Nonlinear Functional Analysis and Applications
Vol. 29, No. 2 (2024), pp. 419-434
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2024.29.02.05
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON KIRCHHOFF TYPE EQUATIONS WITH SINGULAR NONLINEARITIES, SUB-CRITICAL AND CRITICAL EXPONENT

Mohammed El Mokhtar Ould El Mokhtar[1]
and Saleh Fahad Aljurbua[2]

[1]Departement of Mathematics, College of Science, Qassim University BO 6644,
Buraidah, 51452, Kingdom of Saudi Arabia
e-mail: `med.mokhtar66@yahoo.fr; M.labdi@qu.edu.sa`

[2]Departement of Mathematics, College of Science, Qassim University BO 6644,
Kingdom of Saudi Arabia
e-mail: `S.aljurbua@qu.edu.sa`

Abstract. This paper is devoted to the existence of solutions for Kirchhoff type equations with singular nonlinearities, sub-critical and critical exponent. By using the Nehari manifold and Maximum principle theorem, the existence of at least two distinct positive solutions is obtained.

1. Introduction

This paper deals with the existence and multiplicity of nontrivial solutions to the following Kirchhoff problem:

$$\begin{cases} L(u)\left[-\Delta u + bu\right] = |u|^{p-1} u + \mu \frac{|u|^{-1-\beta}}{|x|^{\alpha}} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $L(u) := \left(a + \lambda \int_{\Omega} |\nabla u|^2 + \lambda b \int_{\Omega} u^2\right)$, Ω is a smooth bounded domain of $\mathbb{R}^3$, $a, b > 0$, $p \in (3, 5]$, $\lambda > 0$, $\mu > 0$, $0 < \alpha < 3(p+\beta)/p$ and $0 < \beta < 1$.

[0]Received June 7, 2023. Revised September 8, 2023. Accepted October 2, 2023.
[0]2020 Mathematics Subject Classification: 35J66, 35B40.
[0]Keywords: Singular nonlinearities, Kirchhoff type equations, Nehari manifold, critical exponent.
[0]Corresponding author: M. El Mokhtar Ould El Mokhtar(`med.mokhtar66@yahoo.fr`).

In recent years, many authors have paid much attention to the following Kirchhoff type problem:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^{2}\right)\Delta u=f\left(x,u\right) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $0 \in \Omega, a, b > 0$ and f is a suitable function containing singularities on x. See [1, 3, 4, 9, 13, 17, 19] and the references therein for the existence and multiplicity of positive solutions to (1.2). Mao and Luan [14, 15], Zhang and Perera [20] studied the existence of sign-changing solutions of (1.2). When $\Omega = \mathbb{R}^N$, in [11] Li and Shi showed the existence of nontrivial solutions of the following problem with zero mass:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^{2}\right)\Delta u=h\left(x\right)f\left(u\right) \text{ in } \mathbb{R}^N, \\ u\in D^{1,2}\left(\mathbb{R}^N\right), \end{cases} \tag{1.3}$$

where $D^{1,2}\left(\mathbb{R}^N\right)$ is the closure of the compactly supported smooth functions with respect to the norm $\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{1}{2}}$, the potential function $h\left(x\right)$ is a nonnegative continuous function, $h \in \left[L^s\left(\mathbb{R}^N\right)\cap L^\infty\left(\mathbb{R}^N\right)\right]\setminus\{0\}$ for some $s \geq 2N/\left(N+2\right)$ and $|x.\nabla h\left(x\right)| \leq \alpha h\left(x\right)$ for a.e. $x \in \mathbb{R}^N$ and some $\alpha \in (0,2)$.

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the term $\int_{\mathbb{R}^3}|\nabla u|^2dx$ which implies that the equation in (1.1) is no longer a pointwise identity. It is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^{2}dx\right)\Delta u=f\left(x,u\right), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain ($N \geq 3$), u denotes the displacement, $f(x,u)$ is the external force and a is the initial tension while b is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string (see [10, 18]).

For $N \geq 3, a = 1, \lambda = 0,$ $p = \left(N+2\right)/\left(N-2\right)$ in the problem (1.1), El Mokhtar and Matallah [8] have shown the existence of multiple positive solutions.

It is clear that these problems contribute to the transition from the academic world to that of application. Indeed, very taken for its physical motivations, the problem (1.1) is nothing other than a stationary version of the following model which governs the behavior an elastic thread whose ends are fixed and which is subjected to non-linear vibrations (1.4) in $\Omega \times (0,T)$ where $T > 0$, a is the initial tension, b represents the Young's modulus of the wire material and L its length. The latter is known to be an extension of the Alembert wave equation. Indeed, Kirchhoff took into account the changes caused by the transverse oscillations along the length of the wire.

Strengthened by their implications in other disciplines, and given the extent of their fields of application, non-local problems will be used to model several physical phenomena. They also occur in biological systems where u describes a process depending on its mean such as the density of population.

Due to this significant impact reinforcing the field of applications, this type of problem has perceived the interest of mathematicians and many works aiming at the existence of solutions have emerged. In particular after the coup de grace provided by the famous article by Lions [12] where the latter adopted an approach based on functional analysis. Nevertheless, in most of these articles, the favored method is purely topological.

It is only in recent decades that this approach has been abandoned in favor of variational methods when Alves and his colleagues [1] obtained for the first time times of existence results via these methods. Since then, there has been a very fruitful boom which has given rise to a lot of work founding this advantageous axis see [7, 11, 12].

Nonlocal effect also finds its applications in biological systems. A parabolic version of (1.1) can, in theory, be used to describe the growth and movement of a particular species. The movement, modelled by the integral term, is assumed dependent on the "energy" of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type (1.4). Chipot and Lovat [5] and Correa et al. [6], for examples, studied the existence of solutions and their uniqueness for such nonlocal problems as well as their corresponding elliptic problems.

Before giving our main results, we state here some definitions, notations and known results.

The space $\mathcal{H} = \mathcal{H}_0^1(\Omega)$ is equipped with the norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + b\,|u|^2\right) dx\right)^{1/2}.$$

Let S be the best Sobolev constant. Then

$$S = \inf_{u \in \mathcal{H}_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega |u|^{p+1}\, dx\right)^{2/(p+1)}}. \tag{1.5}$$

From [9], S is achieved.

The functional energy J of (1.1) is defined by

$$J(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \frac{1}{p+1}\int_\Omega |u|^{p+1}dx - \frac{\mu}{1-\beta}\int_\Omega \frac{|u|^{1-\beta}}{|x|^\alpha}dx. \quad (1.6)$$

We consider the following approximation equation:

$$\begin{cases} L(u)[-\Delta u + bu] = |u|^{p-1}u + \mu\frac{|u+\theta|^{-1-\beta}}{|x|^\alpha}(u+\theta) & \text{in } \Omega, \\ u = 0 \quad \text{in } \partial\Omega, \end{cases} \quad (1.7)$$

for any $\theta > 0$ (small another). The energy functional of (1.7) J_θ is defined by

$$J_\theta(u) := \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \frac{1}{p+1}\int_\Omega |u|^{p+1}dx - \frac{\mu}{1-\beta}\int_\Omega \frac{|u+\theta|^{1-\beta}}{|x|^\alpha}dx. \quad (1.8)$$

According to an algebraic relation of Simon [16], the Holder inequality and by (2.2), we obtain that J_θ is a C^1-function on $\mathcal{H} = \mathcal{H}^1(\Omega)$.

A point $u \in \mathcal{H}$ is a weak solution of the equation (1.1) if it satisfies

$$\begin{aligned} \left\langle J_\theta'(u), \varphi\right\rangle : &= \left(\lambda\|u\|^2 + a\right)\int_\Omega (\nabla u \nabla\varphi + bu\varphi)\,dx \\ &\quad - \int_\Omega |u|^{p-1}u\varphi dx - \mu\int_\Omega \frac{|u+\theta|^{1-\beta}}{|x|^\alpha}\varphi dx \quad (1.9) \\ &= 0 \end{aligned}$$

for all $\varphi \in \mathcal{H}$, where $\langle .,. \rangle$ denotes the product in the duality $\mathcal{H}'$ and $\mathcal{H}$.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our problem.

Let μ_* and μ_{**} be positive numbers such that $\mu_* = \max(\mu_1, \mu_2)$ and μ_{**} be positive numbers such that

$$\begin{aligned} \mu_1 &:= \frac{(p-1)a}{(p+\beta)A}S^{(1-\beta/2)}\left[\frac{2\sqrt{(1+\beta)(3+\beta)a\lambda}}{p+\beta}S^{(p+1)/2}\right]^{\frac{\beta+1}{p-2}}, \\ \mu_2 &:= \frac{2\sqrt{(p-1)(p-3)a\lambda}}{(p+\beta)A}S^{(1-\beta)/2}\left[\frac{a(1+\beta)}{p+\beta}S^{(p+1)/2}\right]^{\frac{\beta+1}{p-1}} \end{aligned}$$

and

$$\mu_{**} := \frac{(p-1)A}{(p+\beta)a}\left(\frac{1-\beta}{2}\right)\left(\frac{1+\beta}{p+\beta}\right)^{\frac{3-\beta}{p-1}} S^{\frac{2(p+1)^2-4(p-1)^2+(1+\beta)(p-1)^2}{2(p-1)(p+1)}},$$

where

$$A=\left[\frac{4\pi\left(p+\beta\right)}{3\left(p+\beta\right)-\alpha\left(p+1\right)}\right]^{\frac{p+\beta}{p+1}}R_0^{\frac{3}{p+1}\left(p+\beta\right)-\alpha}>0$$

with

$$0\leq\alpha<\frac{3}{p+1}\left(p+\beta\right).$$

2. Preliminaries

Definition 2.1. Let $c\in\mathbb{R}$, E a Banach space and $I\in C^1\left(E,\mathbb{R}\right)$.

(i) $\{u_n\}_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I\left(u_n\right)=c+o_n\left(1\right)\text{ and }I^{'}\left(u_n\right)=o_n\left(1\right),$$

where $o_n\left(1\right)$ tends to 0 as n goes at infinity.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

Nehari manifold: It is well known that J is of class C^1 in $\mathcal{H}$ and the solutions of (1.1) are the critical points of J which is not bounded below on $\mathcal{H}$. Consider the following Nehari manifold (see, [1])

$$\mathcal{M}=\left\{u\in\mathcal{H}\backslash\left\{0\right\}:\ \left\langle J_\theta^{'}\left(u\right),u\right\rangle=0\right\}.$$

Thus, $u\in\mathcal{M}$ if and only if $u\in\mathcal{H}\backslash\{0\}$ and

$$a\left\|u\right\|^2+\lambda\left\|u\right\|^4-\int_\Omega\left|u\right|^{p+1}dx-\mu\int_\Omega\frac{\left|u^++\theta\right|^{1-\beta}}{\left|x\right|^\alpha}dx=0. \tag{2.1}$$

Note that $\mathcal{M}$ contains every nontrivial solution of the problem (1.1). In order to obtain the first positive solution, we give the following important lemmas.

Lemma 2.2. *J is coercive and bounded from below on $\mathcal{M}$.*

Proof. Let $R_0>0$ such that $\Omega\subset B\left(0,R_0\right)=\left\{x\in\mathbb{R}^3:\left|x\right|<R_0\right\}$. If $u\in\mathcal{M}$, then by (2.1), the Hölder inequality and for any $\theta>0$ (small another), we obtain

$$\begin{aligned}\int_\Omega\frac{\left|u^++\theta\right|^{1-\beta}}{\left|x\right|^\alpha}dx &\leq \left[\frac{4\pi\left(p+\beta\right)}{3\left(p+\beta\right)-\alpha\left(p+1\right)}\right]^{\frac{p+\beta}{p+1}} \\ &\quad\times R_0^{\frac{N}{p+1}\left(p+\beta\right)-\alpha}\left\|u\right\|^{1-\beta}\left(S\right)^{\frac{-(1-\beta)}{2}},\end{aligned} \tag{2.2}$$

and we deduce that

$$\begin{aligned} J_\theta(u) &= ((p-1)/2(p+1))\, a\|u\|^2 + ((p-3)/4(p+1))\,\lambda\|u\|^4 \\ &\quad -\mu((p+\beta)/(p+1)(1-\beta))\int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha}dx, \\ &\geq ((p-1)/2(p+1))\, a\|u\|^2 + ((p-3)/4(p+1))\,\lambda\|u\|^4 \\ &\quad -\mu((p+\beta)/(p+1)(1-\beta))\, A\|u\|^{1-\beta}(S)^{\frac{-(1-\beta)}{2}} \end{aligned}$$

with

$$A = \left[\frac{4\pi(p+\beta)}{3(p+\beta)-\alpha(p+1)}\right]^{\frac{p+\beta}{p+1}} R_0^{\frac{3}{p+1}(p+\beta)-\alpha} > 0$$

for $0 \leq \alpha < \frac{3}{p+1}(p+\beta)$. Thus, J_θ is coercive and bounded from below on $\mathcal{M}$. □

Define

$$\phi(u) = \left\langle J_\theta'(u), u\right\rangle.$$

Then, for $u \in \mathcal{M}$,

$$\begin{aligned} \left\langle \phi'(u), u\right\rangle &= 2a\|u\|^2 + 4\lambda\|u\|^4 - (p+1)\int_\Omega |u|^{p+1}dx \\ &\quad -\mu(1-\beta)\int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha}dx \\ &= (1+\beta)\, a\|u\|^2 + (3+\beta)\,\lambda\|u\|^4 - (p+\beta)\int_\Omega |u|^{p+1}dx \\ &= \mu(p+\beta)\int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha}dx - \left[(p-2)\, a\|u\|^2 + (p-3)\,\lambda\|u\|^4\right]. \end{aligned} \tag{2.3}$$

Now, we split $\mathcal{M}$ in three parts:

$$\begin{aligned} \mathcal{M}^+ &= \left\{u \in \mathcal{M} : \left\langle \phi'(u), u\right\rangle > 0\right\}, \\ \mathcal{M}^0 &= \left\{u \in \mathcal{M} : \left\langle \phi'(u), u\right\rangle = 0\right\}, \\ \mathcal{M}^- &= \left\{u \in \mathcal{M} : \left\langle \phi'(u), u\right\rangle < 0\right\}. \end{aligned}$$

We have the following results.

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_θ on $\mathcal{M}$. If $u_0 \notin \mathcal{M}^0$ then, u_0 is a critical point of J_θ.*

Proof. If u_0 is a local minimizer for J_θ on $\mathcal{M}$, then u_0 is a solution of the optimization problem

$$\min_{\{u\in\mathcal{H}\setminus\{0\}/\ \phi(u)=0\}} J_\theta\left(u\right).$$

Hence, there exists a Lagrange multipliers $\sigma\in\mathbb{R}$ such that

$$J_\theta^{'}\left(u_0\right)=\sigma\phi^{'}\left(u_0\right)\ \text{ in } \mathcal{H}^{'}.$$

Thus,

$$\left\langle J_\theta^{'}\left(u_0\right),u_0\right\rangle=\sigma\left\langle \phi^{'}\left(u_0\right),u_0\right\rangle.$$

But $\left\langle \phi^{'}\left(u_0\right),u_0\right\rangle\neq 0$, since $u_0\notin\mathcal{M}^0$. Hence $\sigma=0$. This completes the proof. □

Lemma 2.4. *There exists a positive number μ_* such that for all μ verifying*

$$0<\mu<\mu_*,$$

we have $\mathcal{M}^0=\emptyset$.

Proof. Let us reason by contradiction. Suppose that $\mathcal{M}^0\neq\emptyset$ such that $0<\mu<\mu_*$. Then, by (2.3) and for $u\in\mathcal{M}^0$, we have

$$(1+\beta)\,a\left\|u\right\|^2+(3+\beta)\,\lambda\left\|u\right\|^4-(p+\beta)\int_\Omega\left|u\right|^{p+1}dx \quad=\quad 0 \tag{$*$}$$

and

$$\mu\left(p+\beta\right)\int_\Omega\frac{\left|u^++\theta\right|^{1-\beta}}{\left|x\right|^\alpha}dx-\left[(p-1)\,a\left\|u\right\|^2+(p-3)\,\lambda\left\|u\right\|^4\right] \quad=\quad 0. \tag{$**$}$$

Moreover, from (2.2) and since $2ab\leq a^2+b^2$, by using Hölder and Sobolev inequalities, we obtain

$$\left\|u\right\|\geq\left[\frac{2\sqrt{(1+\beta)\,(3+\beta)\,a\lambda}}{p+\beta}S^{(p+1)/2}\right]^{\frac{1}{p-2}} \tag{2.4}$$

and

$$\left\|u\right\|\leq\left[\frac{\mu\left(p+\beta\right)A}{2\sqrt{(p-1)\,(p-3)\,a\lambda}}S^{-(1-\beta)/2}\right]^{\frac{1}{2+\beta}}. \tag{2.5}$$

On the other hand, from ($*$) and ($**$), the Hölder inequality and the Sobolev embedding theorem, we also have, respectively

$$\left\|u\right\|\geq\left[\frac{a\left(1+\beta\right)}{p+\beta}S^{-(p+1)/2}\right]^{\frac{1}{p-1}} \tag{2.6}$$

and
$$\|u\| \leq \left[\frac{\mu\left(p+\beta\right)A}{2\left(p-1\right)a}S^{-(1-\beta)/2}\right]^{\frac{1}{1+\beta}}. \tag{2.7}$$
From (2.4) and (2.7), we obtain
$$\mu \geq \mu_1 = \frac{(p-1)\,a}{(p+\beta)\,A}S^{(1-\beta/2)}\left[\frac{2\sqrt{(1+\beta)\,(3+\beta)\,a\lambda}}{p+\beta}S^{(p+1)/2}\right]^{\frac{p+1}{p-2}},$$
and from (2.5) and (2.6), we obtain
$$\mu \geq \mu_2 = \frac{2\sqrt{(p-1)\,(p-3)\,a\lambda}}{(p+\beta)\,A}S^{(1-\beta)/2}\left[\frac{a\,(1+\beta)}{p+\beta}S^{(p+1)/2}\right]^{\frac{\beta+1}{p-1}}.$$
Thus, $\mu \geq \mu_* = \max\left(\mu_1, \mu_2\right)$, which contradicts our hypothesis. □

We know that s $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$. Define
$$c := \inf_{u\in\mathcal{M}} J_\theta\left(u\right),\ c^+ := \inf_{u\in\mathcal{M}^+} J_\theta\left(u\right) \text{ and } c^- := \inf_{u\in\mathcal{M}^-} J_\theta\left(u\right).$$

For the sequel, we need the following lemma.

Lemma 2.5. (i) *For all* μ *such that* $0<\mu<\mu_*$, *one has* $c \leq c^+ < 0$.

(ii) *There exists* μ_{**} *defined in Theorem* 3.7 *such that for all* λ *such that* $0<\mu<\mu_{**}$, *one has*
$$c^- > C_0 = C_0\left(\lambda, S, \beta\right).$$

Proof. (i) Let $u \in \mathcal{M}^+$. By (2.3), we have
$$\frac{1+\beta}{p+\beta}a\left\|u\right\|^2 + \frac{3+\beta}{p+\beta}\lambda\left\|u\right\|^4 > \int_\Omega |u|^{p+1}\,dx$$
and so, since $p>3$ and $0<\beta<1$,
$$\begin{aligned} J_\theta\left(u\right) &< \frac{-\left(1+\beta\right)}{2\left(1-\beta\right)}a\left\|u\right\|^2 - \frac{\left(3+\beta\right)}{4\left(1-\beta\right)}\lambda\left\|u\right\|^4 + \frac{p+\beta}{\left(p+1\right)\left(1-\beta\right)}\int_\Omega |u|^{p+1}\,dx \\ &< -\left[\frac{\left(p-1\right)\left(1+\beta\right)}{2\left(p+1\right)\left(1-\beta\right)}a\left\|u\right\|^2 + \frac{\left(p-3\right)\left(3+\beta\right)}{4\left(p+1\right)\left(1-\beta\right)}\lambda\left\|u\right\|^4\right] \\ &< 0 \end{aligned}$$
Then, we conclude that $c \leq c^+ < 0$.

(ii) Let $u \in \mathcal{M}^-$. By (2.3), we get
$$\frac{1+\beta}{p+\beta}a\left\|u\right\|^2 + \frac{3+\beta}{p+\beta}\lambda\left\|u\right\|^4 < \int_\Omega |u|^{p+1}\,dx.$$

Moreover, by Sobolev embedding theorem, we have

$$\int_{\Omega} |u|^{p+1}\,dx \leq S^{-(p+1)/2}\,\|u\|^{p+1}.$$

This implies

$$\|u\| > S^{\frac{p+1}{2(p-1)}}\left[\frac{1+\beta}{p+\beta}a\right]^{\frac{1}{p-1}} \quad \text{for all } u\in\mathcal{M}^{-}. \tag{2.8}$$

By (2.2), we get $J(u)\geq C_0$ for all μ such that

$$\begin{aligned} 0 &< \mu<\mu_{**} \\ &= \frac{(p-1)\,A}{(p+\beta)\,a}\left(\frac{1-\beta}{2}\right)\left(\frac{1+\beta}{p+\beta}\right)^{\frac{3-\beta}{p-1}} S^{\frac{2(p+1)^2-4(p-1)^2+(1+\beta)(p-1)^2}{2(p-1)(p+1)}}. \end{aligned}$$

□

Proposition 2.6. ([2]) *(i) For all μ such that $0<\mu<\mu_*$, there exists a $(PS)_{c^+}$ sequence in $\mathcal{M}^+$.*

(ii) For all μ such that $0<\mu<\mu_{**}$, there exists a $(PS)_{c^-}$ sequence in $\mathcal{M}^-$ and for each $u\in\mathcal{H}\setminus\{0\}$, we write

$$t_M := t_{\max}(u) = \left[\frac{(p+1)(1+\beta)\,a\,\|u\|^2}{(p+\beta)\int_\Omega |u|^{p+1}\,dx}\right]^{\frac{1}{p-1}} > 0.$$

Lemma 2.7. *Let λ real parameter such that $0<\mu<\mu_*$. For each $u\in\mathcal{H}\setminus\{0\}$, there exist unique t^+ and t^- such that $0<t^+<t_M<t^-$,*

$(t^+u)\in\mathcal{M}^+$, $(t^-u)\in\mathcal{M}^-$

$$J_\theta\left(t^+u\right) = \inf J_\theta(tu) \quad \textit{for} \quad 0\leq t\leq t_M,$$

and

$$J_\theta\left(t^-u\right) = \sup J_\theta(tu) \quad \textit{for} \quad t\geq 0.$$

Proof. With minor modifications, we refer to [4]. □

3. Main Results

Proposition 3.1. *For all μ such that $0<\mu<\mu_*$, the functional J_θ has a minimizer $u_0^+\in\mathcal{M}^+$ and it satisfies:*

(i) $J_\theta\left(u_0^+\right)=c=c^+$,

(ii) u_0^+ *is a nontrivial solution of* (1.1).

Proof. If $0 < \mu < \mu_*$, then by Proposition 2.6 (i) there exists a $(PS)_{c^+}$ sequence $\{u_n\}$ in $\bar{B}_R \subset \mathcal{M}^+$ and it bounded by Lemma 2.2. Then, there exists $u_0^+ \in \mathcal{H}$ and we can extract a subsequence which will denoted by $\{u_n\}$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \text{ weakly in } \mathcal{H}, \\ u_n &\to u_0^+ \text{ strongly in } L^{1-\beta}\left(\Omega, |x|^{-\alpha}\right), \\ u_n &\to u_0^+ \text{ a.e in } \Omega. \end{aligned} \tag{3.1}$$

By (2.2) and (3.1), we have

$$\lim_{n\longrightarrow\infty} \int_\Omega \frac{|u_n+\theta|^{1-\beta} u_n}{|x|^\alpha} dx = \int_\Omega \frac{\left|u_0^+ + \theta\right|^{1-\beta} u_0^+}{|x|^\alpha} dx + o(1).$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of (1.1).

Now, we show that $\{u_n\}$ converges to u_0^+ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\| < \liminf\limits_{n\longrightarrow\infty} \|u_n\|$ and we obtain

$$\begin{aligned} c &\leq J_\theta\left(u_0^+\right) \\ &= \frac{p-1}{2(p+1)} a \left\|u_0^+\right\|^2 + \frac{p-3}{4(p+1)} \lambda \left\|u_0^+\right\|^4 \\ &\quad -\mu \frac{p+\beta}{(p+1)(1-\beta)} \int_\Omega \frac{\left|u_0^+ + \theta\right|^{1-\beta} u_0^+}{|x|^\alpha} dx \\ &< \liminf_{n\longrightarrow\infty} J(u_n) \\ &= c. \end{aligned}$$

We get a contradiction. Therefore, $\{u_n\}$ converge to u_0^+ strongly in $\mathcal{H}$. Moreover, we have $u_0^+ \in \mathcal{M}^+$. If not, then by Lemma 2.7, there are two numbers t_0^+ and t_0^-, uniquely defined so that $t_0^+ u_0^+ \in \mathcal{M}^+$ and $t^- u_0^+ \in \mathcal{M}^-$. In particular, we have $t_0^- < t_0^+ = 1$. Since

$$\frac{d}{dt} J_\theta\left(tu_0^+\right)_{\downarrow t=t_0^+} = 0 \text{ and } \frac{d^2}{dt^2} J_\theta\left(tu_0^+\right)_{\downarrow t=t_0^+} > 0,$$

there exists $t_0^- < t^- \leq t_0^+$ such that $J_\theta\left(t_0^- u_0^+\right) < J_\theta\left(t^+ u_0^+\right)$. By Lemma 2.7, we get

$$J_\theta\left(t_0^- u_0^+\right) < J_\theta\left(t^- u_0^+\right) < J_\theta\left(t_0^+ u_0^+\right) = J_\theta\left(u_0^+\right),$$

which contradicts the fact that $J_\theta\left(u_0^+\right) = c^+$. Since $J_\theta\left(u_0^+\right) = J_\theta\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right| \in \mathcal{M}^+$, by Lemma 2.3, we may assume that u_0^+ is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+ > 0$, see for example [18]. □

Our first main result is follow:

Theorem 3.2. *Assume that* $p \in (3,5]$, $0 \leq \alpha < \frac{3}{p+1}(p+\beta)$, $0 < \beta < 1$, $a, b > 0$, $\lambda > 0$ *and* μ *verifying* $0 < \mu < \mu_*$. *Then the system* (1.1) *has at least one positive solutions.*

Proof. Now, taking as a starting point the work of Tarantello [19], we establish the existence of a local minimum for J_θ on $\mathcal{M}^+$. □

Next, we establish the existence of a local minimum for J_θ on $\mathcal{M}^-$. For this, we require the following lemmas.

Lemma 3.3. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence for* J_θ *for some* $c \in \mathbb{R}$ *with* $u_n \rightharpoonup u$ *in* $\mathcal{H}$. *Then,* $J_\theta'(u) = 0$ *and* $J_\theta(u) \geq -\mu^{\frac{2}{1+\beta}} C(a,p,\beta,A,S)$, *with* $C(a,p,\beta,A,S) > 0$, *where*

$$C(a,p,\beta,A,S) = \frac{p-1}{2(p+1)} a \Big[\frac{(p+\beta)A}{(p-1)a}\Big]^{\frac{2}{1+p}} \frac{2}{1-\beta} S^{\frac{-(1-\beta)}{1+\beta}}.$$

Proof. It easy to prove that $J_\theta'(u) = 0$, which implies that $\left\langle J_\theta'(u), u\right\rangle = 0$, and

$$a\|u\|^2 + \lambda\|u\|^4 - \int_\Omega |u|^{p+1} dx - \mu \int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha} dx = 0.$$

Therefore,

$$\begin{aligned} J_\theta(u^+) &= \frac{p-1}{2(p+1)} a\|u^+\|^2 + \frac{p-3}{4(p+1)}\lambda\|u^+\|^4 \\ &\quad -\mu\frac{p+\beta}{(p+1)(1-\beta)}\int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha} dx. \end{aligned}$$

From (2.2) and considering $\|u\|$ small another, we get

$$\begin{aligned} \int_\Omega \frac{|u^+ + \theta|^{1-\beta}}{|x|^\alpha} dx \leq & \left[\frac{4\pi(p+\beta)}{3(p+\beta)-\alpha(p+1)}\right]^{\frac{p+\beta}{p+1}} \\ & \times R_0^{\frac{N}{p+1}(p+\beta)-\alpha} \|u\|^{1-\beta} (S)^{\frac{-(1-\beta)}{2}}, \end{aligned} \tag{3.2}$$

which implies that

$$\begin{aligned} J_\theta(u) &\geq \frac{p-1}{2(p+1)} a\|u^+\|^2 + \frac{p-3}{4(p+1)}\lambda\|u^+\|^4 \\ &\quad -\mu\frac{p+\beta}{(p+1)(1-\beta)} A\|u\|^{1-\beta}(S)^{\frac{-(1-\beta)}{2}} \\ &\geq \frac{p-1}{2(p+1)} a\|u\|^2 - \mu\frac{p+\beta}{(p+1)(1-\beta)} A\|u\|^{1-\beta}(S)^{\frac{-(1-\beta)}{2}} \end{aligned}$$

with

$$A=\left[\frac{4\pi\left(p+\beta\right)}{3\left(p+\beta\right)-\alpha\left(p+1\right)}\right]^{\frac{p+\beta}{p+1}}R_0^{\frac{N}{p+1}(p+\beta)-\alpha}.$$

Using (3.2) and function $f\left(t\right)=Dt^2-\mu Et^{1-\beta}$, we obtain that

$$f\left(t\right)\geq-\mu^{\frac{2}{1+\beta}}C\left(a,p,\beta,A,S\right)$$

for all $t>0$ small another, where

$$C\left(a,p,\beta,A,S\right)=D\left[\frac{\left(1-\beta\right)E}{2D}\right]^{\frac{2}{1+\beta}}\frac{\beta}{\left(1-\beta\right)}$$

with

$$D=\frac{p-1}{2\left(p+1\right)}a \text{ and } E=\frac{p+\beta}{\left(p+1\right)\left(1-\beta\right)}S^{\frac{-(1-\beta)}{2}}A.$$

Since $0<\beta<1$, we have $C\left(a,p,\beta,A,S\right)>0$. Then we conclude that

$$J_\theta\left(u\right)\geq-\mu^{\frac{2}{1+\beta}}C\left(a,p,\beta,A,S\right).$$

□

Lemma 3.4. *Let $\mu\in(0,\mu_{**})$. Then the functional J satisfies the $(PS)_c$ condition in $\mathcal{H}$ with $c<c^*$, where*

$$c^*=\frac{\left(p-1\right)\left(p+\beta\right)}{2\left(1+\beta\right)\left(p+1\right)}S^{\frac{p+1}{p-1}}-\mu^{\frac{2}{1+\beta}}C\left(a,p,\beta,A,S\right).$$

Proof. If $0<\mu<\mu_{**}$, then by Proposition 2.6 (ii) there exists a $(PS)_c$ sequence $\{u_n\}$ in $\mathcal{M}$, and it is bounded by Lemma 2.2. Then, there exists $u\in\mathcal{H}$ and we can extract a subsequence which will denoted by $\{u_n\}$ such that

$$\begin{aligned}u_n &\rightharpoonup u \text{ weakly in } \mathcal{H},\\ u_n &\rightharpoonup u \text{ weakly in } L^{p+1}\left(\Omega\right),\\ u_n &\to u \text{ a.e in } \Omega.\end{aligned}$$

Then, u is a weak solution of (1.1). Let $v_n=u_n-u$. Then, by Brezis-Lieb [2] we obtain

$$\left\|v_n\right\|^2=\left\|u_n\right\|^2-\left\|u\right\|^2+o_n\left(1\right), \tag{3.3}$$

$$\left\|v_n\right\|^4=\left\|u_n\right\|^4-\left\|u\right\|^4+o_n\left(1\right) \tag{3.4}$$

and

$$\int_\Omega\left|v_n\right|^{p+1}dx=\int_\Omega\left|u_n\right|^{p+1}-dx\int_\Omega\left|u\right|^{p+1}dx+o_n\left(1\right). \tag{3.5}$$

Since

$$J_\theta\left(u_n\right)=c+o_n\left(1\right),\ J_\theta^{'}\left(u_n\right)=o_n\left(1\right),$$

and by (3.3) and (3.5) we deduce that

$$\begin{aligned} \frac{1}{2}a\left\|v_n\right\|^2+\frac{1}{4}\lambda\left\|v_n\right\|^4-\frac{1}{p+1}\int_\Omega\left|v_n\right|^{p+1}dx &= c-J_\theta\left(u\right)+o_n\left(1\right), \quad (3.6)\\ a\left\|v_n\right\|^2+\lambda\left\|v_n\right\|^4-\int_\Omega\left|v_n\right|^{p+1}dx &= o_n\left(1\right). \end{aligned}$$

Hence, we may assume that

$$\left\|v_n\right\|^2\longrightarrow l,\quad \int_\Omega\left|v_n\right|^{p+1}dx\longrightarrow l. \tag{3.7}$$

Moreover, by Sobolev inequality we have

$$\left\|v_n\right\|^2\geq S\int_\Omega\left|v_n\right|^{p+1}dx. \tag{3.8}$$

Combining (3.8) and (3.7), we obtain

$$l\geq l^{\frac{2}{p+1}}S.$$

Either

$$l=0 \quad\text{or}\quad l\geq S^{\frac{p+1}{p-1}}.$$

Then from (3.6), (3.7), Lemma 3.3 and Lemma 3.4 we obtain

$$c\geq\frac{p-1}{2\left(p+1\right)}l+J_\theta\left(u_n\right)\geq c^*,$$

which is a contradiction. Therefore, $l=0$ and we conclude that $\{u_n\}$ converges to u strongly in $\mathcal{H}$. Thus, $\{J_\theta\left(u_n\right)\}$ converges to $J_\theta\left(u\right)=c$ as $n\to+\infty$. □

Lemma 3.5. *There exist $v\in\mathcal{H}$ and $\Lambda_*>0$ such that for $\mu\in(0,\Lambda_*)$, one has*

$$\sup_{t\geq0}J_\theta\left(tv\right)<c^*.$$

In particular, $c<c^*$ for all $\mu\in(0,\Lambda_*)$.

Proof. Let $\varphi_\varepsilon\left(x\right)$ satisfies (1.3). Then, we have

$$\frac{\lambda}{1-\beta}\int_\Omega\frac{\left|\varphi_\varepsilon\right|^{1-\beta}}{\left|x\right|^\alpha}dx>0.$$

We consider the two functions:

$$f\left(t\right):=J_\theta\left(t\varphi_\varepsilon\right)\quad\text{and}\quad g\left(t\right)=\frac{t^2}{2}a\left\|\varphi_\varepsilon\right\|^2-\frac{t^{p+1}}{p+1}\int_\Omega\left|\varphi_\varepsilon\right|^{p+1}dx.$$

Then, for all $\mu\in(0,\mu_{**})$,

$$f\left(0\right)=0<c^*.$$

By the continuity of f, there exists $t_0 > 0$ such that

$$f(t) < c^*, \quad \forall t \in (0, t_0).$$

On the other hand, we have

$$\max_{t \geq 0} g(t) := \frac{(p-1)(p+\beta)}{2(1+\beta)(p+1)} a S^{\frac{p+1}{p-1}}.$$

Then, we obtain

$$\sup_{t \geq 0} J_\theta(t\varphi_\varepsilon) < \frac{(p-1)(p+\beta)}{2(1+\beta)(p+1)} S^{\frac{p+1}{p-1}} - \mu^{\frac{2}{1+\beta}} C(a, p, \beta, A, S).$$

Now, taking $\mu > 0$ such that

$$-\frac{\mu t_0^{1-\beta}}{1-\beta} \int_\Omega \frac{|\varphi_\varepsilon|^{1-\beta}}{|x|^\alpha} dx < -\mu^{\frac{2}{1+\beta}} C(a, p, \beta, A, S),$$

we obtain

$$0 < \mu < \frac{t_0^{1+\beta}}{[(1-\beta) C(a, p, \beta, A, S)]^{\frac{1+\beta}{1-\beta}}} \left[\int_\Omega \frac{|\varphi_\varepsilon|^{1-\beta}}{|x|^\alpha} dx\right]^{\frac{1+\beta}{1-\beta}} = \Lambda_1.$$

Set

$$\Lambda_* = \min\{\mu_{**}, \Lambda_1\}.$$

We deduce that $c^- < c^*$ for all $\mu \in (0, \Lambda_*)$, then there exists $t_n > 0$ such that $t_n w_n \in \mathcal{M}^-$ with w_n satisfying (1.3),

$$c^- \leq J_\theta(t_n w_n) \leq \sup_{t \geq 0} J_\theta(t w_n) < c^*.$$

□

Lemma 3.6. *For all μ such that $0 < \mu < \Lambda_* = \min\{\mu_{**}, \Lambda_1\}$, the functional J_θ has a minimizer u_0^- in $\mathcal{M}^-$ and it satisfies*
(i) $J_\theta(u_0^-) = c^- > 0$,
(ii) u_0^- is a nontrivial solution of (1.1) *in $\mathcal{H}$.*

Proof. By (ii) in Proposition 2.6, there exists a $(PS)_{c^-}$ sequence $\{u_n\}$ for J_θ, in $\mathcal{M}^-$ for all $\mu \in (0, \mu_{**})$. From Lemmas 3.4, 3.5 and by (ii) in Lemma 2.5, for $\mu \in (0, \Lambda_1)$, J_θ satisfies $(PS)_{c^-}$ condition and $c^- > 0$. Then, we get that $\{u_n\}$ is bounded in $\mathcal{H}$. Therefore, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $u_0^- \in \mathcal{M}^-$ such that $\{u_n\}$ converges to u_0^- strongly in $\mathcal{H}$ and $J_\theta(u_0^-) = c^- > 0$ for all $\mu \in (0, \Lambda_*)$.

Finally, by using the same arguments as in the proof of the Proposition 3.1 for all $\mu \in (0, \mu_*)$, we have that u_0^- is a solution of (1.1). □

Theorem 3.7. *In addition to the assumptions of the Theorem* 3.2*, there exists* $\Lambda_* \in (0, \mu_{**})$ *such that if* μ *satisfying* $0 < \mu < \Lambda_*$*, then* (1.1) *has at least two positive solutions.*

Proof. For the complete proof of this theorem, by Proposition 3.1 and Lemma 3.6, we obtain that (1.1) has two positive solutions $u_0^+ \in \mathcal{M}^+$ and $u_0^- \in \mathcal{M}^-$. Since $\mathcal{M}^+ \cap \mathcal{M}^- = \emptyset$, u_0^+ and u_0^- are distinct.

Finally, for every $\theta \in (0,1)$, problem (1.7) has solution $u_\theta \in \mathcal{H} \setminus \{0\}$ such that $J_\theta(u_\theta) = c_\theta$ and $J'_\theta(u_\theta) = 0$. Thus there exist $\{\theta_n\} \subset (0,1)$ with $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then we get $u = \lim_{n \longrightarrow \infty} u_{\theta_n}$. □

4. Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem on the constraint defined by the Nehari manifold $\mathcal{M}$, which are solutions of our problem. Under some sufficient conditions on coefficients of equation of (1.1) such that $a, b > 0$, $p \in (3, 5]$, $\lambda \geq 0$, $\mu > 0$, $0 < \alpha < 3(p+\beta)/p$ and $0 < \beta < 1$, we split $\mathcal{M}$ in two disjoint subsets $\mathcal{M}^+$ and $\mathcal{M}^-$ thus we consider the minimization problems on $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively.

Acknowledgement: The authors gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number(4524) during the academic year 1445AH/2024AD.

References

[1] Z. I. Almuhiameed, *Existence results for p-Laplacian problems involving singular cylindrical potential* , Nonlinear Funct. Anal. Appl., **28**(4) (2023), 1005-1015.

[2] C. Alves, F. Correa and T. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type,* Comput. Math. Appl., **49** (2005), 85–93.

[3] K.J. Brown and Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign changing weight function,* J. Diff. Equ., **2** (2003), 481–499.

[4] B. Cheng, *New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems,* J. Math. Anal. Appl., **394** (2012), 488–495.

[5] B. Cheng and X. Wu, *Existence results of positive solutions of Krichhoff problems,* Nonlinear Anal., **71** (2009), 4883–4892.

[6] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems,* Nonlinear Anal., **30** (1997), 4619–4627.

[7] F.J S.A. Correa, S.D.B. Menezes and J. Ferreira, *On a class of problems involving a nonlocal operator,* Appl. Math. Comput., **147** (2004), 475–489.

[8] P. D'Ancona and S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data,* Invent. Math., **108** (1992), 247–262.

[9] M.E.O. El Mokhtar and A. Matallah, *Existence of Multiple Positive Solutions for Brezis–Nirenberg-Type Problems Involving Singular Nonlinearities,* J. Math., **2021** (2021) 1–8.

[10] M. Haddaoui, N. Tsouli and A. Zaki, *Study of a critical Φ-Kirchhoff type equations in Orlicz-Sobolev spaces*, Nonlinear Funct. Anal. Appl., **27**(3) (2022), 641-648.
[11] D. Kang and S. Peng, *Positive solutions for singular elliptic problems,* Appl. Math. Lett., **17** (2004), 411–416.
[12] C. Lei, J. Liao and C. Tang, *Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents,* J. Math. Anal. Appl., **421** (2015), 521–538.
[13] Y. Li, F. Li and J. Shi, *Existence of positive solutions to Kirchhoff type problems with zero mass,* J. Math. Anal. Appl., **410** (2014), 361–374.
[14] J.L. Lions, *On some questions in boundary value problems of mathematical physics,* in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations in: North-HollandMath. Stud. North-Holland. Amsterdam, **30** (1978), 284–346.
[15] X. Liu and Y. Sun, *Multiple positive solutions for Kirchhoff type problems with singularity,* Commun. Pure Appl. Anal., **12** (2013), 721–733.
[16] A. Mao and S. Luan, *Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems,* J. Math. Anal. Appl., **383** (2011), 239–243.
[17] A. Mao and Z. Zhang, *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition,* Nonlinear Anal., **70** (2009), 1275–1287.
[18] K. Sabri, M. El Mokhtar Ould El Mokhtar and A. Matallah, *Multiple nontrivial solutions for critical p-Kirchhoff type problems in R^N*, Nonlinear Funct. Anal. Appl., **29**(1) (2024), 35-45.
[19] J. Simon, *Sur des équations aux dérivées partielles nonlinéaires,* Thèse, Paris, 1977.
[20] J. Sun and C. Tang, *Existence and multiplicity of solutions for Kirchhoff type equations,* Nonlinear Anal., **74** (2011), 1212–1222.
[21] S. Terracini, *On positive entire solutions to a class of equations with singular coefficient and critical exponent,* Adv. Diff. Equ., **1** (1996), 241–264.
[22] Q. Xie, X. Wu and C. Tang, *Existence and multiplicity of solutions for Kirchhoff type problem with critical exponent,* Commun. Pure Appl. Anal., **12** (2013), 2773–2786
[23] Z. Zhang and K. Perera, *Sign-changing solutions of Kirchhoff type problems via invariant sets of descent flow,* J. Math. Anal. Appl., **317** (2006), 456–463.