\varrho,\varsigma)F(d(\Omega(\varrho),\Omega(\varsigma))) \leq F(\mathcal{N}_\Omega(\varrho,\varsigma)),$$

when $d(\Omega(\varrho),\Omega(\varsigma))) > 0$. In particular that $\alpha(\varrho,\varsigma) = \frac{1}{3}$ and we choose $\tau \in (0, \frac{1}{5})$. Therefore, Ω is a modified extension α-F-contraction.

Example 3.10. Suppose that $X = \{\frac{-1}{2}, 0, \frac{1}{2}\}$ and $\Omega : X \to X$ is a self-map and defined on X as the following:

$$\Omega(\frac{-1}{2}) = \Omega(0) = 0, \;\; \Omega(\frac{1}{2}) = \frac{-1}{2}$$

and

$$d(\varrho,\varsigma) = \begin{cases} 0, & \text{iff } \varrho = \varsigma, \\ \frac{1}{3}, & (\varrho,\varsigma) \in \{(\frac{1}{2}, \frac{-1}{2}), (\frac{-1}{2}, \frac{1}{2})\}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Let (X, d) be a complete metric space and $d(\Omega(\varrho),\Omega(\varsigma)) > 0$, for $(\varrho,\varsigma) = (0, \frac{1}{2})$ and $(\varrho,\varsigma) = (\frac{-1}{2}, \frac{1}{2})$, there are two cases:

Case (1):

Let $(\varrho,\varsigma) = (0, \frac{1}{2})$. So that $d(\Omega(0), \Omega(\frac{1}{2})) = d(0, \frac{-1}{2}) = \frac{1}{2}$ and we get

$$\begin{aligned}\mathcal{M}_\Omega(0, \frac{1}{2}) &= \max\{d(0, \frac{1}{2}), d(0, \Omega(\frac{1}{2})), d(\frac{1}{2}, \Omega(0)), d(0, \Omega(0))+d(\frac{1}{2}, \Omega(\frac{1}{2}))\} \\ &= \max\{d(0, \frac{1}{2}), d(0, \frac{-1}{2}), d(\frac{1}{2}, 0), d(0, 0)+d(\frac{1}{2}, \frac{-1}{2})\} \\ &= \max\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}.\end{aligned}$$

And

$$\begin{aligned}\mathcal{N}_\Omega(0, \frac{1}{2}) &= \max\{d(0, \frac{1}{2}), d(0, \Omega(0)), d(\frac{1}{2}, \Omega(\frac{1}{2})), d(0, \Omega(0))+d(\frac{1}{2}, \Omega(0))\} \\ &= \max\{d(0, \frac{1}{2}), d(0, 0), d(\frac{1}{2}, \frac{-1}{2}), d(0, 0)+d(\frac{1}{2}, 0)\} \\ &= \max\{\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{2}\} = \frac{1}{2}.\end{aligned}$$

Case (2):
Let $(\varrho,\varsigma)=(\frac{-1}{2},\frac{1}{2})$. So that $d(\Omega(\frac{-1}{2}),\Omega(\frac{1}{2}))=d(0,\frac{-1}{2})=\frac{1}{2}$ and we get

$$\begin{aligned}\mathcal{M}_\Omega(\frac{-1}{2},\frac{1}{2}) &= \max\{d(\frac{-1}{2},\frac{1}{2}),d(\frac{-1}{2},\Omega(\frac{1}{2})),d(\frac{1}{2},\Omega(\frac{-1}{2})),\\ &\qquad d(\frac{-1}{2},\Omega(\frac{-1}{2}))+d(\frac{1}{2},\Omega(\frac{1}{2}))\}\\ &= \max\{d(\frac{-1}{2},\frac{1}{2}),d(\frac{-1}{2},\frac{-1}{2}),d(\frac{1}{2},0),d(\frac{-1}{2},0)+d(\frac{1}{2},\frac{-1}{2})\}\\ &= \max\{\frac{1}{3},0,\frac{1}{2},\frac{5}{6}\}=\frac{5}{6}.\end{aligned}$$

And

$$\begin{aligned}\mathcal{N}_\Omega(\frac{-1}{2},\frac{1}{2}) &= \max\{d(\frac{-1}{2},\frac{1}{2}),d(\frac{-1}{2},\Omega(\frac{-1}{2})),d(\frac{1}{2},\Omega(\frac{1}{2})),\\ &\qquad d(\frac{-1}{2},\Omega(\frac{-1}{2}))+d(\frac{1}{2},\Omega(\frac{-1}{2}))\}\\ &= \max\{d(\frac{-1}{2},\frac{1}{2}),d(\frac{-1}{2},0),d(\frac{1}{2},\frac{-1}{2}),d(\frac{-1}{2},0)+d(\frac{1}{2},0)\}\\ &= \max\{\frac{1}{3},\frac{1}{2},\frac{1}{3},1\}=1.\end{aligned}$$

We choose $F(\varrho)=Ln(\varrho)$, for all $\varrho>0$ and $\alpha(\varrho,\varsigma)\geq 0$.
Since,

$$\tau+\alpha(0,\frac{1}{2})F(d(\Omega(0),\Omega(\frac{1}{2})))\leq F(\mathcal{N}_\Omega(0,\frac{1}{2}))\Rightarrow \tau+1.Ln(\frac{1}{2})\leq Ln(\frac{1}{2})\Rightarrow \tau\leq 0,$$

which is a contradiction because $\tau>0$, therefore Ω is not an extension α-F-contraction. We replace $\mathcal{M}_\Omega(0,\frac{1}{2})$ with $\mathcal{N}_\Omega(0,\frac{1}{2})$ in case (1), we get Ω which is not a modified extension α-F-contraction, but in case (2), when $(\varrho,\varsigma)=(\frac{-1}{2},\frac{1}{2})$, we get Ω which is an extension α-F-contraction and Ω is a modified extension α-F-contraction.

Note: In a similar way, we can re-examine the previous examples given that $\lambda\in(0,1)$ in equations (3.2) and (3.4) to prove our results. We also follow the same methods in order to prove the theorems above. We get similar results to what was previously proven when $\lambda=1$. We can also replace $\lambda\in(0,1)$ with any value within this period in the previous examples to get similar solutions to what, we obtained in the previous examples.

4. Implications and applications

The extension α-F-contractions and the modified extension α-F-contractions have diverse implications and applications in mathematics and scientific research. These concepts are fundamental in fixed point theory, providing a broader framework for studying the existence and uniqueness of fixed points. In addition to fixed point theory, these concepts find applications in functional analysis, topological dynamics and optimization theory. Moreover, their relevance extends to various scientific fields including computer science, physics, biology and engineering. These mappings serve as crucial tools in modeling and analyzing dynamic systems, stability analysis and the design of algorithms for solving complex optimization problems. Furthermore, the flexibility offered by a modified extension α-F-contractions allows for tailored approaches to specific mathematical and scientific problems. By appropriately choosing the function α and $\theta = \tau$, mathematicians and researchers can obtain more accurate and efficient solutions, as well as gain deeper insights into the behavior and stability of dynamic systems.

5. Conclusions

The extension α-F-contractions and the modified extension α-F-contractions had provided valuable extensions to traditional contraction mapping principles, enabling a more flexible and adaptable framework for studying fixed points in metric spaces. These concepts had widespread implications and applications in various areas of mathematics and scientific research, including fixed point theory, functional analysis and optimization. The introduction of these extensions had enhanced our ability to analyze and prove the existence and uniqueness of fixed points, stability and the behavior of dynamic systems. Continued exploration and application of extension α-F-contractions and a modified extension α-F-contractions would undoubtedly have contributed to advancements in diverse fields, allowing us to tackle complex problems and gain deeper insights into the dynamics and stability of systems.

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