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ON FIXED POINT THEOREMS SATISFYING COMPATIBILITY PROPERTY IN MODULAR G-METRIC SPACES

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Abstract. In this paper, a pair of ω-compatible self mappings in the setting of modular G-metric space is defined. We prove the existence and uniqueness of common fixed point of pairs of ω-compatible self mappings in a G-complete modular G-metric space. Furthermore, we give an example to justify our claims. The results established in this paper extend, improve, generalize and complement some existing results in literature.

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1. Introduction

In 1986, Jungck [14] introduced a specified treatment of common fixed points in metric spaces and defined compatibility of two self-mappings f and g in a metric space (X, d) in its rough sense as $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X. Shrivastava *et al.* [37] proved some common fixed point theorems for compatible mappings in metric spaces following the ideas of Jungck [14].

The concept of 2-metric spaces was initiated by [8] and Gahler [13]. Baskaran *et al* [10] established some common fixed point theorems for expansive mapping by using compatibility and sequentially continuous mappings in 2 metric space. Dhage [12], generalized the work in [13] to D-metric spaces. These authors claimed that their results generalized the concept of metric spaces.

In 2003, Mustafa and Sims [20] pointed out that the fundamental topological properties of D-metric spaces introduced by Dhage [12] were incorrect. To overcome these drawbacks about D-metric spaces, Mustafa and Sims [21] introduced a generalization of metric spaces, which they called G-metric space and proved some fixed point theorems in this framework. Mustafa *et al.* [19] proved some fixed point results on complete G-metric spaces. Mustafa [18], proved several common fixed points results for pair of weakly compatible mappings satisfying certain contractive conditions on G-metric space. Abbas *et al.* [1] proved common fixed point theorems for three mappings in generalized metric spaces and their results do not rely on continuity and commutativity of any mappings involved therein. In the same sense, Abbas and Rhodes [2], obtained several fixed point theorems for occasionally weakly compatible mappings defined on a symmetric space satisfying a generalized contractive condition.

In 2010, Chistyakov [11] introduced a generalized classical metric spaces called modular metric space or parameterized metric space with the time parameter ($\lambda > 0$, say) and his anticipated outcome were to define the notion of a modular on an arbitrary set, and developed the theory of metric spaces generated by modular(s), called modular metric spaces. The results of Chistyakov [11] extended the results given by Nakano [22], Musielak and Orlicz [34], Musielak [17] to modular metric spaces. Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions. Abdou [3] studied the existence of fixed points for contractive and nonexpansive Kannan mappings in the setting of modular metric spaces. These are related to the successive approximations of fixed points (via orbits) which converge to the fixed points in the modular sense, which is weaker than the metric convergence and other fixed point results in modular metric spaces can be found in [27], [31], [32], [33] and [35] and the references therein.

Azadifar *et al.* [5] initiated the idea of modular G-metric spaces and obtained some fixed point theorems for contractive mappings defined on modular G-metric spaces. Azadifar *et al.* [6] proved the existence of the unique common fixed point of a pair of weakly compatible mappings satisfying Φ-mappings in modular G-metric spaces and Okeke and Francis [23] proved the existence and uniqueness of fixed point of mappings satisfying Geraghty-type contractions in the setting of preordered modular G-metric spaces. The authors applied their results in solving nonlinear Volterra-Fredholm-type integral equations. Furthermore, Okeke and Francis [24] proved some interesting fixed point theorems for the class of asymptotically T-regular mappings in the framework of preordered modular G-metric spaces and their result were used in solving nonlinear integral equations. For other interesting results see ([26]-[29]) and the references therein.

Our aim in this paper is to define a pair of ω-compatible self-mappings in the setting of modular G-metric space is define. We prove the existence and uniqueness of some common fixed point theorems for this class of ω-compatible self-mappings in a G-complete modular G-metric space. An example will be given to justify our claim.

2. Preliminaries

We begin this section by recalling some definitions and results which will be useful in this paper.

Theorem 2.1. ([9]) *Let* $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ *be three sequences in* $\mathbb{R}$ *such that*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ell$,

(ii) *for some positive integer* N, $a_n \leq c_n \leq b_n$ *for all* $n \geq N$.

Then $\lim_{n\to\infty} c_n = \ell$.

Definition 2.2. ([14]) Self-mappings f and g of a metric space (X, d) are compatible if $\lim_{n\to\infty} d(gf(x_n), fg(x_n)) = 0$, whenever $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, for some t in X.

Definition 2.3. ([15]) Let f and g be mappings from a G-metric space (X, G) into itself. The mappings f and g are said to be compatible if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$$

or

$$\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.4. ([5]) Let X be a nonempty set and for $\lambda > 0$, $\omega_\lambda^G : (0,\infty) \times X \times X \times X \to [0,\infty]$ be a function satisfying;

(1) $\omega_\lambda^G(x,y,z) = 0$ for all $x,y,z \in X$ and $\lambda > 0$ if $x = y = z$,
(2) $\omega_\lambda^G(x,x,y) > 0$ for all $x,y \in X$ and $\lambda > 0$ with $x \neq y$,
(3) $\omega_\lambda^G(x,x,y) \leq \omega_\lambda^G(x,y,z)$ for all $x,y,z \in X$ and $\lambda > 0$ with $z \neq y$,
(4) $\omega_\lambda^G(x,y,z) = \omega_\lambda^G(x,z,y) = \omega_\lambda^G(y,z,x) = \cdots$ for all $\lambda > 0$ (symmetry in all three variables),
(5) $\omega_{\lambda+\mu}^G(x,y,z) \leq \omega_\lambda^G(x,a^*,a^*) + \omega_\mu^G(a^*,y,z)$, for all $x,y,z,a^* \in X$ and $\lambda,\mu > 0$.

Then, the function ω_λ^G is called a modular G-metric on X.

Definition 2.5. ([5]) Let $(X_\omega, \omega_\lambda^G)$ be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{ω^G} is modular G-convergent to x^*, if it converges to x^* in the topology $\tau(\omega_\lambda^G)$.

A function $T : X_\omega \to X_\omega$ at $x^* \in X_{\omega^G}$ is called modular G-continuous if $\omega_\lambda^G(x_n, x^*, x^*) \to 0$ then $\omega_\lambda^G(Tx_n, Tx^*, Tx^*) \to 0$, for all $\lambda > 0$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ modular G-converges to x^* as $n \to \infty$, if $\lim_{n\to\infty} \omega_\lambda^G(x_n, x_m, x^*) = 0$. That is for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x^*) < \epsilon$ for all $n, m \geq n_0$. Here we say that x^* is modular G-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Without any confusion we will take X_{ω^G} as a modular ω^G-metric space.

Definition 2.6. ([5]) Let (X_ω, ω^G) be a modular ω^G-metric space. Then $\{x_n\} \subseteq X_{\omega^G}$ is said to be modular ω^G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega_\lambda^G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq n_\epsilon$ and $\lambda > 0$.

A modular G-metric space X_{ω^G} is said to be modular G-complete if every modular ω^G-Cauchy sequence in X_{ω^G} is modular ω^G-convergent in X_{ω^G}.

Proposition 2.7. ([5]) *Let (X_ω, ω^G) be a modular ω^G-metric space, for any $x, y, z, a \in X_{\omega^G}$, it follows that:*

(1) *If $\omega_\lambda^G(x,y,z) = 0$ for all $\lambda > 0$, then $x = y = z$.*
(2) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,x,y) + \omega_{\frac{\lambda}{2}}^G(x,x,z)$ *for all $\lambda > 0$.*
(3) $\omega_\lambda^G(x,y,y) \leq 2\omega_{\frac{\lambda}{2}}^G(y,x,x)$ *for all $\lambda > 0$.*
(4) $\omega_\lambda^G(x,y,z) \leq \omega_{\frac{\lambda}{2}}^G(x,a,z) + \omega_{\frac{\lambda}{2}}^G(a,y,z)$ *for all $\lambda > 0$.*

(5) $\omega_\lambda^G(x,y,z) \le \frac{2}{3}(\omega_{\frac{\lambda}{2}}^G(x,y,a) + \omega_{\frac{\lambda}{2}}^G(x,a,z) + \omega_{\frac{\lambda}{2}}^G(a,y,z))$ *for all* $\lambda > 0$.
(6) $\omega_\lambda^G(x,y,z) \le \omega_{\frac{\lambda}{2}}^G(x,a,a) + \omega_{\frac{\lambda}{4}}^G(y,a,a) + \omega_{\frac{\lambda}{4}}^G(z,a,a)$ *for all* $\lambda > 0$.

Proposition 2.8. ([5]) *Let* (X_ω, ω^G) *be a modular* ω^G*-metric space and* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_ω. *Then the following are equivalent:*

(1) $\{x_n\}_{n\in\mathbb{N}}$ *is* ω^G*-convergent to* x,
(2) $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *relative to modular metric* ω_λ^G,
(3) $\omega_\lambda^G(x_n, x_n, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$,
(4) $\omega_\lambda^G(x_n, x, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$,
(5) $\omega_\lambda^G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$ *for all* $\lambda > 0$.

Definition 2.9. ([7]) Let X_ω be a modular metric space induced by metric modular ω. Two self-mappings T, h of X_ω are called ω-compatible if $\omega_\lambda(Thx_n, hTx_n) \to 0$, whenever $\{x_n\}_{n=1}^\infty$ is a sequence in X_ω such that $hx_n \to z$ and $Tx_n \to z$ for some $z \in X_\omega$ and for $\lambda > 0$.

Next, we state the definition below following [7], which will play some vital roles in Section 3 of this paper.

Definition 2.10. ([30]) Let (X_{ω^G}, ω^G) be a modular G-metric space. A pair $\{T_1, T_2\}$ is said to be ω-compatible if for all $\lambda > 0$,

$$\lim_{n\to\infty} \omega_\lambda^G(T_1T_2x_n, T_1T_2x_n, T_2T_1x_n) = 0$$

or

$$\lim_{n\to\infty} \omega_\lambda^G(T_2T_1x_n, T_2T_1x_n, T_1T_2x_n) = 0,$$

whenever, $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X_{ω^G} such that $\lim_{n\to\infty} T_1x_n = \lim_{n\to\infty} T_2x_n = x$, for $x \in X_{\omega^G}$.

Proposition 2.11. ([30]) *Let* (X_{ω^G}, ω^G) *be a modular* G*-metric space. Let* $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ *be two sequences in* X_{ω^G} *for which* $\lim_{n\to\infty} \omega_\lambda^G(a_n, a_n, b_n) = 0$ *if and only if* $\lim_{n\to\infty} \omega_\lambda^G(a_n, b_n, b_n) = 0$ *for all* $\lambda > 0$. *If* $\lim_{n\to\infty} a_n = a$ *for some* $a \in X_{\omega^G}$, *then* $\lim_{n\to\infty} b_n = a \in X_{\omega^G}$.

A point $x \in M$ is said to be a fixed point of a mapping T if $x = Tx$. And the set of fixed points of T will be denoted by $Fix(T)$, that is, $Fix(T) = \{x \in M : x = Tx\}$.

3. Main results

We begin this section with the following results.

Theorem 3.1. *Let* (X_{ω^G}, ω^G) *be a* G*-complete modular* G*-metric space and let* $T_i : X_{\omega^G} \to X_{\omega^G}$ *for* $i = 1, 2, 3, 4,$ *be four self* ω*-compatible mappings with* $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ *in which* T_3, T_4 *are continuous and that the pairs* $\{T_1, T_3\}$ *and* $\{T_2, T_4\}$ *are compatible so that there is a point* $y_0 \in X_{\omega^G}, \lambda > 0,$ *such that* $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, *for which the following condition is satisfied;*

$$\omega^G_\lambda(T_1x, T_1y, T_2z) \leq k\omega^G_\lambda(T_3x, T_3y, T_4z), \tag{3.1}$$

for each $x, y, z \in X_{\omega^G}$, *with* $k < 1$. *Then* T_i *have a unique common fixed point in* X_{ω^G} *for* $i = 1, 2, 3, 4$.

Proof. Let $x_0 \in X_{\omega^G}$. Since $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, there exists $x_1 \in X_{\omega^G}$ such that $T_1x_0 = T_4x_1$, and also as $T_2x_1 \in T_3(X_{\omega^G})$, we choose $x_2 \in X_{\omega^G}$ such that $T_2x_1 = T_3x_2$. In general, $x_{2n+1} \in X_{\omega^G}$ is chosen such that $T_1x_{2n} = T_4x_{2n+1}$ and $x_{2n+2} \in X_{\omega^G}$ such that $T_2x_{2n+1} = T_3x_{2n+2}$, we obtain a sequence $\{y_n\}_{n\geq 1}$ such that $y_{2n} = T_1x_{2n} = T_4x_{2n+1}$ and $y_{2n+1} = T_2x_{2n+1} = T_3x_{2n+2}$.

Now we show that $\{y_n\} \subseteq X_\omega$ is a modular G-Cauchy sequence. Indeed we proceed as follows;

$$\begin{aligned}\omega^G_\lambda(y_{2n}, y_{2n}, y_{2n+1}) &= \omega^G_\lambda(T_1x_{2n}, T_1x_{2n}, T_2x_{2n+1})\\ &\leq k\omega^G_\lambda(T_3x_{2n}, T_3x_{2n}, T_4x_{2n+1})\\ &= \omega^G_\lambda(y_{2n-1}, y_{2n-1}, y_{2n}).\end{aligned} \tag{3.2}$$

Therefore,

$$\omega^G_\lambda(y_{2n}, y_{2n}, y_{2n+1}) \leq k\omega^G_\lambda(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.3}$$

Using the above procedure and condition (3) of Proposition 2.7, we have

$$\begin{aligned}&\omega^G_\lambda(y_{2n-1}, y_{2n-1}, y_{2n})\\ &\quad\leq 2\omega^G_{\frac{\lambda}{2}}(y_{2n}, y_{2n}, y_{2n-1})\\ &\quad\leq 2\omega^G_\lambda(y_{2n}, y_{2n}, y_{2n-1})\\ &\quad= 2\omega^G_\lambda(T_1x_{2n}, T_1x_{2n}, T_2x_{2n-1})\\ &\quad\leq \frac{k}{2}\max\{\omega^G_\lambda(T_3x_{2n}, T_3x_{2n}, T_4x_{2n-1}), \omega^G_\lambda(T_2x_{2n}, T_2x_{2n}, T_2x_{2n-1}),\\ &\qquad\qquad \omega^G_\lambda(T_1x_{2n-1}, T_1x_{2n-1}, T_3x_{2n-1}), \omega^G_\lambda(T_1x_{2n}, T_1x_{2n}, T_2x_{2n-1})\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{2}\max\{\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}), \omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1}), \\
&\qquad \omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}), \omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1})\} \\
&= \frac{k}{2}\max\{\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}), \omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1})\}.
\end{aligned} \tag{3.4}
$$

Then

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1}) \leq \frac{k}{2}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}). \tag{3.5}$$

By the above processes, we get

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \leq \frac{k}{2}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.6}$$

Therefore, for all n and $\lambda > 0$, we have

$$
\begin{aligned}
\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) &\leq \frac{k}{2}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}) \\
&\ \vdots \\
&\leq (\frac{k}{2})^{n-1}\omega_\lambda^G(y_0, y_0, y_1)
\end{aligned} \tag{3.7}
$$

for $\lambda > 0$ and $n \geq 2$.

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, that is, condition (5) of Definition 2.4 we have

$$
\begin{aligned}
\omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) &\leq \omega_{\frac{\lambda}{m-n}}^G(y_{2n}, y_{2n+1}, y_{2n+1}) + \omega_{\frac{\lambda}{m-n}}^G(y_{2n+1}, y_{2n+2}, y_{2n+2}) \\
&\quad + \omega_{\frac{\lambda}{m-n}}^G(y_{2n+2}, y_{2n+3}, y_{2n+3}) + \omega_{\frac{\lambda}{m-n}}^G(y_{2n+3}, y_{2n+4}, y_{2n+4}) \\
&\quad + \cdots + \omega_{\frac{\lambda}{m-n}}^G(y_{2m-1}, y_{2m}, y_{2m}) \\
&\leq \omega_{\frac{\lambda}{n}}^G(y_{2n}, y_{2n+1}, y_{2n+1}) + \omega_{\frac{\lambda}{n}}^G(y_{2n+1}, y_{2n+2}, y_{2n+2}) \\
&\quad + \omega_{\frac{\lambda}{n}}^G(y_{2n+2}, y_{2n+3}, y_{2n+3}) + \omega_{\frac{\lambda}{n}}^G(y_{2n+3}, y_{2n+4}, y_{2n+4}) \\
&\quad + \cdots + \omega_{\frac{\lambda}{n}}^G(y_{2m-1}, y_{2m}, y_{2m}) \\
&\leq ((\frac{k}{2})^n + (\frac{k}{2})^{n+1} + \cdots + (\frac{k}{2})^{m-1})\omega_\lambda^G(y_1, y_1, y_0) \\
&\leq \frac{(\frac{k}{2})^n}{1-(\frac{k}{2})}\omega_\lambda^G(y_1, y_1, y_0)
\end{aligned} \tag{3.8}
$$

for all $m > n \geq N \in \mathbb{N}$, then

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) \leq \frac{(\frac{k}{2})^n}{1-(\frac{k}{2})}\omega_\lambda^G(y_1, y_1, y_0) \tag{3.9}$$

for all $m, l, n \geq N$ for some $N \in \mathbb{N}$, so that by condition (2) of Proposition 2.7, we have

$$\omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) \leq \omega_{\frac{\lambda}{2}}^G(y_{2n}, y_{2n}, y_{2m}) + \omega_{\frac{\lambda}{2}}^G(y_{2l}, y_{2m}, y_{2m}), \tag{3.10}$$

so that

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) &\leq \omega_{\frac{\lambda}{2}}^G(y_{2n}, y_{2n}, y_{2m}) + \omega_{\frac{\lambda}{2}}^G(y_{2l}, y_{2m}, y_{2m}) \\ &\leq \omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) + \omega_\lambda^G(y_{2l}, y_{2m}, y_{2m}) \\ &\leq \frac{(\frac{k}{2})^n}{1-(\frac{k}{2})}\omega_\lambda^G(y_1, y_1, y_0) + \frac{(\frac{k}{2})^n}{1-(\frac{k}{2})}\omega_\lambda^G(y_1, y_1, y_0) \\ &= \left(\frac{(\frac{k}{2})^n}{1-(\frac{k}{2})} + \frac{(\frac{k}{2})^n}{1-(\frac{k}{2})}\right)\omega_\lambda^G(y_1, y_1, y_0). \end{aligned} \tag{3.11}$$

Thus, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) = 0 \tag{3.12}$$

or

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(y_n, y_m, y_l) = 0. \tag{3.13}$$

Therefore, we can easily see that $\{y_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence in X_{ω^G}. The modular G-completeness of (X_{ω^G}, ω^G) implies that for any $\lambda > 0$, $\lim\limits_{n,m\to\infty} \omega_\lambda^G(y_n, y_m, u) = 0$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(y_n, y_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim\limits_{n\to\infty} y_n = u \in X_{\omega^G}$ as $n \to \infty$, or by applying condition (5) of Proposition 2.8, such that

$$\lim_{n\to\infty} T_1x_{2n} = \lim_{n\to\infty} T_4x_{2n+1} = \lim_{n\to\infty} T_2x_{2n+1} = \lim_{n\to\infty} T_3x_{2n+2} = u.$$

Now we show that u is a common fixed point of the mappings, T_1, T_2, T_3 and T_4. Recall that T_3 is continuous, then it follows that $\lim\limits_{n\to\infty} T_3^2x_{2n+2} = T_3(\lim\limits_{n\to\infty} T_3x_{2n+2}) = T_3u$ and $\lim\limits_{n\to\infty} T_3(T_1x_{2n}) = T_3u$. Since $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are ω-compatible and for all $\lambda > 0$, we have

$$\lim_{n\to\infty} \omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_3T_1x_{2n}) = 0.$$

Thus by Proposition 2.11, we have $\lim\limits_{n\to\infty} T_1(T_3x_{2n}) = T_3u$. On putting $x = y = T_3x_{2n}$ and $z = x_{2n+1}$ into inequality (3.1), we have

$$\begin{aligned}\omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) &\leq k\omega_\lambda^G(T_3T_3x_{2n}, T_3T_3x_{2n}, T_4x_{2n+1}) \\ &= k\omega_\lambda^G(T_3^2x_{2n}, T_3^2x_{2n}, T_4x_{2n+1}). \end{aligned} \tag{3.14}$$

Taking the limit of both sides of inequality (3.14) as n tends to infinity, we have

$$\begin{aligned}\omega_\lambda^G(T_3u, T_3u, u) &= \lim_{n\to\infty} \omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1})\\ &\le k \lim_{n\to\infty} \omega_\lambda^G(T_3T_3x_{2n}, T_3T_3x_{2n}, T_4x_{2n+1})\\ &= k \lim_{n\to\infty} \omega_\lambda^G(T_3^2x_{2n}, T_3^2x_{2n}, T_4x_{2n+1})\\ &= k\omega_\lambda^G(T_3u, T_3u, u). \end{aligned}\tag{3.15}$$

Hence,

$$(1-k)\omega_\lambda^G(T_3u, T_3u, u) \le 0, \tag{3.16}$$

where, $k < 1$ for all $\lambda > 0$, thus $T_3u = u$. Again, in a similar way, note that T_4 is continuous then $\lim_{n\to\infty} T_4^2x_{2n+1} = T_4u$ and $\lim_{n\to\infty} T_4T_2x_{2n+1} = T_4u$ and $\lim_{n\to\infty} T_1T_4x_{2n+1} = T_4u$. Since $\{T_2, T_4\}$ is ω-compatible and for all $\lambda > 0$, we have that

$$\lim_{n\to\infty} \omega_\lambda^G(T_2T_4x_{2n+1}, T_2T_4x_{2n+1}, T_4T_2x_{2n+1}) = 0.$$

Thus by Proposition 2.11, we have that $\lim_{n\to\infty} T_2(T_4x_{2n+1}) = T_4u$. On putting $x = y = x_{2n}$ and $z = T_4x_{2n+1}$ into inequality (3.1), we have

$$\begin{aligned}\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1}) &\le k\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4T_4x_{2n+1})\\ &= k\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4^2x_{2n+1}),\end{aligned}\tag{3.17}$$

on taking the limit of both sides of inequality (3.17) as n tends to infinity, we have

$$\begin{aligned}\omega_\lambda^G(u, u, T_4u) &= \lim_{n\to\infty} \omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1})\\ &\le k \lim_{n\to\infty} \omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4T_4x_{2n+1})\\ &= k\omega_\lambda^G(u, u, T_4u),\end{aligned}\tag{3.18}$$

then we have that $T_4u = u$ for all $\lambda > 0$ and $k < 1$.

Furthermore, if we put $x = x_{2n}, y = u$ and $z = x_{2n+1}$, then from inequality (3.1), we get

$$\omega_\lambda^G(T_1x_{2n}, T_1u, T_2x_{2n+1}) \le k\omega_\lambda^G(T_3x_{2n}, T_3u, T_4x_{2n+1}) \tag{3.19}$$

as $n \longrightarrow \infty$, we have

$$\omega_\lambda^G(u, T_1u, u) \le k\omega_\lambda^G(u, T_3u, u),$$

so that

$$\omega_\lambda^G(u, u, T_1u) \le k\omega_\lambda^G(u, T_3u, u), \tag{3.20}$$

$$\omega_\lambda^G(u, u, T_1u) \le k\omega_\lambda^G(u, T_3u, u) = k\omega_\lambda^G(u, u, u) = 0. \tag{3.21}$$

Hence, $T_1u = u$.

Finally, using the fact that $T_1u = T_3u = T_4u = u$, then inequality (3.1), becomes

$$\begin{aligned}\omega_\lambda^G(u,u,T_2u) &= \omega_\lambda^G(T_1u,T_1u,T_2u)\\ &\le k\omega_\lambda^G(T_3u,T_3u,T_4u)\\ &= k\omega_\lambda^G(u,u,u) = 0. \end{aligned} \tag{3.22}$$

Hence, $T_2u = u$. Therefore, we have that

$$T_1u = T_2u = T_3u = T_4u = u, \tag{3.23}$$

which shows that u is a common fixed point of T_1, T_2, T_3 and T_4.

To prove uniqueness, suppose that there exists another common fixed point of T_1, T_2, T_3 and T_4 that is, there is a $u^* \in X_{\omega^G}$ such that $u^* = T_1u^* = T_2u^* = T_3u^* = T_4u^*$. If $u \neq u^*$, and for all $\lambda > 0$, again inequality (3.1) becomes;

$$\begin{aligned}\omega_\lambda^G(u,u,u^*) &= \omega_\lambda^G(T_1u,T_1u,T_2u^*)\\ &\le k\omega_\lambda^G(T_3u,T_3u,T_4u^*)\\ &= k\omega_\lambda^G(u,u,u^*).\end{aligned}$$

Therefore,

$$\omega_\lambda^G(u,u,u^*) \le k\omega_\lambda^G(u,u,u^*), \tag{3.24}$$

so that

$$(1-k)\omega_\lambda^G(u,u,u^*) \le 0, \tag{3.25}$$

where, $k < 1$ and $\lambda > 0$, thus $u = u^*$. Therefore, the proof of Theorem 3.1 is now completed. □

Remark 3.2. Theorem 3.1 is a generalization of Theorem 3.1 in Agarwal and Karapinar [4]. Suppose we allow $T_1 = T_2$ and $T_3 = T_4$, then we get inequality (6) of Agarwal and Karapinar [4] in the setting of modular G-metric spaces. Again, if $y = z$ and $T_1 = T_2$ and $T_3 = T_4$, in inequality (3.1), then we get inequality (10) of Theorem 3.2 in Agarwal and Karapinar [4] in the setting of modular G-metric spaces.

The example below follows from Example 3.1 in Okeke et al. [30].

Example 3.3. Let $X_{\omega^G} = \mathbb{R}^+ \cup \{\infty\}$. Define ω-compatible mappings $T_1, T_2, T_3, T_4 : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by $T_1x = \left(\frac{x}{2}\right)^{8p}, T_2x = \left(\frac{x}{2}\right)^{4p}, T_3x = \left(\frac{x}{2}\right)^{2p}$ and $T_4x = \left(\frac{x}{2}\right)^{p}$ for all $x \in \mathbb{R}^+ \cup \{\infty\}, p \ge 1$ and $n \in \mathbb{N}$. Then the mappings T_1, T_2, T_3, T_4 satisfies inequality (3.1) of Theorem 3.1.

Corollary 3.4. *Let* (X_{ω^G}, ω^G) *be a G-complete modular G-metric space. Let* $T_i : X_{\omega^G} \to X_{\omega^G}$ *for* $i = 1, 2, 3, 4$*, be four self* ω*-compatible mappings with* $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ *in which* T_3, T_4 *are continuous and that the pairs* $\{T_1, T_3\}$ *and* $\{T_2, T_4\}$ *are compatible so that there is a point* $y_0 \in X_{\omega^G}, \lambda > 0$*, such that* $\omega^G_\lambda(y_1, y_1, y_0)$ *is finite, for which the following condition is satisfied for some positive integer,* $m \geq 1$

$$\omega^G_\lambda(T_1^m x, T_1^m y, T_2^m z) \leq k\omega^G_\lambda(T_3^m x, T_3^m y, T_4^m z), \tag{3.26}$$

for each $x, y, z \in X_{\omega^G}$*, with* $k < 1$*. Then* T_i *have a unique common fixed point for some positive integer,* $m \geq 1$ *in* X_{ω^G} *for* $i = 1, 2, 3, 4$*.*

Proof. By Theorem 3.1, $T_1^m, T_2^m, T_3^m, T_4^m$ has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.26).

Now $T_1^m(T_1 u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1 u^*$, so $T_1 u^*$ is a fixed point of $T_1^m u^*$. Similarly, $T_2 u^*$ is a fixed point of $T_2^m u^*$, $T_3 u^*$ is a fixed point of $T_3^m u^*$ and $T_4 u^*$ is a fixed point of $T_4^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of $T_1^m, T_2^m, T_3^m, T_4^m$ say $v^* \in X_{\omega^G}$, that is, $T_1^m v^* = T_2^m v^* = T_3^m v^* = T_4^m v^* = v^*$. Now, we show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega^G_\lambda(u^*, u^*, v^*) > 0$, from inequality (3.26), we have

$$\begin{aligned} \omega^G_\lambda(u^*, u^*, v^*) &= \omega^G_\lambda(T_1^m u^*, T_1^m u^*, T_2^m v^*) \\ &\leq k\omega^G_\lambda(T_3^m u^*, T_3^m u^*, T_4^m v^*) \\ &= k\omega^G_\lambda(u^*, u^*, v^*), \end{aligned} \tag{3.27}$$

hence $u^* = v^*$ for $k < 1$. Therefore, T_i have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G} for $i = 1, 2, 3, 4$. □

Corollary 3.5. *Let* (X_{ω^G}, ω^G) *be a G-complete modular G-metric space and let* $T_i : X_{\omega^G} \to X_{\omega^G}$ *for* $i = 1, 2$*, be two self* ω*-compatible mappings with a point* $y_0 \in X_{\omega^G}, \lambda > 0$*, such that* $\omega^G_\lambda(y_1, y_1, y_0) < \infty$*, for which the following condition is satisfied*

$$\omega^G_\lambda(T_1 x, T_1 y, T_2 z) \leq k\omega^G_\lambda(x, y, z), \tag{3.28}$$

for each $x, y, z \in X_{\omega^G}$*, with* $k < 1$*. Then* T_i *have a unique common fixed point in* X_{ω^G} *for* $i = 1, 2$*.*

Proof. Now, if we take T_3 and T_4 as identity mappings on X_{ω^G}, which we are sure that it is continuous, then we conclude quickly from Theorem 3.1 that and set T_1, T_2, have a unique common fixed point in X_{ω^G}. Hence the proof of Corollary 3.5 is completed. □

Corollary 3.6. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2$, be two self ω-compatible mappings with a point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \geq 1$;*

$$\omega^G_\lambda(T_1^m x, T_1^m y, T_2^m z) \leq k\omega^G_\lambda(x, y, z), \tag{3.29}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_i have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G} for $i = 1, 2$.

Proof. By Corollary 3.5, T_1^m, T_2^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.29). Now $T_1^m(T_1 u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1 u^*$, so $T_1 u^*$ is a fixed point of $T_1^m u^*$. Similarly, $T_2 u^*$ is a fixed point of $T_2^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of T_1^m, T_2^m say $v^* \in X_{\omega^G}$, that is, $T_1^m v^* = T_2^m v^* = v^*$. Now, we show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega^G_\lambda(u^*, u^*, v^*) > 0$, from inequality (3.29), we have

$$\begin{aligned} \omega^G_\lambda(u^*, u^*, v^*) &= \omega^G_\lambda(T_1^m u^*, T_1^m u^*, T_2^m v^*) \\ &\leq k\omega^G_\lambda(u^*, u^*, v^*), \end{aligned} \tag{3.30}$$

hence $u^* = v^*$ for $k < 1$. We can say that T_i have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G} for $i = 1, 2$. □

Corollary 3.7. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_1 : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2$, be two self ω-compatible mappings with a point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega^G_\lambda(T_1 x, T_1 y, T_1 z) \leq k\omega^G_\lambda(x, y, z), \tag{3.31}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_1 have a unique fixed point in X_{ω^G}.

Proof. If we take T_3 and T_4 as identity mappings on X_{ω^G}, which we are sure that it is continuous, and set $T_1 = T_2$, then we conclude quickly from Theorem 3.1 that T_1 have a unique fixed point in X_{ω^G}. □

Remark 3.8. Corollary 3.7 is a generalization of Theorem 3.2 in [16] which is also Corollary 13 in [36]. To see it, take $y = z$ in

$$\omega^G_\lambda(T_1 x, T_1 y, T_1 z) \leq k\omega^G_\lambda(x, y, z). \tag{3.32}$$

Corollary 3.9. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_1 : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2$, be two self ω-compatible mappings with a* *point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \geq 1$*

$$\omega_\lambda^G(T_1^m x, T_1^m y, T_1^m z) \leq k\omega_\lambda^G(x, y, z), \tag{3.33}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_1 have a unique fixed point for some positive integer, $m \geq 1$ in X_{ω^G}.

Proof. By Corollary 3.7, T_1^m has a fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.33). Now

$$T_1^m(T_1 u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1 u^*,$$

so $T_1 u^*$ is a fixed point of $T_1^m u^*$.

For the uniqueness, suppose that there exists another fixed point of T_1^m say $v^* \in X_{\omega^G}$ that is $T_1^m v^* = v^*$. We claim that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, u^*, v^*) > 0$, from inequality (3.33), we have

$$\omega_\lambda^G(u^*, u^*, v^*) = \omega_\lambda^G(T_1^m u^*, T_1^m u^*, T_1^m v^*) \leq k\omega_\lambda^G(u^*, u^*, v^*). \tag{3.34}$$

Hence, T_1 have a unique fixed point for some positive integer, $m \geq 1$ in X_{ω^G}. □

Corollary 3.10. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_3, T_4 : X_{\omega^G} \to X_{\omega^G}$ be two continuous self ω-compatible mappings with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega_\lambda^G(x, y, z) \leq k\omega_\lambda^G(T_3 x, T_3 y, T_4 z), \tag{3.35}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_3, T_4 have a unique common fixed point in X_{ω^G}.

Proof. We set T_1 and T_2 to be identity mappings. Let $x_0 \in X_{\omega^G}$. Since $I(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, there exists $x_1 \in X_{\omega^G}$ such that $Ix_0 = T_4 x_1$, and also as $Ix_1 \in T_3(X_{\omega^G})$, we choose $x_2 \in X_{\omega^G}$ such that $Ix_1 = T_3 x_2$. In general, $x_{2n+1} \in X_{\omega^G}$ is chosen such that $Ix_{2n} = T_4 x_{2n+1}$ and $x_{2n+2} \in X_{\omega^G}$ such that $Ix_{2n+1} = T_3 x_{2n+2}$, we obtain a sequence $\{y_n\}_{n\geq 1}$ such that $y_{2n} = Ix_{2n} = T_4 x_{2n+1}$ and $y_{2n+1} = Ix_{2n+1} = T_3 x_{2n+2}$.

Now we show that $\{y_n\} \subseteq X_{\omega^G}$ is a modular G-Cauchy sequence. Indeed we proceed as follows

$$\begin{aligned} \omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) &= \omega_\lambda^G(x_{2n}, x_{2n}, x_{2n+1}) \\ &\leq k\omega_\lambda^G(T_3 x_{2n}, T_3 x_{2n}, T_4 x_{2n+1}) \\ &= \omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \end{aligned} \tag{3.36}$$

Therefore,

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \leq k\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.37}$$

By Theorem 3.1, we conclude that T_3, T_4 have a unique common fixed point in X_{ω^G}. □

Corollary 3.11. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_3, T_4 : X_{\omega^G} \to X_{\omega^G}$ be two continuous self ω-compatible mappings with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \geq 1$*

$$\omega_\lambda^G(x, y, z) \leq k\omega_\lambda^G(T_3^m x, T_3^m y, T_4^m z), \tag{3.38}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_3, T_4 have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G}.

Proof. By Corollary 3.10, T_3^m, T_4^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.38). Now $T_3^m(T_3u^*) = T_3^{m+1}u^* = T_3(T_3^m u^*) = T_3u^*$, so T_3u^* is a fixed point of $T_3^m u^*$. Similarly, T_4u^* is a fixed point of $T_4^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of T_3^m, T_4^m say $v^* \in X_{\omega^G}$, that is, $T_3^m v^* = T_4^m v^* = v^*$. Now, we show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, u^*, v^*) > 0$, from inequality (3.38), we have

$$\begin{aligned} \omega_\lambda^G(u^*, u^*, v^*) &= \omega_\lambda^G(u^*, u^*, v^*) \\ &\leq k\omega_\lambda^G(T_3u^*, T_3u^*, T_4v^*) \\ &= k\omega_\lambda^G(u^*, u^*, v^*), \end{aligned} \tag{3.39}$$

hence $u^* = v^*$ for $k < 1$. Therefore, T_3, T_4 have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G}. □

Corollary 3.12. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_3 : X_{\omega^G} \to X_{\omega^G}$ be a continuous self ω-compatible mapping with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega_\lambda^G(x, y, z) \leq k\omega_\lambda^G(T_3x, T_3y, T_3z), \tag{3.40}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_3 have a unique fixed point in X_{ω^G}.

Proof. Set T_4 as an identity mapping, then Corollary 3.10 completes the proof of Corollary 3.12. Hence T_3 have a unique fixed point in X_{ω^G}. □

Corollary 3.13. *Let (X_{ω^G}, ω^G) be a G-complete modular G−metric space and let $T_3 : X_{\omega^G} \to X_{\omega^G}$ be a continuous self ω-compatible mapping with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \geq 1$*

$$\omega_\lambda^G(x, y, z) \leq k\omega_\lambda^G(T_3^m x, T_3^m y, T_3^m z), \tag{3.41}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_3 have a unique fixed point for some positive integer, $m \geq 1$ in X_{ω^G}.

Proof. From Corollary 3.12, we conclude that T_3 have a unique fixed point for some positive integer, $m \geq 1$ in X_{ω^G}. □

Corollary 3.14. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3$, be three self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_1(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_1 are continuous and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_1\}$ are compatible so that there is a point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega_\lambda^G(T_1 x, T_1 y, T_2 z) \leq k\omega_\lambda^G(T_1 z, T_1 z, T_3 z), \tag{3.42}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3$.

Proof. Let $x_0 \in X_{\omega^G}$. Since $T_1(X_{\omega^G}) \subseteq T_1(X_{\omega^G})$, there exists $x_1 \in X_{\omega^G}$ such that $T_1 x_0 = T_1 x_1$, and also as $T_2 x_1 \in T_3(X_{\omega^G})$, we choose $x_2 \in X_{\omega^G}$ such that $T_2 x_1 = T_3 x_2$. In general, $x_{2n+1} \in X_{\omega^G}$ is chosen such that $T_1 x_{2n} = T_1 x_{2n+1}$ and $x_{2n+2} \in X_{\omega^G}$ such that $T_2 x_{2n+1} = T_3 x_{2n+2}$, we obtain a sequence $\{y_n\}_{n \geq 1}$ such that $y_{2n} = T_1 x_{2n} = T_1 x_{2n+1}$ and $y_{2n+1} = T_2 x_{2n+1} = T_3 x_{2n+2}$. Now we show that $\{y_n\} \subseteq X_\omega$ is a modular G-Cauchy sequence. Indeed we proceed as follows

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) &= \omega_\lambda^G(T_1 x_{2n}, T_1 x_{2n}, T_2 x_{2n+1}) \\ &\leq k\omega_\lambda^G(T_1 x_{2n+1}, T_1 x_{2n+1}, T_3 x_{2n+1}) \\ &= \omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \end{aligned}\tag{3.43}$$

Therefore,

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \leq k\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.44}$$

Following the proof of Theorem 3.1, we have that T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3$. □

Remark 3.15. We can deduce analogue of Banach contraction mapping principle as follows; inequality (3.42) of Corollary 3.14 says that

$$\omega_\lambda^G(T_1 x, T_1 y, T_2 z) \leq k\omega_\lambda^G(T_1 z, T_1 z, T_3 z), \tag{3.45}$$

take $y = x$, then

$$\omega_\lambda^G(T_1x, T_1x, T_2z) \le k\omega_\lambda^G(T_1z, T_1z, T_3z), \tag{3.46}$$

which implies

$$\omega_\lambda(T_1x, T_2z) \le k\omega_\lambda(T_1z, T_3z). \tag{3.47}$$

Put $T_1 = I$, so that

$$\omega_\lambda(x, T_2z) \le k\omega_\lambda(z, T_3z). \tag{3.48}$$

Now take again $T_2 = T_3$, we get

$$\omega_\lambda(x, T_2z) \le k\omega_\lambda(z, T_2z). \tag{3.49}$$

Corollary 3.16. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3$, be three self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_1(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_1 are continuous and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_1\}$ are compatible so that there is a point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \ge 1$*

$$\omega_\lambda^G(T_1^m x, T_1^m y, T_2^m z) \le k\omega_\lambda^G(T_1^m z, T_1^m z, T_3z), \tag{3.50}$$

for each $x, y, z \in X_{\omega^G}$, with $k < 1$. Then T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3$.

Proof. By Corollary 3.14 , T_1^m, T_2^m, T_3^m, has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \ge 1$ by using inequality (3.50). Now $T_1^m(T_1u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1u^*$, so T_1u^* is a fixed point of $T_1^m u^*$. Similarly, T_2u^* is a fixed point of $T_2^m u^*$, T_3u^* is a fixed point of $T_3^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of T_1^m, T_2^m, T_3^m say $v^* \in X_{\omega^G}$ that is $T_1^m v^* = T_2^m v^* = T_3^m v^* = v^*$. Now, we show that $u^* = v^*$. Indeed, suppose that $u^* \ne v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, u^*, v^*) > 0$, from inequality (3.50), we have

$$\begin{aligned} \omega_\lambda^G(u^*, u^*, v^*) &= \omega_\lambda^G(T_1^m u^*, T_1^m u^*, T_2^m v^*) \\ &\le k\omega_\lambda^G(T_1^m u^*, T_1^m u^*, T_3^m v^*) \\ &= k\omega_\lambda^G(u^*, u^*, v^*), \end{aligned} \tag{3.51}$$

hence $u^* = v^*$ for $k < 1$. Therefore, T_i have a unique common fixed point for some positive integer, $m \ge 1$ in X_{ω^G} for $i = 1, 2, 3$. □

Remark 3.17. We now reduced inequality (3.50) of Theorem 3.16 to modular metric space in which Corollary 10 of [3] becomes a special case. Indeed, from inequality (3.50) of Theorem 3.16 we have that

$$\omega_\lambda^G(T_1^m x, T_1^m y, T_2^m z) \le k\omega_\lambda^G(T_1^m z, T_1^m z, T_3z), \tag{3.52}$$

take $y = x$, then

$$\omega_\lambda^G(T_1^m x, T_1^m x, T_2^m z) \leq k\omega_\lambda^G(T_1^m z, T_1^m z, T_3 z), \tag{3.53}$$

which implies

$$\omega_\lambda(T_1^m x, T_2^m z) \leq k\omega_\lambda^G(T_1^m z, T_3 z). \tag{3.54}$$

Take $T_1^m = I$ for all $m \geq 1$, we get

$$\omega_\lambda(T_2^m z, x) \leq k\omega_\lambda^G(z, T_3 z). \tag{3.55}$$

Inequality (3.55) is an extension of Corollary 10 of [3]. Indeed, as $T_2 = T_3$, inequality (3.55) becomes

$$\omega_\lambda(T_2^m z, x) \leq k\omega_\lambda^G(z, T_2 z). \tag{3.56}$$

Corollary 3.18. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4$, be four self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_4 are continuous and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are ω-compatible mappings, so that there is an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\begin{aligned}\omega_\lambda^G(T_1 x, T_1 y, T_2 z) \leq\ & a\omega_\lambda^G(T_3 x, T_3 y, T_4 z) + b\omega_\lambda^G(T_2 x, T_2 x, T_2 z) \\ & + c\omega_\lambda^G(T_1 z, T_1 z, T_3 z) + d\omega_\lambda^G(T_1 y, T_1 y, T_2 z), \end{aligned}\tag{3.57}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3, 4$.

Proof. Let $x_0 \in X_{\omega^G}$. Since $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, there exists $x_1 \in X_{\omega^G}$ such that $T_1 x_0 = T_4 x_1$, and also as $T_2 x_1 \in T_3(X_{\omega^G})$, we choose $x_2 \in X_{\omega^G}$ such that $T_2 x_1 = T_3 x_2$. In general, $x_{2n+1} \in X_{\omega^G}$ is chosen such that $T_1 x_{2n} = T_4 x_{2n+1}$ and $x_{2n+2} \in X_{\omega^G}$ such that $T_2 x_{2n+1} = T_3 x_{2n+2}$, we obtain a sequence $\{y_n\}_{n\geq 1}$ such that

$$y_{2n} = T_1 x_{2n} = T_4 x_{2n+1}$$

and

$$y_{2n+1} = T_2 x_{2n+1} = T_3 x_{2n+2}.$$

Now we show that $\{y_n\} \subseteq X_{\omega^G}$ is a modular G-Cauchy sequence. Indeed from inequality (3.57)

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) &= \omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2x_{2n+1}) \\ &\le a\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4x_{2n+1}) \\ &\quad + b\omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2x_{2n+1}) \\ &\quad + c\omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\ &\quad + d\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2x_{2n+1}) \\ &= a\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}) + b\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \\ &\quad + c\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}) + d\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \\ &= (a+c)\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}) \\ &\quad + (b+d)\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}). \end{aligned} \tag{3.58}$$

Hence,

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \le \frac{a+c}{1-(b+d)}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.59}$$

Take $k := \dfrac{a+c}{1-(b+d)}$, then

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) \le k\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{3.60}$$

Using the above procedure and condition (3) of Proposition 2.7, we have

$$\begin{aligned}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n}) &\le 2\omega_{\frac{\lambda}{2}}^G(y_{2n}, y_{2n}, y_{2n-1}) \\ &\le 2\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1}) \\ &= 2\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2x_{2n-1}) \\ &\le \frac{k}{2}\{a\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4x_{2n-1}) \\ &\quad + b\omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2x_{2n-1}) \\ &\quad + c\omega_\lambda^G(T_1x_{2n-1}, T_1x_{2n-1}, T_3x_{2n-1}) \\ &\quad + d\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2x_{2n-1})\} \\ &= \frac{k}{2}\{a\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}) + b\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1}) \\ &\quad + c\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}) + d\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1})\} \\ &= \frac{k}{2}\{(a+c)\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\quad + (b+d)\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1})\}. \end{aligned} \tag{3.61}$$

So that for $n \geq 2$,

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n-1}) \leq \frac{k(a+c)}{2\big(1-\frac{1}{2}(k(b+d))\big)}\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n-2}), \tag{3.62}$$

where $k_1 := \frac{k(a+c)}{2\left(1-\frac{1}{2}(k(b+d))\right)}$. Take $h := \max\{k, k_1\}$, therefore, for all n and $\lambda > 0$, we have

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2n}, y_{2n+1}) &\leq h\omega_\lambda^G(y_{2n-1}, y_{2n-1}, y_{2n})\\ &\vdots\\ &\leq h^{n-1}\omega_\lambda^G(y_1, y_1, y_0),\end{aligned} \tag{3.63}$$

for $\lambda > 0$ and $n \geq 2$.

Suppose that $m, n \in \mathbb{N}$ and $m > n \in \mathbb{N}$. Applying rectangle inequality repeatedly, that is, condition (5) of Definition 2.4 we have

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) &\leq \omega^G_{\frac{\lambda}{m-n}}(y_{2n}, y_{2n}, y_{2n+1}) + \omega^G_{\frac{\lambda}{m-n}}(y_{2n+1}, y_{2n+1}, y_{2n+2})\\ &\quad + \omega^G_{\frac{\lambda}{m-n}}(y_{2n+2}, y_{2n+2}, y_{2n+3}) + \omega^G_{\frac{\lambda}{m-n}}(y_{2n+3}, y_{2n+3}, y_{2n+4})\\ &\quad + \cdots + \omega^G_{\frac{\lambda}{m-n}}(y_{2m-1}, y_{2m-1}, y_{2m})\\ &\leq \omega^G_{\frac{\lambda}{n}}(y_{2n}, y_{2n}, y_{2n+1}) + \omega^G_{\frac{\lambda}{n}}(y_{2n+1}, y_{2n+1}, y_{2n+2})\\ &\quad + \omega^G_{\frac{\lambda}{n}}(y_{2n+2}, y_{2n+2}, y_{2n+3})\\ &\quad + \omega^G_{\frac{\lambda}{n}}(y_{2n+3}, y_{2n+3}, y_{2n+4}) + \cdots + \omega^G_{\frac{\lambda}{n}}(y_{2m-1}, y_{2m-1}, y_{2m})\\ &\leq h^n + h^{n+1} + \cdots + h^{m-1})\omega_\lambda^G(y_1, y_1, y_0)\\ &\leq \frac{h^n}{1-h}\omega_\lambda^G(y_1, y_1, y_0),\end{aligned} \tag{3.64}$$

for all $m > n \geq N \in \mathbb{N}$, then

$$\omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) \leq \frac{h^n}{1-h}\omega_\lambda^G(y_1, y_1, y_0), \tag{3.65}$$

for all $m, l, n \geq N$ for some $N \in \mathbb{N}$, so that by condition (2) of Proposition 2.7, we have

$$\omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) \leq \omega^G_{\frac{\lambda}{2}}(y_{2n}, y_{2n}, y_{2m}) + \omega^G_{\frac{\lambda}{2}}(y_{2l}, y_{2m}, y_{2m}), \tag{3.66}$$

so that

$$\begin{aligned}\omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) \leq& \omega_{\frac{\lambda}{2}}^G(y_{2n}, y_{2n}, y_{2m}) + \omega_{\frac{\lambda}{2}}^G(y_{2l}, y_{2m}, y_{2m}) \\ \leq& \omega_\lambda^G(y_{2n}, y_{2n}, y_{2m}) + \omega_\lambda^G(y_{2l}, y_{2m}, y_{2m}) \\ \leq& \frac{h^n}{1-h}\omega_\lambda^G(y_1, y_1, y_0) + \frac{h^n}{1-h}\omega_\lambda^G(y_1, y_1, y_0) \\ =& \left(\frac{h^n}{1-h} + \frac{h^n}{1-h}\right)\omega_\lambda^G(y_1, y_1, y_0). \end{aligned} \tag{3.67}$$

Thus, we have

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(y_{2n}, y_{2m}, y_{2l}) = 0 \tag{3.68}$$

or

$$\lim_{n,m,l\to\infty} \omega_\lambda^G(y_n, y_m, y_l) = 0. \tag{3.69}$$

Therefore, we can easily see that $\{y_n\}_{n\in\mathbb{N}}$ is modular G-Cauchy sequence in X_{ω^G}. The modular G-completeness of (X_{ω^G}, ω^G) implies that for any $\lambda > 0$, $\lim_{n,m\to\infty} \omega_\lambda^G(y_n, y_m, u) = 0$, that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda^G(y_n, y_m, u) < \epsilon$ for all $n, m \in \mathbb{N}$ and $n, m \geq n_0$, which implies that $\lim_{n\to\infty} y_n = u \in X_{\omega^G}$ as $n \to \infty$, or by applying condition (5) of Proposition 2.8, such that $\lim_{n\to\infty} T_1x_{2n} = \lim_{n\to\infty} T_4x_{2n+1} = \lim_{n\to\infty} T_2x_{2n+1} = \lim_{n\to\infty} T_3x_{2n+2} = u$. Now we show that u is a common fixed point of the mappings, T_1, T_2, T_3 and T_4. Recall that T_3 is continuous, then it follows that $\lim_{n\to\infty} T_3^2x_{2n+2} = T_3(\lim_{n\to\infty} T_3x_{2n+2}) = T_3u$ and $\lim_{n\to\infty} T_3(T_1x_{2n}) = T_3u$. Since $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are ω-compatible mappings and for all $\lambda > 0$, we have

$$\lim_{n\to\infty} \omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_3T_1x_{2n}) = 0.$$

Thus by Proposition 2.11, we have $\lim_{n\to\infty} T_1(T_3x_{2n}) = T_3u$. On putting $x = y = T_3x_{2n}$ and $z = x_{2n+1}$ into inequality (3.57), we have

$$\begin{aligned}\omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \leq& a\omega_\lambda^G(T_3T_3x_{2n}, T_3T_3x_{2n}, T_4x_{2n+1}) \\ &+ b\omega_\lambda^G(T_2T_3x_{2n}, T_2T_3x_{2n}, T_2x_{2n+1}) \\ &+ c\omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\ &+ d\omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \\ =& a\omega_\lambda^G(T_3^2x_{2n}, T_3^2x_{2n}, T_4x_{2n+1}) \\ &+ b\omega_\lambda^G(T_2T_3x_{2n}, T_2T_3x_{2n}, T_2x_{2n+1}) \\ &+ c\omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\ &+ d\omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \end{aligned} \tag{3.70}$$

Taking the limit of both sides of inequality (3.70) as n tends to infinity, we have

$$\begin{aligned}
\omega_\lambda^G(T_3u, T_3u, u) &= \lim_{n\to\infty} \omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \\
&\leq a \lim_{n\to\infty} \omega_\lambda^G(T_3T_3x_{2n}, T_3T_3x_{2n}, T_4x_{2n+1}) \\
&\quad + b \lim_{n\to\infty} \omega_\lambda^G(T_2T_3x_{2n}, T_2T_3x_{2n}, T_2x_{2n+1}) \\
&\quad + c \lim_{n\to\infty} \omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\
&\quad + d \lim_{n\to\infty} \ omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \\
&= a \lim_{n\to\infty} \omega_\lambda^G(T_3^2x_{2n}, T_3^2x_{2n}, T_4x_{2n+1}) \\
&\quad + b \lim_{n\to\infty} \omega_\lambda^G(T_2T_3x_{2n}, T_2T_3x_{2n}, T_2x_{2n+1}) \\
&\quad + c \lim_{n\to\infty} \omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\
&\quad + d \lim_{n\to\infty} \ omega_\lambda^G(T_1T_3x_{2n}, T_1T_3x_{2n}, T_2x_{2n+1}) \\
&= a\omega_\lambda^G(T_3u, T_3u, u) + b\omega_\lambda^G(T_3u, T_3u, u), \\
&\quad + c\omega_\lambda^G(u, u, u) + d\omega_\lambda^G(T_3u, T_3u, u) \\
&= a\omega_\lambda^G(T_3u, T_3u, u) + b\omega_\lambda^G(T_3u, T_3u, u), \\
&\quad + d\omega_\lambda^G(T_3u, T_3u, u) \\
&= (a + b + d)\omega_\lambda^G(T_3u, T_3u, u). \qquad (3.71)
\end{aligned}$$

Hence $T_3u = u$ as $a+b+d < 1$. Again, in a similar way, note that T_4 is continuous then $\lim_{n\to\infty} T_4^2x_{2n+1} = T_4u$ and $\lim_{n\to\infty} T_4T_2x_{2n+1} = T_4u$, $\lim_{n\to\infty} T_1T_4x_{2n+1} = T_4u$. Since $\{T_2, T_4\}$ is ω-compatible mapping and for all $\lambda > 0$, we have

$$\lim_{n\to\infty} \omega_\lambda^G(T_2T_4x_{2n+1}, T_2T_4x_{2n+1}, T_4T_2x_{2n+1}) = 0.$$

Thus by Proposition 2.11, we have $\lim_{n\to\infty} T_2(T_4x_{2n+1}) = T_4u$. On putting $x = y = x_{2n}$ and $z = T_4x_{2n+1}$ into inequality (3.57), we have

$$\begin{aligned}
\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1}) &\leq a\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4T_4x_{2n+1}) \\
&\quad + b\omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2T_4x_{2n+1}) \\
&\quad + c\omega_\lambda^G(T_1T_4x_{2n+1}, T_1T_4x_{2n+1}, T_3T_4x_{2n+1}) \\
&\quad + d\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1})
\end{aligned}$$

$$\begin{aligned}
&= a\omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4^2x_{2n+1}) \\
&\quad + b\omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2T_4x_{2n+1}), \\
&\quad + c\omega_\lambda^G(T_1T_4x_{2n+1}, T_1T_4x_{2n+1}, T_3T_4x_{2n+1}) \\
&\quad + d\omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1}). \qquad (3.72)
\end{aligned}$$

On taking the limit of both sides of inequality (3.72) as n tends to infinity, we have

$$\begin{aligned}
\omega_\lambda^G(u, u, T_4u) &= \lim_{n\to\infty} \omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1}) \\
&\leq a \lim_{n\to\infty} \omega_\lambda^G(T_3x_{2n}, T_3x_{2n}, T_4T_4x_{2n+1}) \\
&\quad + b \lim_{n\to\infty} \omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2T_4x_{2n+1}) \\
&\quad + c \lim_{n\to\infty} \omega_\lambda^G(T_1T_4x_{2n+1}, T_1T_4x_{2n+1}, T_3T_4x_{2n+1}) \\
&\quad + d \lim_{n\to\infty} \omega_\lambda^G(T_1x_{2n}, T_1x_{2n}, T_2T_4x_{2n+1}) \\
&= a\omega_\lambda^G(u, u, T_4u) + b\omega_\lambda^G(u, u, T_4u) + c\omega_\lambda^G(u, u, u) \\
&\quad + d\omega_\lambda^G(u, u, T_4u) \\
&= a\omega_\lambda^G(u, u, T_4u) + b\omega_\lambda^G(u, u, T_4u) \\
&\quad + d\omega_\lambda^G(u, u, T_4u) \\
&= (a + b + d)\omega_\lambda^G(u, u, T_4u), \qquad (3.73)
\end{aligned}$$

so that $T_4u = u$ for all $\lambda > 0$ and $a + b + d < 1$.

Furthermore, if we put $x = x_{2n}, y = u$ and $z = x_{2n+1}$, then from inequality (3.57)

$$\begin{aligned}
\omega_\lambda^G(T_1x_{2n}, T_1u, T_2x_{2n+1}) &\leq a\omega_\lambda^G(T_3x_{2n}, T_3u, T_4x_{2n+1}) \\
&\quad + b\omega_\lambda^G(T_2x_{2n}, T_2x_{2n}, T_2x_{2n+1}) \\
&\quad + c\omega_\lambda^G(T_1x_{2n+1}, T_1x_{2n+1}, T_3x_{2n+1}) \\
&\quad + d\omega_\lambda^G(T_1u, T_1u, T_2x_{2n+1}) \qquad (3.74)
\end{aligned}$$

as $n \longrightarrow \infty$, we have

$$\begin{aligned}
\omega_\lambda^G(u, T_1u, u) &\leq a\omega_\lambda^G(u, T_3u, u) + b\omega_\lambda^G(u, u, u) \\
&\quad + c\omega_\lambda^G(u, u, u) + d\omega_\lambda^G(T_1u, T_1u, u), \qquad (3.75)
\end{aligned}$$

so that

$$\begin{aligned}\omega_\lambda^G(u,u,T_1u) &\le a\omega_\lambda^G(u,u,T_3u)+b\omega_\lambda^G(u,u,u),\\ &\quad + c\omega_\lambda^G(u,u,u)+d\omega_\lambda^G(T_1u,T_1u,u)\\ &= d\omega_\lambda^G(T_1u,T_1u,u). \end{aligned} \tag{3.76}$$

Now, using condition (3) of Proposition 2.7, we get

$$\begin{aligned}\omega_\lambda^G(u,u,T_1u) &\le d\omega_\lambda^G(T_1u,T_1u,u)\\ &\le 2d\omega_{\frac{\lambda}{2}}^G(u,u,T_1u)\\ &\le 2\omega_\lambda^G(u,u,T_1u). \end{aligned} \tag{3.77}$$

Hence, $T_1u = u$ for all $\lambda > 0$ and $2d < 1$. Finally, using the fact that $T_1u = T_3u = T_4u = u$, then inequality (3.57), becomes

$$\begin{aligned}\omega_\lambda^G(u,u,T_2u) &= \omega_\lambda^G(T_1u,T_1u,T_2u)\\ &\le a\omega_\lambda^G(T_3u,T_3u,T_4u)+b\omega_\lambda^G(T_2u,T_2u,T_2u)\\ &\quad + c\omega_\lambda^G(T_1u,T_1u,T_3u)+d\omega_\lambda^G(T_1u,T_1u,T_2u)\\ &= a\omega_\lambda^G(u,u,u)+b\omega_\lambda^G(T_2u,T_2u,T_2u)\\ &\quad + c\omega_\lambda^G(u,u,u)+d\omega_\lambda^G(u,u,T_2u)\\ &= d\omega_\lambda^G(u,u,T_2u). \end{aligned} \tag{3.78}$$

So that

$$\omega_\lambda^G(u,u,T_2u) \le d\omega_\lambda^G(u,u,T_2u), \tag{3.79}$$

hence,

$$(1-d)\omega_\lambda^G(u,u,T_2u) \le 0, \tag{3.80}$$

where, $d < 1$ and for all $\lambda > 0$. Hence, $T_2u = u$. Therefore, we have that

$$T_1u = T_2u = T_3u = T_4u = u, \tag{3.81}$$

which shows that u is a common fixed point of T_1, T_2, T_3 and T_4.

To prove uniqueness, suppose that there exists another common fixed point of T_1, T_2, T_3 and T_4, that is, there is a $u^* \in X_{\omega^G}$ such that

$$u^* = T_1u^* = T_2u^* = T_3u^* = T_4u^*.$$

If $u \neq u^*$, and for all $\lambda > 0$, again inequality (3.57) becomes

$$\begin{aligned}\omega_\lambda^G(u,u,u^*) &= \omega_\lambda^G(T_1u,T_1u,T_2u^*)\\ &\leq a\omega_\lambda^G(T_3u,T_3u,T_4u^*) + b\omega_\lambda^G(T_2u,T_2u,T_2u^*)\\ &\quad + c\omega_\lambda^G(T_1u^*,T_1u^*,T_3u^*) + d\omega_\lambda^G(T_1u,T_1u,T_2u^*)\\ &= a\omega_\lambda^G(u,u,u^*) + b\omega_\lambda^G(u,u,u^*)\\ &\quad + c\omega_\lambda^G(u^*,u^*,u^*) + d\omega_\lambda^G(u,u,u^*)\\ &= a\omega_\lambda^G(u,u,u^*) + b\omega_\lambda^G(u,u,u^*)\\ &\quad + d\omega_\lambda^G(u,u,u^*)\\ &= (a+b+d)\omega_\lambda^G(u,u,u^*). \end{aligned} \tag{3.82}$$

Therefore,

$$\omega_\lambda^G(u,u,u^*) \leq (a+b+d)\omega_\lambda^G(u,u,u^*), \tag{3.83}$$

so that

$$(1-(a+b+d))\omega_\lambda^G(u,u,u^*) \leq 0, \tag{3.84}$$

where, $a+b+d<1$ and $\lambda > 0$, thus $u = u^*$. Therefore, the proof of Theorem 3.18 is now completed. □

Remark 3.19. Inequality (3.57) of Corollary 3.18 can be reduced to modular metric which may generalized or complements other existing results as follows:

$$\begin{aligned}\omega_\lambda^G(T_1x,T_1y,T_2z) &\leq a\omega_\lambda^G(T_3x,T_3y,T_4z) + b\omega_\lambda^G(T_2x,T_2x,T_2z)\\ &\quad + c\omega_\lambda^G(T_1z,T_1z,T_3z) + d\omega_\lambda^G(T_1y,T_1y,T_2z), \end{aligned} \tag{3.85}$$

taking $x = y$, then

$$\begin{aligned}\omega_\lambda^G(T_1x,T_1x,T_2z) &\leq a\omega_\lambda^G(T_3x,T_3x,T_4z) + b\omega_\lambda^G(T_2x,T_2x,T_2z)\\ &\quad + c\omega_\lambda^G(T_1z,T_1z,T_3z) + d\omega_\lambda^G(T_1x,T_1x,T_2z). \end{aligned} \tag{3.86}$$

Again, put $z = y$, we get

$$\begin{aligned}\omega_\lambda^G(T_1x,T_1x,T_2y) &\leq a\omega_\lambda^G(T_3x,T_3x,T_4y) + b\omega_\lambda^G(T_2x,T_2x,T_2y)\\ &\quad + c\omega_\lambda^G(T_1y,T_1y,T_3y) + d\omega_\lambda^G(T_1x,T_1x,T_2y), \end{aligned} \tag{3.87}$$

which gives

$$\begin{aligned}\omega_\lambda(T_1x,T_2y) &\leq a\omega_\lambda(T_3x,T_4y) + b\omega_\lambda(T_2x,T_2y)\\ &\quad + c\omega_\lambda(T_1y,T_3y) + d\omega_\lambda(T_1x,T_2y). \end{aligned} \tag{3.88}$$

Inequality (3.88) is a modification of condition (3) of Theorem 15 in [36]. Now from inequality (3.85) put $T_3 = T_4 = I$, we get

$$\omega_\lambda^G(T_1x, T_1y, T_2z) \leq a\omega_\lambda^G(x, y, z) + b\omega_\lambda^G(T_2x, T_2x, T_2z) + c\omega_\lambda^G(T_1z, T_1z, z) + d\omega_\lambda^G(T_1y, T_1y, T_2z), \tag{3.89}$$

letting $T_2 = T_1$,

$$\omega_\lambda^G(T_1x, T_1y, T_1z) \leq a\omega_\lambda^G(x, y, z) + b\omega_\lambda^G(T_1x, T_1x, T_1z) + c\omega_\lambda^G(T_1z, T_1z, z) + d\omega_\lambda^G(T_1y, T_1y, T_1z). \tag{3.90}$$

Now inequality (3.90) is a modified inequality (3.6) of Theorem 3.3 of [5], if $a = 0$ in inequality (3.90), then we get modified condition $(I1)$ of Theorem 3.2 in [5]. Furthermore, Theorem 3.4 follows from inequality (3.90) as $a = 0$. Lastly, if $a = 0$ and $b = c = d = k$, then inequality (3.90) modified condition (II-1) of Theorem 3.6 of [5].

Corollary 3.20. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2$, be two self ω-compatible mappings with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega_\lambda^G(T_1x, T_1y, T_2z) \leq a\omega_\lambda^G(x, y, z) + b\omega_\lambda^G(T_2x, T_2x, T_2z) + c\omega_\lambda^G(T_1z, T_1z, z) + d\omega_\lambda^G(T_1y, T_1y, T_2z), \tag{3.91}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2$.

Proof. Take T_3 and T_4 to be an identity mapping, then by Corollary 3.18, we conclude that T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2$. □

Remark 3.21. As remarked in Remark 3.19 above. Again, we can deduce from inequality (3.91) of Corollary 3.20 an analogue of Banach contraction mapping principle in modular metric space as pointed out in Theorem 3.2 in [16] as follows take $z = y$, then inequality (3.91) becomes

$$\omega_\lambda^G(T_1x, T_1y, T_2y) \leq a\omega_\lambda^G(x, y, y) + b\omega_\lambda^G(T_2x, T_2x, T_2y) + c\omega_\lambda^G(T_1y, T_1y, y) + d\omega_\lambda^G(T_1y, T_1y, T_2y), \tag{3.92}$$

so that on taking $T_2 = T_1$, we get

$$\omega_\lambda^G(T_1x, T_1y, T_1y) \leq a\omega_\lambda^G(x, y, y) + b\omega_\lambda^G(T_1x, T_1x, T_1y) + c\omega_\lambda^G(T_1y, T_1y, y) + d\omega_\lambda^G(T_1y, T_1y, T_1y). \tag{3.93}$$

This implies

$$\omega_\lambda(T_1x, T_1y) \leq a\omega_\lambda(x, y) + b\omega_\lambda(T_1x, T_1y) + c\omega_\lambda(T_1y, y). \tag{3.94}$$

Therefore,

$$\omega_\lambda(T_1x, T_1y) \le \frac{1}{1-b}\bigg(a\omega_\lambda(x,y) + c\omega_\lambda(T_1y, y)\bigg). \tag{3.95}$$

Hence,

$$\omega_\lambda(T_1x, T_1y) \le \frac{a}{1-b}\omega_\lambda(x,y) + \frac{c}{1-b}\omega_\lambda(y, T_1y),\ \forall\ \lambda > 0. \tag{3.96}$$

Corollary 3.22. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2$, be two self ω-compatible mappings with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \ge 1$*

$$\begin{aligned}\omega^G_\lambda(T_1^m x, T_1^m y, T_2^m z) &\le a\omega^G_\lambda(x,y,z) + b\omega^G_\lambda(T_2^m x, T_2^m x, T_2^m z)\\ &\quad + c\omega^G_\lambda(T_1^m z, T_1^m z, z) + d\omega^G_\lambda(T_1^m y, T_1^m y, T_2^m z),\end{aligned} \tag{3.97}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_i have a unique common fixed point for some positive integer, $m \ge 1$ in X_{ω^G} for $i = 1, 2$.

Proof. By Corollary 3.20, T_1^m, T_2^m has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \ge 1$ by using inequality (3.97). Now $T_1^m(T_1u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1u^*$, so T_1u^* is a fixed point of $T_1^m u^*$. Similarly, T_2u^* is a fixed point of $T_2^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of T_1^m, T_2^m say $v^* \in X_{\omega^G}$ that is $T_1^m v^* = T_2^m v^* = v^*$. Now, we show that $u^* = v^*$. Indeed, suppose that $u^* \ne v^*$ implies that for any $\lambda > 0$, $\omega^G_\lambda(u^*, u^*, v^*) > 0$, from inequality (3.97), we have

$$\omega^G_\lambda(u^*, u^*, v^*) \le (a+b+d)\omega^G_\lambda(u^*, u^*, v^*), \tag{3.98}$$

so that

$$(1-(a+b+d))\omega^G_\lambda(u^*, u^*, v^*) \le 0, \tag{3.99}$$

where, $a+b+d < 1$ and $\lambda > 0$, thus $u^* = v^*$. Therefore, the proof of Corollary 3.22 is completed. □

Corollary 3.23. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_1 : X_{\omega^G} \to X_{\omega^G}$ be a self ω-compatible mapping with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\begin{aligned}\omega^G_\lambda(T_1x, T_1y, T_1z) &\le a\omega^G_\lambda(x,y,z) + b\omega^G_\lambda(T_1x, T_1x, T_1z)\\ &\quad + c\omega^G_\lambda(T_1z, T_1z, z) + d\omega^G_\lambda(T_1y, T_1y, T_1z),\end{aligned} \tag{3.100}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_1 have a unique fixed point in X_{ω^G}.

Proof. We set T_3 and T_4 as an identity mappings, $T_1 = T_2$, then by Corollary 3.18, we conclude that T_1 have a unique fixed point in X_{ω^G}. □

Corollary 3.24. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_1 : X_{\omega^G} \to X_{\omega^G}$ be a self ω-compatible mapping with an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied for some positive integer, $m \geq 1$*

$$\begin{aligned}\omega^G_\lambda(T_1^m x, T_1^m y, T_1^m z) \leq a\omega^G_\lambda(x, y, z) + b\omega^G_\lambda(T_1^m x, T_1^m x, T_1^m z) \\ + c\omega^G_\lambda(T_1^m z, T_1^m z, z) + d\omega^G_\lambda(T_1^m y, T_1^m y, T_1^m z),\end{aligned} \tag{3.101}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_1 have a unique fixed point in for some positive integer, $m \geq 1$ X_{ω^G}.

Proof. By Corollary 3.23, T_1^m has a fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.101). Now $T_1^m(T_1 u^*) = T_1^{m+1} u^* = T_1(T_1^m u^*) = T_1 u^*$, so $T_1 u^*$ is a fixed point of $T_1^m u^*$.

For the uniqueness, suppose that there exists another fixed point of T_1^m say $v^* \in X_{\omega^G}$ that is $T_1^m v^* = v^*$. We claim that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega^G_\lambda(u^*, u^*, v^*) > 0$, from inequality (3.101), we have

$$\omega^G_\lambda(u^*, u^*, v^*) \leq (a + b + d)\omega^G_\lambda(u^*, u^*, v^*), \tag{3.102}$$

so that

$$(1 - (a + b + d))\omega^G_\lambda(u^*, u^*, v^*) \leq 0, \tag{3.103}$$

where, $a + b + d < 1$ and $\lambda > 0$, thus $u^* = v^*$. Hence, T_1 have a unique fixed point in for some positive integer, $m \geq 1$ X_{ω^G}. □

Corollary 3.25. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4$, be four self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_4 are continuous for all positive integer, $m \geq 1$ and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are ω-compatible mappings, so that there is an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega^G_\lambda(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\begin{aligned}\omega^G_\lambda(T_1^m x, T_1^m y, T_2^m z) \leq a\omega^G_\lambda(T_3^m x, T_3^m y, T_4^m z) + b\omega^G_\lambda(T_2^m x, T_2^m x, T_2^m z) \\ + c\omega^G_\lambda(T_1^m z, T_1^m z, T_3^m z) + d\omega^G_\lambda(T_1^m y, T_1^m y, T_2^m z),\end{aligned} \tag{3.104}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+c+d < 1, b+d < 1, 2d < 1$ and $a+b+d < 1$. Then T_i have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G} for $i = 1, 2, 3, 4$.

Proof. By Corollary 3.18, $T_1^m, T_2^m, T_3^m, T_4^m$ has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.104). Now $T_1^m(T_1u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1u^*$, so T_1u^* is a fixed point of $T_1^m u^*$. Similarly, T_2u^* is a fixed point of $T_2^m u^*$, T_3u^* is a fixed point of $T_3^m u^*$ and T_4u^* is a fixed point of $T_4^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of $T_1^m, T_2^m, T_3^m, T_4^m$ say $v^* \in X_{\omega^G}$, that is, $T_1^m v^* = T_2^m v^* = T_3^m v^* = T_4^m v^* = v^*$. We want to show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, u^*, v^*) > 0$, from inequality (3.104), we have

$$\omega_\lambda^G(u^*, u^*, v^*) \leq (a+b+d)\omega_\lambda^G(u^*, u^*, v^*), \tag{3.105}$$

so that

$$(1-(a+b+d))\omega_\lambda^G(u^*, u^*, v^*) \leq 0, \tag{3.106}$$

where, $a+b+d < 1$ and $\lambda > 0$, thus $u^* = v^*$. Therefore, the proof of Theorem 3.25 is now completed. □

Remark 3.26. Corollary 3.25 is a variant form of Corollary 3.18 above.

Corollary 3.27. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space and let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4$, be four self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_4 are continuous and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are compatible so that there is an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\begin{aligned}\omega_\lambda^G(T_1x, T_1y, T_2z) \leq\ & a\omega_\lambda^G(T_3x, T_3y, T_4z)\\ &+ b\omega_\lambda^G(T_2x, T_2x, T_2z)\\ &+ d\omega_\lambda^G(T_1y, T_1y, T_2z),\end{aligned} \tag{3.107}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+d < 1, b+d < 1, 2d < 1$. Then T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3, 4$.

Proof. Observe that if $c = 0$, then from Theorem 3.18, T_i have a unique common fixed point in X_{ω^G} for $i = 1, 2, 3, 4$. □

Corollary 3.28. *Let (X_{ω^G}, ω^G) be a G-complete modular G-metric space. Let $T_i : X_{\omega^G} \to X_{\omega^G}$ for $i = 1, 2, 3, 4$, be four self ω-compatible mappings with $T_1(X_{\omega^G}) \subseteq T_4(X_{\omega^G})$, $T_2(X_{\omega^G}) \subseteq T_3(X_{\omega^G})$ in which T_3, T_4 are continuous for all positive integer, $m \geq 1$ and that the pairs $\{T_1, T_3\}$ and $\{T_2, T_4\}$ are ω-compatible mappings, so that there is an arbitrary point $y_0 \in X_{\omega^G}, \lambda > 0$, such that $\omega_\lambda^G(y_1, y_1, y_0) < \infty$, for which the following condition is satisfied*

$$\omega_\lambda^G(T_1^m x, T_1^m y, T_2^m z) \leq a\omega_\lambda^G(T_3^m x, T_3^m y, T_4^m z) + b\omega_\lambda^G(T_2^m x, T_2^m x, T_2^m z) + d\omega_\lambda^G(T_1^m y, T_1^m y, T_2^m z), \tag{3.108}$$

for each $x, y, z \in X_{\omega^G}$, with $a+b+d < 1, b+d < 1, 2d < 1$. Then T_i have a unique common fixed point for some positive integer, $m \geq 1$ in X_{ω^G} for $i = 1, 2, 3, 4$.

Proof. By Corollary 3.27, $T_1^m, T_2^m, T_3^m, T_4^m$ has a common fixed point say $u^* \in X_{\omega^G}$ for some positive integer $m \geq 1$ by using inequality (3.108). Now $T_1^m(T_1 u^*) = T_1^{m+1}u^* = T_1(T_1^m u^*) = T_1 u^*$, so $T_1 u^*$ is a fixed point of $T_1^m u^*$. Similarly, $T_2 u^*$ is a fixed point of $T_2^m u^*$, $T_3 u^*$ is a fixed point of $T_3^m u^*$ and $T_4 u^*$ is a fixed point of $T_4^m u^*$.

For the uniqueness, suppose that there exists another common fixed point of $T_1^m, T_2^m, T_3^m, T_4^m$ say $v^* \in X_{\omega^G}$ that is $T_1^m v^* = T_2^m v^* = T_3^m v^* = T_4^m v^* = v^*$. We want to show that $u^* = v^*$. Indeed, suppose that $u^* \neq v^*$ implies that for any $\lambda > 0$, $\omega_\lambda^G(u^*, u^*, v^*) > 0$, from inequality (3.108), we have

$$\omega_\lambda^G(u^*, u^*, v^*) \leq (a+b+d)\omega_\lambda^G(u^*, u^*, v^*), \tag{3.109}$$

so that

$$(1-(a+b+d))\omega_\lambda^G(u^*, u^*, v^*) \leq 0, \tag{3.110}$$

where, $a+b+d < 1$ and $\lambda > 0$, thus $u^* = v^*$. Therefore, the proof of Corollary 3.28 is now completed. □

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