Nonlinear Functional Analysis and Applications
Vol. 29, No. 2 (2024), pp. 451-460
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.02.07
http://nfaa.kyungnam.ac.kr/journal-nfaa

L^r INEQUALITIES FOR POLYNOMIALS

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Abstract. If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$ and $p'(z)$ its derivative, then Qazi [19] proved

$$\max_{|z|=1} |p'(z)| \le n \frac{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| (k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |p(z)|.$$

In this paper, we not only obtain the L^r version of the polar derivative of the above inequality for $r > 0$, but also obtain an improved L^r extension in polar derivative.

[0]Received August 10, 2023. Revised January 10, 2024. Accepted January 13, 2024.
[0]2020 Mathematics Subject Classification: 30A10, 30C10, 30D15.
[0]Keywords: Polynomial, polar derivative, zero, L^r inequalities.
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1. Introduction and preliminaries

Let $p(z)$ be a polynomial of degree n. Then, according to a well-known classical result due to Bernstein [4],

$$\max_{|z|=1}|p'(z)| \leq n\max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds if $p(z)$ has all its zeros at the origin.

Now, for a polynomial $p(z)$ of degree n, we define for $r > 0$

$$\|p\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.2}$$

We let $r \to \infty$ in (1.2) and make use of the well-known fact from analysis [22] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z=1}|p(z)|, \tag{1.3}$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1}|p(z)|. \tag{1.4}$$

Similarly, one can define

$$\|p\|_0 = exp\left\{\frac{1}{2\pi}\int_0^{2\pi} log|p(e^{i\theta})|d\theta\right\}$$

and show that $\lim_{r\to 0^+}\|p\|_r = \|p\|_0$. It would be of further interest that by taking limit as $r \to 0^+$, the stated results concerning L^r inequalities holding for $r > 0$, hold for $r = 0$ as well. Inequality (1.1) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \leq n\|p\|_r,\ r > 0. \tag{1.5}$$

Inequality (1.5) for $r \geq 1$ is due to Zygmund [24]. Arestov [1] proved that (1.5) remains valid for $0 < r < 1$ as well.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1.1) and (1.5) can be respectively improved by

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty \tag{1.6}$$

and

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r}\|p\|_\infty,\ r > 0. \tag{1.7}$$

Inequality (1.6) was conjectured by Erdös and later verified by Lax [13], whereas inequality (1.7) was proved by de-Brujin [8] for $r \geq 1$, Rahman and Schmeisser [20] showed that (1.7) remains true for $0 < r < 1$.

Let $P_{n,\mu}$ be the class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, of degree n. As a generalization of (1.6), Malik [14] proved that if $p \in P_{n,1}$ and $p(z) \ne 0$ in $|z| < k$, $k \ge 1$, then

$$\|p'\|_\infty \le \frac{n}{1+k}\|p\|_\infty. \tag{1.8}$$

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number α as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

This polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z),$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

Aziz [2] was among the first who extended some of the above inequalities to polar versions. He, in fact, extended inequality (1.8) to polar derivative of a polynomial by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \ge 1$, then for any complex number α with $|\alpha| \ge 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \le n\left(\frac{|\alpha| + k}{1+k}\right) \max_{|z|=1} |p(z)|. \tag{1.9}$$

As an L^r analogue of polar derivative of (1.8), Rather [21] proved that for any complex number α with $|\alpha| \ge 1$ and for every $r > 0$,

$$\|D_\alpha p(z)\|_r \le n\left(\frac{|\alpha| + k}{\|z+k\|_r}\right)\|p\|_r. \tag{1.10}$$

Over the last four decades, many different authors produced a large number of different versions and generalizations of the above inequalities. Many of these generalizations involve the comparison of polar derivative $D_\alpha p(z)$ with various choices of $p(z)$, α and other parameters. More information on this topic can be found in the books of Milovanović et al. [17] and Marden [15], and in the literatures [6, 7, 10, 12, 16, 23].

Using the class of Lacunary-type polynomial, Mir [18] recently extended (1.9) by proving that if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k \ge 1$, then for every $r > 0$ and for every complex number α with $|\alpha| \geq 1$,

$$\|D_\alpha p(z)\|_r \leq n\frac{(|\alpha| + k^\mu)}{\|z + k^\mu\|_r}\|p\|_r. \tag{1.11}$$

Further, Qazi [19] improved (1.8) by proving:

Theorem 1.1. *If $p \in P_{n,\mu}$, $1 \leq \mu \leq n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{\|A_0 + z\|_\infty}\|p\|_\infty, \tag{1.12}$$

where

$$A_0 = \frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}}. \tag{1.13}$$

Dewan et al. [9] also improved Theorem 1.1 by involving $\min\limits_{|z|=k} |p(z)|$.

Theorem 1.2. *If $p \in P_{n,\mu}$, $1 \leq \mu \leq n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then*

$$\|p'\|_\infty \leq \frac{n}{\|A_0 + z\|_\infty}\|p\|_\infty - \frac{n}{k^n}\left\{1 - \frac{1}{\|A_0 + z\|_\infty}\right\}\min_{|z|=k}|p(z)|, \tag{1.14}$$

where A_0 is as defined in (1.13).

Dewan et al. [9] extended Theorem 1.1 to L^r analogue for $r \geq 1$ and for $r > 0$ by Chanam [5].

Theorem 1.3. *If $p \in P_{n,\mu}$, $1 \leq \mu \leq n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for each $r > 0$,*

$$\|p'\|_r \leq \frac{n}{\|A_0 + z\|_r}\|p\|_r, \tag{1.15}$$

where A_0 is as defined in (1.13).

2. Lemmas

For the proof of the theorem, we require the following lemmas. The first lemma is due to Qazi [19].

Lemma 2.1. *If $p \in P_{n,\mu}$, $1 \leq \mu \leq n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then*

$$k^{\mu+1}\frac{\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1} + 1}{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}|p'(z)| \leq |q'(z)| \text{ on } |z| = 1 \tag{2.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|k^\mu \leq 1, \tag{2.2}$$

where $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

The next lemma is due to Govil and Kumar [11].

Lemma 2.2. *Let p, q be any two positive real numbers such that $p \geq qx$, where $x \geq 1$. If γ is any real such that $0 \leq \gamma \leq 2\pi$, then for any $y \geq 1$*

$$\frac{p+qy}{x+y} \leq \left|\frac{p+qe^{i\gamma}}{x+e^{i\gamma}}\right|. \tag{2.3}$$

Lemma 2.3. *Let z_1, z_2 be two complex numbers independent of α, where α being real. Then for $r > 0$,*

$$\int_0^{2\pi} |z_1 + z_2 e^{i\alpha}|^r d\alpha = \int_0^{2\pi} \left||z_1| + |z_2|e^{i\alpha}\right|^r d\alpha. \tag{2.4}$$

The above lemma is due to Govil and Kumar [11].

Lemma 2.4. *Let $p(z)$ be a polynomial of degree n. Then for every γ with $0 \leq \gamma < 2\pi$ and $r > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} |p'(e^{i\theta}) + e^{i\gamma}q'(e^{i\theta})|^r d\theta d\gamma \leq 2\pi n^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta, \tag{2.5}$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

The above result is due to Aziz and Rather [3].

3. Main results

In this paper, we obtain L^r analogue of the polar derivative version of Theorem 1.2 for $r > 0$ which further extends both Theorems 1.1 and 1.3. More precisely, we prove:

Theorem 3.1. *If $p \in P_{n,\mu}$, $1 \leq \mu \leq n$ and $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every real or complex number α and β with $|\alpha| \geq 1$ and $|\beta| < \frac{1}{k^n}$ and for each $r > 0$,*

$$\|D_\alpha p(z) + n\alpha m\beta z^{n-1}\|_r \leq \frac{n(|\alpha| + A)}{\|A + z\|_r}\|p(z) + m\beta z^n\|_r, \tag{3.1}$$

where

$$A = \frac{n|a_0|k^{\mu+1} + \mu|a_\mu + m\beta|k^{2\mu}}{n|a_0| + \mu|a_\mu + m\beta|k^{\mu+1}} \quad \textit{and} \quad m = \min_{|z|=k}|p(z)|. \tag{3.2}$$

Proof. We have for any $r > 0$

$$\begin{aligned}
&\left\{\int_0^{2\pi} |A+e^{i\gamma}|^r d\gamma\right\}\left[\int_0^{2\pi} |D_\alpha\{p(e^{i\theta})+m\beta e^{in\theta}\}|^r d\theta\right] \\
&= \left\{\int_0^{2\pi} |A+e^{i\gamma}|^r d\gamma\right\}\left[\int_0^{2\pi} |n\{p(e^{i\theta})+m\beta e^{in\theta}\} \right.\\
&\quad \left. +(\alpha-e^{i\theta})\{p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\}|^r d\theta\right] \\
&= \left\{\int_0^{2\pi} |A+e^{i\gamma}|^r d\gamma\right\}\left\{\int_0^{2\pi} |D_\alpha p(e^{i\theta})+n\alpha m\beta e^{i(n-1)\theta}|^r d\theta\right\}. \qquad (3.3)
\end{aligned}$$

If $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$, then it can be easily verified that for $0 \le \theta < 2\pi$,

$$n\{p(e^{i\theta})+m\beta e^{in\theta}\} - e^{i\theta}\{p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\} = e^{i(n-1)\theta}\overline{q'(e^{i\theta})}.$$

Using the above inequality, we have for $0 \le \theta < 2\pi$,

$$\begin{aligned}
D_\alpha\{p(e^{i\theta})+m\beta e^{in\theta}\} &= n\{p(e^{i\theta})+m\beta e^{in\theta}\} \\
&\quad +(\alpha-e^{i\theta})\{p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\} \\
&= e^{i(n-1)\theta}\overline{q'(e^{i\theta})}+\alpha\{p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\},
\end{aligned}$$

which implies

$$|D_\alpha\{p(e^{i\theta})+m\beta e^{in\theta}\}| \le |q'(e^{i\theta})| + |\alpha||p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}|. \qquad (3.4)$$

Using inequality (3.4) in inequality (3.3), we get

$$\begin{aligned}
&\left\{\int_0^{2\pi} |A+e^{i\gamma}|^r d\gamma\right\}\left\{\int_0^{2\pi} |D_\alpha p(e^{i\theta})+n\alpha m\beta e^{i(n-1)\theta}|^r d\theta\right\} \\
&\le \left\{\int_0^{2\pi} |A+e^{i\gamma}|^r d\gamma\right\}\left[\int_0^{2\pi} \left\{|q'(e^{i\theta})| + |\alpha||p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}|\right\}^r d\theta\right]. \qquad (3.5)
\end{aligned}$$

By Rouche's theorem, the polynomial $P(z) = p(z) + m\beta z^n$ has no zero in $|z| < k, k \ge 1$ and if we apply Lemma 2.1 to the polynomial $P(z)$, we have

$$\left\{\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|k^{2\mu}}{n|a_0|+\mu|a_\mu|k^{\mu+1}}\right\}|p'(e^{i\theta})+nm\beta e^{i(n-1)\theta}| \le |q'(e^{i\theta})|. \qquad (3.6)$$

Taking $p = |q'(e^{i\theta})|,\ q = |p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}|,\ x = A$ and $y = |\alpha|$ in Lemma 2.2, we have for all $\gamma \in [0, 2\pi]$,

$$
\begin{aligned}
&|A + e^{i\gamma}| \left\{ |q'(e^{i\theta})| + |\alpha||p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| \right\} \\
&\leq (A + |\alpha|) \left| |q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| \right|. \qquad (3.7)
\end{aligned}
$$

Further, it can be easily verified that

$$
\begin{aligned}
&\left| |q'(e^{i\theta})| + e^{i\gamma}|p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| \right| \\
&= \left| |p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| + e^{i\gamma}|q'(e^{i\theta})| \right|. \qquad (3.8)
\end{aligned}
$$

Now, inequality (3.7) and inequality (3.8) give

$$
\begin{aligned}
&\left|A + e^{i\gamma}\right| \left\{ |q'(e^{i\theta})| + |\alpha||p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| \right\} \\
&\leq (A + |\alpha|) \left| |p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| + e^{i\gamma}|q'(e^{i\theta})| \right|. \qquad (3.9)
\end{aligned}
$$

Applying inequality (3.9) to the right hand side of inequality (3.5), we have for any $r > 0$

$$
\begin{aligned}
&\left\{ \int_0^{2\pi} |A + e^{i\gamma}|^r d\gamma \right\} \left\{ \int_0^{2\pi} |D_\alpha p(e^{i\theta}) + n\alpha m\beta e^{i(n-1)\theta}|^r d\theta \right\} \\
&\leq (A+|\alpha|)^r \left\{ \int_0^{2\pi}\int_0^{2\pi} \left| |p'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}| + e^{i\gamma}|q'(e^{i\theta})| \right|^r d\gamma d\theta \right\}. \qquad (3.10)
\end{aligned}
$$

Using Lemma 2.3 and then applying Lemma 2.4 to the right hand side of inequality (3.10), we get

$$
\begin{aligned}
&\left\{ \int_0^{2\pi} |A + e^{i\gamma}|^r d\gamma \right\} \left\{ \int_0^{2\pi} |D_\alpha p(e^{i\theta}) + n\alpha m\beta e^{i(n-1)\theta}|^r d\theta \right\} \\
&\leq (A + |\alpha|)^r 2\pi n^r \left\{ \int_0^{2\pi} |p(e^{i\theta}) + m\beta e^{in\theta}|^r d\gamma d\theta \right\},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\left\{ \frac{1}{2\pi} \int_0^{2\pi} |A + e^{i\gamma}|^r d\gamma \right\}^{\frac{1}{r}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |D_\alpha p(e^{i\theta}) + n\alpha m\beta e^{i(n-1)\theta}|^r d\theta \right\}^{\frac{1}{r}} \\
&\leq (A + |\alpha|) n \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta}) + m\beta e^{in\theta}|^r d\theta \right\}^{\frac{1}{r}},
\end{aligned}
$$

which completes the proof. □

Remark 3.2. If we take $\beta = 0$ and divide both sides of inequality (3.1), by $|\alpha|$ and letting $|\alpha| \to \infty$, Theorem 3.1 reduces to the integral analogue of Theorem 1.1.

If we let $r \to \infty$ in (3.1), we get the following result.

Corollary 3.3. *If $p \in P_{n,\mu}$, $1 \le \mu \le n$ and $p(z) \neq 0$ in $|z| < k$, $k \ge 1$, then for every real or complex numbers α and β with $|\alpha| > 1$ and $|\beta| < \frac{1}{k^n}$,*

$$\max_{|z|=1} |D_\alpha p(z) + n\alpha m\beta z^{n-1}| \le \frac{n(|\alpha| + A)}{(A+1)} \max_{|z|=1} |p(z) + m\beta z^n|. \tag{3.11}$$

Remark 3.4. If we take $\beta = 0$, Theorem 3.1 reduces to the following interesting result which provides the polar version of Theorem 1.3.

Corollary 3.5. *If $p \in P_{n,\mu}$, $1 \le \mu \le n$ and $p(z) \neq 0$ in $|z| < k$, $k \ge 1$, then for every real or complex number α with $|\alpha| > 1$ and for each $r > 0$,*

$$\|D_\alpha p(z)\|_r \le \frac{n(|\alpha| + A)}{\|A + z\|_r} \|p\|_r, \tag{3.12}$$

where A is as defined in Theorem 3.1.

Remark 3.6. If we divide both sides of (3.12) of Corollary 3.5 by $|\alpha|$ and take limit as $|\alpha| \to \infty$, we obtain inequality (1.15) of Theorem 1.3, which corresponds the L^r analogue of Theorem 1.1.

Remark 3.7. By (2.2) of Lemma 2.1, it is evident that

$$A = \frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0| + \mu|a_\mu|k^{\mu+1}} \ge k^\mu$$

for $1 \le \mu \le n$ and hence for $|\alpha| \ge 1$ and for each $r > 0$,

$$\frac{(|\alpha| + A)}{\|A + z\|_r} \le \frac{(|\alpha| + k^\mu)}{\|k^\mu + z\|_r}. \tag{3.13}$$

Using (3.13) to Corollary 3.5, we get inequality (1.11).

Remark 3.8. Dividing both sides of inequality (3.1) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we have the following result independently proved by Chanam [5].

Corollary 3.9. *If $p \in P_{n,\mu}$, $1 \le \mu \le n$ and $p(z) \neq 0$ in $|z| < k$, $k \ge 1$, then for every real or complex number β with $|\beta| < \frac{1}{k^n}$ and for each $r > 0$,*

$$\|p'(z) + mn\beta z^{n-1}\|_r \le \frac{n}{\|A + z\|_r} \|p(z) + m\beta z^n\|_r, \tag{3.14}$$

where A and m are as defined in Theorem 3.1.

Remark 3.10. If we let $r \to \infty$ on both sides of (3.14), we have

$$\max_{|z|=1} |p'(z) + mn\beta z^{n-1}| \leq \frac{n}{1+A} \max_{|z|=1} |p(z) + m\beta z^n| \leq \frac{n}{1+A} \left\{ \max_{|z|=1} |p(z)| + m|\beta| \right\}, \tag{3.15}$$

where A and m are as defined in Theorem 3.1.

Let z_0 on $|z| = 1$ be such that

$$\max_{|z|=1} |p'(z)| = |p'(z_0)|.$$

Then, in particular, inequality (3.15) becomes

$$|p'(z_0) + mn\beta z_0^{n-1}| \leq \frac{n}{1+A} \left\{ \max_{|z|=1} |p(z)| + m|\beta| \right\}.$$

Choosing the argument of β suitably such that

$$|p'(z_0) + mn\beta z_0^{n-1}| = |p'(z_0)| + mn|\beta|,$$

and finally making limit as $|\beta| \to \frac{1}{k^n}$, we get as cited earlier, the best possible inequality due to Dewan et al. [9].

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