Bull. Korean Math. Soc. **61** (2024), No. 3, pp. 867–873
https://doi.org/10.4134/BKMS.b230497
pISSN: 1015-8634 / eISSN: 2234-3016

# NILPOTENCY OF THE RICCI OPERATOR OF PSEUDO-RIEMANNIAN SOLVMANIFOLDS

Huihui An, Shaoqiang Deng, and Zaili Yan

Abstract. A pseudo-Riemannian solvmanifold is a solvable Lie group endowed with a left invariant pseudo-Riemannian metric. In this short note, we investigate the nilpotency of the Ricci operator of pseudo-Riemannian solvmanifolds. We focus on a special class of solvable Lie groups whose Lie algebras can be expressed as a one-dimensional extension of a nilpotent Lie algebra $\mathbb{R}D \ltimes \mathfrak{n}$, where $D$ is a derivation of $\mathfrak{n}$ whose restriction to the center of $\mathfrak{n}$ has at least one real eigenvalue. The main result asserts that every solvable Lie group belonging to this special class admits a left invariant pseudo-Riemannian metric with nilpotent Ricci operator. As an application, we obtain a complete classification of three-dimensional solvable Lie groups which admit a left invariant pseudo-Riemannian metric with nilpotent Ricci operator.

## 1. Introduction

Ricci curvature is an important quantity in differential geometry. In 1976, Milnor [15] studied Ricci curvature of a left invariant Riemannian metric on a Lie group and obtained several interesting results. For instance, a nilpotent Lie group admits a left invariant Riemannian metric with vanishing Ricci curvature if and only if it is Abelian. Later, in 1979, Nomizu [16] studied left invariant Lorentz metrics on Lie groups and found that the three-dimensional Heisenberg group admits a left invariant Lorentz metric with vanishing Ricci curvature. Recently, Conti, del Barco and Rossi [4–7] studied left invariant pseudo-Riemannian Einstein metrics on nice Lie groups extensively. They showed that there are plenty of nilpotent Lie groups admitting left invariant pseudo-Riemannian metrics with vanishing Ricci curvature. Unlike the Riemannian case, the existence of a left invariant pseudo-Riemannian Ricci flat metric seems to be a more common phenomenon. In fact, the authors in [4]

Received August 29, 2023; Revised January 12, 2024; Accepted January 26, 2024.
2020 *Mathematics Subject Classification.* Primary 53C50, 53C30, 53C24.
*Key words and phrases.* Left invariant pseudo-Riemannian metric, Ricci operator, solvable Lie group.
S. Deng is supported by NSFC (nos. 12131012, 12071228), and the Fundamental Research Funds for the Central Universities. Z. Yan is supported by the Fundamental Research Funds for the Provincial Universities of Zhejiang and K.C. Wong Magna Fund in Ningbo University.

posed a question whether every nilpotent Lie group admits a left invariant pseudo-Riemannian Ricci flat metric. For more results on this topic, we refer the readers to [1, 10–12, 14] for the Riemannian case, and [3, 8] for the pseudo-Riemannian case. In this paper, we investigate the nilpotency of the Ricci operator of a left invariant pseudo-Riemannian metric on a connected solvable Lie group. More precisely, we consider the following problem:

**Problem.** *Which Lie groups admit a left invariant pseudo-Riemannian metric with nilpotent Ricci operator?*

This paper focus on a special class $\mathfrak{G}$ of solvable Lie groups, namely, those Lie groups $G$ whose Lie algebras $\mathfrak{g}$ can be expressed as a semi-product $\mathbb{R}D \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent Lie algebra with center $C(\mathfrak{n})$, and $D$ is a derivation of $\mathfrak{n}$ such that the restriction $D|_{C(\mathfrak{n})}$ has at least one real eigenvalue. We should mention that, this construction is a special kind of the double extension of nilpotent Lie algebras in the sense of [13]. Clearly, every nilpotent Lie group belongs to $\mathfrak{G}$. Our main result is the following

**Theorem 1.1.** *Every solvable Lie group belonging to $\mathfrak{G}$ admits a left invariant pseudo-Riemannian metric with nilpotent Ricci operator.*

As an interesting consequence, we have:

**Corollary 1.2.** *Every nilpotent Lie group admits a left invariant pseudo-Riemannian metric with nilpotent Ricci operator.*

We remark here that this result is significant, since, as we mentioned above, a nilpotent Lie group admits a left invariant Riemannian metric with vanishing scalar curvature (nilpotent Ricci operator) if and only if it is Abelian [15].

Next we consider the Ricci operator of left invariant pseudo-Riemannian metrics on three-dimensional solvable Lie groups. It is well known [9] that, for any three-dimensional real solvable Lie algebra which is not Abelian, there exists a basis $\{h_0, e_1, e_2\}$ such that one of the following conditions holds:

$$\begin{aligned}
&\mathfrak{q}: && [h_0, e_1] = e_1, && [h_0, e_2] = e_2, && [e_1, e_2] = 0; \\
&\mathfrak{r}_\alpha: && [h_0, e_1] = e_2, && [h_0, e_2] = \alpha e_1, && [e_1, e_2] = 0, \quad \alpha = 0, -1 \text{ or } 1; \\
&\mathfrak{s}_\beta: && [h_0, e_1] = e_2, && [h_0, e_2] = \beta e_1 + e_2, && [e_1, e_2] = 0, \quad \beta \in \mathbb{R}.
\end{aligned}$$

*Remark* 1.3. Clearly, the Lie algebras $\mathfrak{q}$, $\mathfrak{r}_0$ and $\mathfrak{r}_1$ are contained in the class $\mathfrak{G}$, $\mathfrak{r}_{-1}$ is not contained in the class $\mathfrak{G}$. For the Lie algebras $\mathfrak{s}_\beta$, $\beta \in \mathbb{R}$, note that the matrix of $\operatorname{ad} h_0$ relative to the basis $\{e_1, e_2\}$ is

$$\begin{pmatrix} 0 & \beta \\ 1 & 1 \end{pmatrix},$$

which has a real eigenvalue if and only if $\beta \geq -\frac{1}{4}$. This asserts that $\mathfrak{s}_\beta$ are contained in the class $\mathfrak{G}$ if and only if $\beta \geq -\frac{1}{4}$.

We prove the following:

**Theorem 1.4.** *A three-dimensional solvable Lie group admits a left invariant pseudo-Riemannian metric with nilpotent Ricci operator if and only if its Lie algebra is either Abelian or is isomorphic to one of the Lie algebras* $\mathfrak{q}$, $\mathfrak{r}_0$, $\mathfrak{r}_{-1}$, $\mathfrak{r}_1$ *and* $\mathfrak{s}_\beta$, *where* $\beta \geq -\frac{1}{4}$.

**Corollary 1.5.** *There are infinitely many three-dimensional solvable Lie groups which are non-isometric to each other and do not admit any left invariant pseudo-Riemannian Ricci flat metrics.*

## 2. Preliminaries

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ consisting of left invariant vector fields and $\langle\cdot,\cdot\rangle$ be a left invariant pseudo-Riemannian metric on $G$. Let $\nabla$ be the Levi-Civita connection associated with $\langle\cdot,\cdot\rangle$, and $x, y, z, u, v \in \mathfrak{g}$. Then

$$[x, y] = \nabla_x y - \nabla_y x,$$
$$\langle \nabla_x y, z\rangle = \frac{1}{2}(\langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle).$$

The curvature tensor is given by

$$R(x, y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z.$$

The (0,2)-Ricci tensor ric associated with $\langle\cdot,\cdot\rangle$ is

$$\mathrm{ric}(u, v) = \mathrm{tr}(x \mapsto R(x, u)v),$$

and the Ricci operator Ric is defined by

$$\langle \mathrm{Ric}(u), v\rangle = \mathrm{ric}(u, v).$$

The mean curvature vector $Z \in \mathfrak{g}$ associated with $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is given by

$$\langle Z, x\rangle = \mathrm{tr}\,(\mathrm{ad}\, x).$$

Note that $\mathfrak{g}$ is unimodular if and only if $Z = 0$.

**Lemma 2.1** ([2, 17])**.** *Let* $\{e_i\}$ *be an orthonormal basis of* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. *Then*

$$\begin{aligned}\mathrm{ric}(x, y) = & -\frac{1}{2}\sum_i \langle [x, e_i], [y, e_i]\rangle \varepsilon_i + \frac{1}{4}\sum_{i,j}\langle [e_i, e_j], x\rangle\langle [e_i, e_j], y\rangle \varepsilon_i\varepsilon_j \\ & -\frac{1}{2}K(x, y) - \frac{1}{2}(\langle [Z, x], y\rangle + \langle [Z, y], x\rangle),\end{aligned}$$

*where* $x, y \in \mathfrak{g}$, $\varepsilon_i = \langle e_i, e_i\rangle \in \{1, -1\}$, $K$ *is the Killing form of* $\mathfrak{g}$, *and* $Z$ *denotes the mean curvature vector associated with* $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$.

## 3. Proof of Theorem 1.1 and Theorem 1.4

*Proof of Theorem 1.1.* We will prove the theorem inductively on $\dim \mathfrak{g}$. In the case of $\dim \mathfrak{g} = 1$, the Lie algebra $\mathfrak{g}$ is Abelian and hence any left invariant pseudo-Riemannian metric on $\mathfrak{g}$ has vanishing Riemann curvature tensor. If $\dim \mathfrak{g} = 2$, then $\mathfrak{g}$ is either Abelian or has a basis $\{x, y\}$ such that $[x, y] = y$. In this case there also exists a left invariant Lorentz flat metric on $\mathfrak{g}$ [16].

Now assume $\dim \mathfrak{g} \geq 3$ and $v$ is a non-zero vector in the center of $\mathfrak{n}$ satisfying $Dv \in \mathbb{R}v$. Let $\mathfrak{n}_0$ be a subspace of $\mathfrak{n}$ complementary to $\mathbb{R}v$. Then the Lie bracket $[\cdot,\cdot]_\mathfrak{n}$ of $\mathfrak{n}$ induces a Lie bracket $[\cdot,\cdot]_0$ on $\mathfrak{n}_0$ defined by

$$[x, y]_\mathfrak{n} = [x, y]_0 + \theta(x, y)v, \ \forall x, y \in \mathfrak{n}_0,$$

where $[x, y]_0 \in \mathfrak{n}_0$ and $\theta : \mathfrak{n}_0 \times \mathfrak{n}_0 \to \mathbb{R}$ is a cocycle satisfying

$$\theta([x, y]_0, z) + \theta([y, z]_0, x) + \theta([z, x]_0, y) = 0, \ \forall x, y, z \in \mathfrak{n}_0.$$

It is easily seen that $(\mathfrak{n}_0, [\cdot,\cdot]_0)$ is isomorphic to $\mathfrak{n}/\mathbb{R}v$ and hence nilpotent. By the assumption, there exists a left invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle_0$ on $(\mathfrak{n}_0, [\cdot,\cdot]_0)$ with nilpotent Ricci operator $\mathrm{Ric}_0$. We now define a left invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g} = \mathbb{R}D \ltimes \mathfrak{n} = \mathbb{R}D + \mathfrak{n}_0 + \mathbb{R}v$ by

$$\langle D, v\rangle = 1, \langle D, D\rangle = \langle v, v\rangle = \langle D, \mathfrak{n}_0\rangle = \langle v, \mathfrak{n}_0\rangle = 0, \langle x, y\rangle = \langle x, y\rangle_0, \ \forall x, y \in \mathfrak{n}_0.$$

It is sufficient to show that the Ricci operator Ric of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is nilpotent. Let $\{e_i\}$ be an orthonormal basis of $(\mathfrak{n}_0, [\cdot,\cdot]_0, \langle\cdot,\cdot\rangle_0)$, and denote $\varepsilon_i = \langle e_i, e_i\rangle_0 \in \{1, -1\}$. Then $\{e_i, \frac{D+v}{\sqrt{2}}, \frac{D-v}{\sqrt{2}}\}$ forms an orthonormal basis of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, where

$$\left\langle \frac{D+v}{\sqrt{2}}, \frac{D+v}{\sqrt{2}} \right\rangle = -\left\langle \frac{D-v}{\sqrt{2}}, \frac{D-v}{\sqrt{2}} \right\rangle = 1.$$

Notice that the mean curvature vector in $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is $Z = (\mathrm{tr}\, D)v$, and hence $\langle [Z, \mathfrak{g}], \mathfrak{n}\rangle = 0$. Moreover, one has $\langle [\mathfrak{g}, \mathfrak{g}], v\rangle = \langle \mathfrak{n}, v\rangle = 0$ and $K(\mathfrak{g}, \mathfrak{n}) = 0$. Now by the formula in Lemma 2.1, given any $x \in \mathfrak{g}$, we have

$$\begin{aligned}\langle \mathrm{Ric}(v), x\rangle = & -\frac{1}{2}\left\langle [v, \frac{D+v}{\sqrt{2}}], [x, \frac{D+v}{\sqrt{2}}] \right\rangle + \frac{1}{2}\left\langle [v, \frac{D-v}{\sqrt{2}}], [x, \frac{D-v}{\sqrt{2}}] \right\rangle \\ & - \frac{1}{2}(\langle [Z, v], x\rangle + \langle [Z, x], v\rangle) \\ = & \ 0.\end{aligned}$$

This implies that $\mathrm{Ric}(v) = 0$ and $\mathrm{Ric}(\mathfrak{g}) \subset \mathfrak{n}$.

Furthermore, for any $x, y \in \mathfrak{n}_0$, we have

$$\begin{aligned}& \langle \mathrm{Ric}(x), y\rangle \\ = & -\frac{1}{2}\left(\left\langle [x, \frac{D+v}{\sqrt{2}}], [y, \frac{D+v}{\sqrt{2}}] \right\rangle - \left\langle [x, \frac{D-v}{\sqrt{2}}], [y, \frac{D-v}{\sqrt{2}}] \right\rangle\right) \\ & - \frac{1}{2}\sum_i \langle [x, e_i], [y, e_i]\rangle\varepsilon_i - \frac{1}{2}\left\langle [\frac{D+v}{\sqrt{2}}, \frac{D-v}{\sqrt{2}}], x \right\rangle \left\langle [\frac{D+v}{\sqrt{2}}, \frac{D-v}{\sqrt{2}}], y \right\rangle\end{aligned}$$

$$+\frac{1}{2}\sum_i\left\langle[\frac{D+v}{\sqrt{2}},e_i],x\right\rangle\left\langle[\frac{D+v}{\sqrt{2}},e_i],y\right\rangle\varepsilon_i$$
$$+\frac{1}{2}\sum_i\left\langle[\frac{D-v}{\sqrt{2}},e_i],x\right\rangle\left\langle[\frac{D-v}{\sqrt{2}},e_i],y\right\rangle(-\varepsilon_i)$$
$$+\frac{1}{4}\sum_{i,j}\langle[e_i,e_j],x\rangle\langle[e_i,e_j],y\rangle\varepsilon_i\varepsilon_j$$
$$= -\frac{1}{2}\sum_i\langle[x,e_i],[y,e_i]\rangle\varepsilon_i+\frac{1}{4}\sum_{i,j}\langle[e_i,e_j],x\rangle\langle[e_i,e_j],y\rangle\varepsilon_i\varepsilon_j$$
$$= -\frac{1}{2}\sum_i\langle[x,e_i]_0,[y,e_i]_0\rangle_0\varepsilon_i+\frac{1}{4}\sum_{i,j}\langle[e_i,e_j]_0,x\rangle_0\langle[e_i,e_j]_0,y\rangle_0\varepsilon_i\varepsilon_j$$
$$= \langle \mathrm{Ric}_0(x),y\rangle_0.$$

This implies that $\mathrm{Ric}|_{\mathfrak{n}_0}=\mathrm{Ric}_0$. Therefore, the matrix of $\mathrm{Ric}:\mathfrak{g}\to\mathfrak{g}$ relative to the basis $\{D,e_i,v\}$ is

$$\begin{pmatrix} 0 & 0 & 0\\ * & \mathrm{Ric}_0 & 0\\ * & * & 0\end{pmatrix},$$

which is a nilpotent matrix. This completes the proof of the theorem. □

*Proof of Theorem 1.4.* Keep the notations as above. Let $(\mathfrak{g}=\mathbb{R}h_0+\mathbb{R}e_1+\mathbb{R}e_2,\langle\cdot,\cdot\rangle)$ be a three-dimensional non-Abelian solvable Lie algebra with a left invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ such that the Ricci operator $\mathrm{Ric}:\mathfrak{g}\to\mathfrak{g}$ is nilpotent. We will show that $\mathfrak{g}=\mathfrak{r}_{-1}$ or $\mathfrak{g}$ is in the class $\mathfrak{G}$, and consequently the theorem follows from Theorem 1.1, Remark 1.3 and a result of Milnor [15] that $\mathfrak{r}_{-1}$ admits a left invariant Riemannian flat metric. Denote $E=\mathbb{R}e_1+\mathbb{R}e_2$. Then we have $[\mathfrak{g},\mathfrak{g}]\subset E$ and $K(\mathfrak{g},E)=0$. According as $\langle\cdot,\cdot\rangle|_E$ is degenerate or not, we have the following two cases:

Case **I**: $E$ is nondegenerate, namely, the restriction of the metric $\langle\cdot,\cdot\rangle$ to $E$ is nondegenerate. In this case, there exists an orthonormal basis $\{H,X_1,X_2\}$ of $\mathfrak{g}$ such that $X_1,X_2\in E$, $\mathrm{ad}\,H(E)\subset E$, $\delta_H=\langle H,H\rangle\in\{1,-1\}$, $\varepsilon_1=\langle X_1,X_1\rangle\in\{1,-1\}$, $\varepsilon_2=\langle X_2,X_2\rangle\in\{1,-1\}$. Notice that the mean curvature vector $Z=(\delta_H\mathrm{tr}\,\mathrm{ad}\,H|_E)H$ and $\langle Z,E\rangle=0$. Since $[E,E]=0$, by Lemma 2.1 (see also Proposition 1.10 of [8]) we have

$$\mathrm{ric}(H,e)=-\frac{1}{2}\langle[H,X_1],[e,X_1]\rangle\varepsilon_1-\frac{1}{2}\langle[H,X_2],[e,X_2]\rangle\varepsilon_2=0,\ \forall e\in E.$$

This implies that $\mathrm{Ric}(H)\in\mathbb{R}H$, $\mathrm{Ric}(E)\subset E$. Furthermore, for any $e,e'\in E$,

$$\mathrm{ric}(e,e') = -\frac{1}{2}\delta_H\mathrm{tr}\,\mathrm{ad}\,H|_E(\langle[H,e],e'\rangle+\langle e,[H,e']\rangle)-\frac{1}{2}\langle[H,e],[H,e']\rangle\delta_H$$
$$+\frac{1}{2}\sum_{i=1}^{2}\langle[H,X_i],e\rangle\langle[H,X_i],e'\rangle\delta_H\varepsilon_i$$

$$
\begin{aligned}
= &-\frac{1}{2}\delta_H \operatorname{tr}\operatorname{ad} H|_E \langle (\operatorname{ad} H|_E + (\operatorname{ad} H|_E)^*)(e), e'\rangle \\
&-\frac{1}{2}\delta_H \left\langle \Big( (\operatorname{ad} H|_E)^* \operatorname{ad} H|_E \Big)(e), e' \right\rangle \\
&+\frac{1}{2}\delta_H \left\langle \Big( (\operatorname{ad} H|_E)(\operatorname{ad} H|_E)^* \Big)(e), e' \right\rangle .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Ric}|_E = &-\frac{1}{2}\delta_H \operatorname{tr}\operatorname{ad} H|_E \Big( \operatorname{ad} H|_E + (\operatorname{ad} H|_E)^* \Big) \\
&-\frac{1}{2}\delta_H \Big( (\operatorname{ad} H|_E)^* \operatorname{ad} H|_E - (\operatorname{ad} H|_E)(\operatorname{ad} H|_E)^* \Big).
\end{aligned}
$$

Since Ric is nilpotent, $\operatorname{Ric}|_E$ is nilpotent and $\operatorname{tr}\operatorname{Ric}|_E = 0$. So we have $(\operatorname{tr}\operatorname{ad} H|_E)^2 = 0$ and $\operatorname{tr}\operatorname{ad} H|_E = 0$. This implies that $\mathfrak{g}$ is unimodular. Hence $\mathfrak{g}$ is isomorphic to $\mathfrak{r}_0$, $\mathfrak{r}_{-1}$ or $\mathfrak{r}_1$.

Case **II**: $E$ is degenerate, namely, the restriction of the metric $\langle \cdot, \cdot \rangle$ to $E$ is degenerate. In this case, there exist two vectors $X, Y \in E$ such that

$$\langle X, X\rangle \in \{1, -1\},\ \langle X, Y\rangle = \langle Y, Y\rangle = 0.$$

In the orthogonal complement $(\mathbb{R}X)^\perp$ of $\mathfrak{g}$ there exists a vector $H \in \mathfrak{g} \setminus E$ such that

$$\langle H, H\rangle = 0, \langle H, Y\rangle = 1.$$

Then $\{\frac{H+Y}{\sqrt{2}}, \frac{H-Y}{\sqrt{2}}, X\}$ forms an orthonormal basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The mean curvature vector $Z = (\operatorname{tr}\operatorname{ad} H)Y \in E$. By a direct calculation, we get

$$
\begin{aligned}
&\operatorname{ric}(X, X) = -\frac{1}{2}\langle [\frac{H+Y}{\sqrt{2}}, \frac{H-Y}{\sqrt{2}}], X\rangle^2 = -\frac{1}{2}\langle [H, Y], X\rangle^2, \\
&\operatorname{ric}(X, Y) = \operatorname{ric}(Y, Y) = 0.
\end{aligned}
$$

Thus the matrix of Ric relative to the basis $\{H, X, Y\}$ is

$$
\mathrm{M}(\operatorname{Ric}; H, X, Y) = \begin{pmatrix} * & 0 & 0 \\ * & \gamma_0 & 0 \\ * & * & * \end{pmatrix},
$$

where $\gamma_0 = \frac{\operatorname{ric}(X,X)}{\langle X,X\rangle} = -\frac{\langle [H,Y],X\rangle^2}{2\langle X,X\rangle}$. As Ric is nilpotent, we have $\gamma_0 = 0$. Therefore $[H, Y] \in \mathbb{R}Y$. Consequently, $\operatorname{ad} h_0|_E$ has a real eigenvalue and $\mathfrak{g}$ is in the class $\mathfrak{G}$. This completes the proof of the theorem. ☐

## References


[1] R. M. Arroyo and R. A. Lafuente, *On the signature of the Ricci curvature on nilmanifolds*, Transf. Groups (2022). https://doi.org/10.1007/s00031-021-09686-5

[2] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, Heidelberg, 1987. https://doi.org/10.1007/978-3-540-74311-8

[3] M. Boucetta and O. Tibssirte, *On Einstein Lorentzian nilpotent Lie groups*, J. Pure Appl. Algebra **224** (2020), no. 12, 106443, 22 pp. https://doi.org/10.1016/j.jpaa.2020.106443

[4] D. Conti, V. del Barco, and F. A. Rossi, *Diagram involutions and homogeneous Ricci-flat metrics*, Manuscripta Math. **165** (2021), no. 3-4, 381–413. https://doi.org/10.1007/s00229-020-01225-y
[5] D. Conti and F. A. Rossi, *Einstein nilpotent Lie groups*, J. Pure Appl. Algebra **223** (2019), no. 3, 976–997. https://doi.org/10.1016/j.jpaa.2018.05.010
[6] D. Conti and F. A. Rossi, *Ricci-flat and Einstein pseudoriemannian nilmanifolds*, Complex Manifolds **6** (2019), no. 1, 170–193. https://doi.org/10.1515/coma-2019-0010
[7] D. Conti and F. A. Rossi, *Indefinite Einstein metrics on nice Lie groups*, Forum Math. **32** (2020), no. 6, 1599–1619. https://doi.org/10.1515/forum-2020-0049
[8] D. Conti and F. A. Rossi, *Indefinite nilsolitons and Einstein solvmanifolds*, J. Geom. Anal. **32** (2022), no. 3, Paper 88, 34 pp.
[9] W. A. de Graaf, *Classification of solvable Lie algebras*, Experiment. Math. **14** (2005), no. 1, 15–25. http://projecteuclid.org/euclid.em/1120145567
[10] M. B. Djiadeu Ngaha, M. Boucetta, and J. Wouafo Kamga, *The signature of the Ricci curvature of left-invariant Riemannian metrics on nilpotent Lie groups*, Differential Geom. Appl. **47** (2016), 26–42. https://doi.org/10.1016/j.difgeo.2016.03.004
[11] J. Heber, *Noncompact homogeneous Einstein spaces*, Invent. Math. **133** (1998), no. 2, 279–352. https://doi.org/10.1007/s002220050247
[12] J. Lauret, *A canonical compatible metric for geometric structures on nilmanifolds*, Ann. Global Anal. Geom. **30** (2006), no. 2, 107–138. https://doi.org/10.1007/s10455-006-9015-y
[13] A. Medina and P. Revoy, *Algèbres de Lie et produit scalaire invariant*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 553–561.
[14] I. D. Miatello, *Ricci curvature of left invariant metrics on solvable unimodular Lie groups*, Math. Z. **180** (1982), no. 2, 257–263. https://doi.org/10.1007/BF01318909
[15] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329. https://doi.org/10.1016/S0001-8708(76)80002-3
[16] K. Nomizu, *Left-invariant Lorentz metrics on Lie groups*, Osaka Math. J. **16** (1979), no. 1, 143–150. http://projecteuclid.org/euclid.ojm/1200771834
[17] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

Huihui An
School of Mathematics
Liaoning Normal University
Dalian, Liaoning Province, 116029, P. R. China
*Email address*: finsleran@163.com

Shaoqiang Deng
School of Mathematical Sciences and LPMC
Nankai University
Tianjin 300071, P. R. China
*Email address*: dengsq@nankai.edu.cn

Zaili Yan
School of Mathematics and Statistics
Ningbo University
Ningbo, Zhejiang Province, 315211, P. R. China
*Email address*: yanzaili@nbu.edu.cn