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NON-UNIFORM DEPENDENCE ON INITIAL DATA FOR THE FORNBERG–WHITHAM EQUATION IN $C^1(\mathbb{R})$

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Abstract. It is shown in [1] that the Cauchy problem for the Fornberg–Whitham equation is well-posed in $C^1(\mathbb{R})$ and the data-to-solution map is Hölder continuous from C^α to $\mathcal{C}([0,T];C^\alpha)$ with $\alpha\in[0,1)$. In this short paper, we further show that the data-to-solution map of the Fornberg–Whitham equation is not uniformly continuous on the initial data in $C^1(\mathbb{R})$.

1. Introduction

In this paper, we focus on the Cauchy problem of the Fornberg–Whitham (FW) equation

$$(1)\quad \begin{cases} u_t-u_{xxt}+\frac{3}{2}uu_x-\frac{9}{2}u_xu_{xx}-\frac{3}{2}uu_{xxx}-u_x=0, & (x,t)\in\mathbb{R}\times\mathbb{R}^+,\\ u(x,t=0)=u_0(x), & x\in\mathbb{R},\end{cases}$$

which was first introduced by Whitham [26] in 1967 and Fornberg and Whitham [7] in 1978, as a shallow water wave model to study the qualitative behaviors of wave breaking (the solution remains bounded while its slope becomes unbounded in finite time). The FW equation was compared with the famous Korteweg-de Vries (KdV) equation [21]

$$u_t+6uu_x=-u_{xxx},$$

and the classical Camassa-Holm (CH) equation [2–6, 8]

$$u_t-u_{xxt}+3uu_x=2u_xu_{xx}+uu_{xxx}.$$

The KdV equation admits solitons or solitary traveling wave solutions. Indeed, the KdV equation in the non-periodic admits the solitary wave solutions with the form $u(t,x)=\frac{c}{2}\mathrm{sech}^2\Big(\frac{\sqrt{c}}{2}(x-ct)\Big)$. And, the CH equation possess exact peaked soliton solutions (peakons) of the form $u(t,x)=ce^{-|x-ct|}$. It is interesting that the FW equation does not only admit solitary traveling wave solutions like the KdV equation, but also possess peakon solutions (or

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peaked traveling wave solutions) [26] as the CH equation which are of the form $u(t,x) = \frac{8}{9}e^{-\frac{1}{2}|x-\frac{4}{3}t|}$. In [20], the KdV equation is shown to be well-posed in $H^s(\mathbb{R})$ with $s > -3/4$, then the solution map is Lipschitz on the same $H^s(\mathbb{R})$. For the CH equation, Himonas-Misiołek [13] obtained the first result on the non-uniform dependence in $H^s(\mathbb{T})$ with $s \geq 2$ using explicitly constructed travelling wave solutions, which was sharpened to $s > \frac{3}{2}$ by Himonas-Kenig [11] on the real-line and Himonas-Kenig-Misiołek [12] on the circle. We should mention that, non-uniform continuity of the CH solution map in $H^1(\mathbb{R}$ or $\mathbb{T})$ was established by Himonas-Misiołek-Ponce [14] by using traveling wave solutions.

The FW equation (1) admits the conserved quantities [23]

$$E_1(u) = \int_{\mathbb{R}} u \mathrm{d}x, \quad E_2(u) = \int_{\mathbb{R}} u^2 \mathrm{d}x, \quad E_3(u) = \int_{\mathbb{R}} \left(u \left(1-\partial_x^2\right)^{-1} u - u^3 \right) \mathrm{d}x.$$

A classification of other traveling wave solutions of the FW equation was presented by Yin, Tian and Fan [29]. It's worth mentioning that the KdV equation and CH equation are integrable, and they possess infinitely many conserved quantities, an infinite hierarchy of quasi-local symmetries, a Lax pair and a bi-Hamiltonian structure. Unlike the KdV and CH equation, the FW equation is not integrable. Although the FW equation is in a simple form, the only useful conservation law we know so far is $\|u\|_{L^2}$. Therefore, the analysis of the FW equation would be somewhat more difficult due to the special structure of this equation and the lower regularity of the conservation law. Particularly, the well-posedness theory for the FW equation is not completely understood. Before recalling the well-posedness results the FW equation, we firstly transform the FW equation (1) equivalently into the following non-local form

$$\begin{cases} \partial_t u + \frac{3}{2} u \partial_x u = \mathbf{P}(u) = \partial_x \left(1-\partial_x^2\right)^{-1} u, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,t=0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

By the Galerkin approximation argument, Holmes [15] proved the well-posed of the FW equation in Sobolev spaces $H^s(\mathbb{T})$ with $s > 3/2$. Holmes and Thompson [16] obtained the well-posedness of the FW equation in Besov spaces $B^s_{2,r}(\mathbb{R}$ or $\mathbb{T})$ ($s > 3/2, 1 < r < \infty$ or $s = 3/2, r = 1$). They also proved that the data-to-solution map is not uniformly continuous but Hölder continuous in some given topology and presented a blow-up criterion for solutions. Later, Haziot [10], Hörmann [17,19], Wei [24,25], Wu-Zhang [27] and Yang [28] sharpened this blowup criterion and presented the sufficient conditions about the initial data to lead the wave-breaking phenomena of the FW equation. The discontinuous traveling waves as weak solutions to the FW equation were investigated in [18]. Recently, Guo [9] established the local well-posedness (existence, uniqueness and continuous dependence) for the FW equation in both supercritical Besov spaces $B^s_{p,r}$ with $s > 1 + 1/p$, $(p,r) \in [1,\infty] \times [1,\infty]$ and critical Besov spaces $B^{1+1/p}_{p,1}$ with $p \in [1,\infty)$. Furthermore, Guo [9] proved the data-to-solution is not uniformly continuous dependence on the initial data in the Besov spaces $B^s_{p,r}$ with $s > 1 + 1/p$, $(p, r) \in [1, \infty] \times [1, \infty)$ and critical Besov spaces $B^{1+1/p}_{p,1}$ with $p \in [1, \infty)$. Li-Wu-Yu-Zhu [22] proved that the Cauchy problem for the FW equation is ill-posed in $B^s_{p,r}(\mathbb{R})$ with $(s, p, r) \in (1, 1 + 1/p) \times [2, \infty) \times [1, \infty]$ or $(s, p, r) \in \{1\} \times [2, \infty) \times [1, 2]$ by showing norm inflation phenomena of the solution for some special initial data.

Very recently, Burkhalter-Thompson-Waldrep [1] proved that the Cauchy problem for the FW equation is well-posed in $C^1(\mathbb{R})$. Meanwhile, they also obtained the data-to-solution map is Hölder continuous from C^α to $\mathcal{C}([0, T]; C^\alpha)$ with $\alpha \in [0, 1)$. Naturally, we would like to ask that whether or not the above Hölder continuous can be improved to be Lipschitz continuous. Our answer is Not. More precisely, we shall prove that the data-to-solution map $u_0 \mapsto \mathbf{S}_t(u_0)$ to the FW equation as function of the initial data is not uniformly continuous on the initial data in $C^1(\mathbb{R})$. From the PDE's point of view, it is crucial to know if an equation which models a physical phenomenon is well-posed in the Hadamard's sense: existence, uniqueness, and continuous dependence of the solutions with respect to the initial data. In particular, continuity properties of the solution map is an important part of the well-posedness theory since the lack of continuous dependence would cause incorrect solutions or non meaningful solutions. Furthermore, the non-uniform continuity of data-to-solution map suggests that the local well-posedness cannot be established by the contraction mappings principle since this would imply Lipschitz continuity for the solution map.

Now, we can formulate the main result.

Theorem 1.1. *Denote $U_R \equiv \big\{u_0 \in C^1(\mathbb{R}) : \|u_0\|_{C^1} \le R\big\}$ for any $R > 0$. Then the data-to-solution map of the Cauchy problem* (2)

$$\mathbf{S}_t : \begin{cases} U_R \to \mathcal{C}([0, T]; C^1) \cap \mathcal{C}^1([0, T]; C), \\ u_0 \mapsto \mathbf{S}_t(u_0), \end{cases}$$

is not uniformly continuous from any bounded subset U_R in C^1 into $\mathcal{C}([0, T]; C^1)$. More precisely, there exists two sequences of solutions $\mathbf{S}_t(f_n + g_n)$ and $\mathbf{S}_t(f_n)$ such that

$$\|f_n\|_{C^1} \lesssim 1 \quad \text{and} \quad \lim_{n\to\infty} \|g_n\|_{C^1} = 0$$

but

$$\liminf_{n\to\infty} \|\mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n)\|_{C^1} \gtrsim t, \quad \forall\, t \in [0, T_0],$$

with small time T_0.

Remark 1.2. Compared with the result in [9] where the regularity $s > 1$ is required, our theorem holds for the lower regularity $s = 1$.

Remark 1.3. The method we used in proving Theorem 1.1 is very general and can be applied equally well to other related system with the simple lower nonlinear term, such as the Degasperis-Procesi equation

$$\begin{cases} \partial_t u + u\partial_x u = -\frac{3}{2}\partial_x \left(1-\partial_x^2\right)^{-1} u^2, & (x,t)\in\mathbb{R}\times\mathbb{R}^+, \\ u(x,t=0)=u_0(x), & x\in\mathbb{R}. \end{cases}$$

Organization of our paper. In Section 2, we establish some technical lemmas which will be used in the sequel. In Section 3, we prove Theorem 1.1.

Notations. Given a Banach space X, we denote its norm by $\|\cdot\|_X$. For $I\subset\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ the set of continuous functions on I with values in X. The symbol $a\lesssim(\gtrsim)b$ means that there is a uniform positive constant C independent of a and b such that $a\leq(\geq)Cb$. $a\approx b$ means that $a\lesssim b$ and $a\gtrsim b$. We let $C^1(\mathbb{R})$ be the Banach space of bounded and continuously differentiable functions which are equipped with the norm $\|f\|_{C^1}=\|f\|_{L^\infty}+\|\partial_x f\|_{L^\infty}$ with $\|f\|_{L^\infty}=\sup_{x\in\mathbb{R}}|f(x)|$. We use $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ to denote Schwartz functions and the tempered distributions spaces on $\mathbb{R}$, respectively. Let us recall that for all $u\in\mathcal{S}'$, the Fourier transform $\mathcal{F}u$, also denoted by $\widehat{u}$, is defined by $\mathcal{F}u(\xi)=\widehat{u}(\xi)=\int_{\mathbb{R}}e^{-\mathrm{i}x\xi}u(x)\mathrm{d}x$ for any $\xi\in\mathbb{R}$. The inverse Fourier transform of any g is given by $(\mathcal{F}^{-1}g)(x)=\frac{1}{2\pi}\int_{\mathbb{R}}g(\xi)e^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi$.

2. Preliminary lemmas

In this section, we establish some useful lemmas.

Lemma 2.1. *Let $u_0\in C^1(\mathbb{R})$. Assume that $u\in L^\infty([0,T];C^1(\mathbb{R}))$ solves* (2). *Then for all $t\in(0,\min\{1,1/(4\|\partial_x u_0\|_{L^\infty})\}]$, we have*

$$\|u(t)\|_{L^\infty(\mathbb{R})}\leq 3\|u_0\|_{L^\infty(\mathbb{R})}, \tag{3}$$

$$\|\partial_x u(t)\|_{L^\infty(\mathbb{R})}\leq 2(1+\|\partial_x u_0\|_{L^\infty(\mathbb{R})}). \tag{4}$$

Proof. From now on, we set $u_x=\partial_x u$ for simplicity. Given a Lipschitz velocity field u, we may solve the following ODE to find the flow induced by u:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x)=\frac{3}{2}u(t,\psi(t,x)), \\ \psi(0,x)=x. \end{cases} \tag{5}$$

From (2), we get that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}u(t,\psi(t,x)) &= u_t(t,\psi(t,x))+u_x(t,\psi(t,x))\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x) \\ &= \left(u_t+\frac{3}{2}uu_x\right)(t,\psi(t,x)) \\ &= \left(\partial_x\left(1-\partial_x^2\right)^{-1}u\right)(t,\psi(t,x)). \end{aligned}$$

Integrating the above with respect to time variable yields that

$$u(t,\psi(t,x))=u_0(x)+\int_0^t\left(\partial_x\left(1-\partial_x^2\right)^{-1}u\right)(\tau,\psi(\tau,x))\mathrm{d}\tau.$$

Using the fact that the L^∞-norm of any function is preserved under the flow ψ, then we have

$$\|u(t,x)\|_{L^\infty} = \|u(t,\psi(t,x))\|_{L^\infty} \le \|u_0(x)\|_{L^\infty} + \int_0^t \|u(\tau,x)\|_{L^\infty} \mathrm{d}\tau, \tag{6}$$

where we have used the following estimate:

$$\left\|\partial_x \left(1-\partial_x^2\right)^{-1} f\right\|_{L^\infty} = \|\partial_x G * f\|_{L^\infty} \le \|f\|_{L^\infty} \quad \text{where} \quad G(x) = \frac{1}{2}e^{-|x|}.$$

Using Gronwall's inequality to (6), we get (3).

Applying ∂_x to (2) yields

$$u_{tx} + \frac{3}{2}uu_{xx} = -\frac{3}{2}(u_x)^2 + \partial_x^2\left(1-\partial_x^2\right)^{-1} u. \tag{7}$$

Combining (5) and (7), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u_x(t,\psi(t,x)) &= u_{tx}(t,\psi(t,x)) + u_{xx}(t,\psi(t,x))\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x) \\
&= \left(u_{tx} + \frac{3}{2}uu_{xx}\right)(t,\psi(t,x)) \\
&= -\left(\frac{3}{2}(u_x)^2 - \partial_x^2\left(1-\partial_x^2\right)^{-1} u\right)(t,\psi(t,x)),
\end{aligned} \tag{8}$$

which means that

$$u_x(t,\psi(t,x)) = \partial_x u_0(x) - \int_0^t \left(\frac{3}{2}(u_x)^2 - \partial_x^2\left(1-\partial_x^2\right)^{-1} u\right)(\tau,\psi(\tau,x))\mathrm{d}\tau.$$

Notice that the L^∞-norm of any function is preserved under the flow ψ again, we have

$$\begin{aligned}
\|u_x(t,x)\|_{L^\infty} &\le \|\partial_x u_0\|_{L^\infty} + \frac{3}{2}\int_0^t \|u_x(\tau,x)\|_{L^\infty}^2 \mathrm{d}\tau \\
&\quad + \int_0^t \left\|\partial_x\left(1-\partial_x^2\right)^{-1} u_x(\tau,x)\right\|_{L^\infty} \mathrm{d}\tau \\
&\le \|\partial_x u_0\|_{L^\infty} + 2\int_0^t \left(\|u_x(\tau,x)\|_{L^\infty}^2 + \|u_x(\tau,x)\|_{L^\infty}\right)\mathrm{d}\tau.
\end{aligned} \tag{9}$$

By Gronwall's inequality, we get

$$\|u_x(t)\|_{L^\infty} \le \|\partial_x u_0\|_{L^\infty} \exp\left(2\int_0^t \left(1+\|u_x(\tau)\|_{L^\infty}\right)\mathrm{d}\tau\right). \tag{10}$$

Setting

$$\lambda(t) := \|\partial_x u_0\|_{L^\infty} \exp\left(2\int_0^t \left(1+\|u_x(\tau)\|_{L^\infty}\right)\mathrm{d}\tau\right) \quad \text{with} \quad \lambda(0) := \|\partial_x u_0\|_{L^\infty},$$

then from (10), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) \le 2(1+\lambda(t))^2 \quad \Leftrightarrow \quad -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1+\lambda(t)}\right) \le 2.$$

Solving the above directly yields for $t \in (0, \min\{1, 1/(4\|\partial_x u_0\|_{L^\infty})\}]$

$$\sup_{\tau\in[0,t]} \|u_x(\tau)\|_{L^\infty} \le \frac{1+\|\partial_x u_0\|_{L^\infty}}{1-2t\|\partial_x u_0\|_{L^\infty}} \le 2(1+\|\partial_x u_0\|_{L^\infty}).$$

This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Assume that $u_0 \in \mathcal{S}$ and $\|\partial_x u_0\|_{L^\infty} \lesssim 1$. The data-to-solution map $u_0 \mapsto \mathbf{S}_t(u_0)$ of the Cauchy problem* (2) *satisfies that for $t \in (0,1]$*

$$\|\mathbf{S}_t(u_0) - u_0\|_{L^\infty} \le Ct\|u_0\|_{L^\infty}, \tag{11}$$

$$\|\partial_x(\mathbf{S}_t(u_0) - u_0)\|_{L^\infty} \le Ct\left(1+\|u_0\|_{L^\infty}\|\partial_x^2 u_0\|_{L^\infty}\right), \tag{12}$$

$$\|\partial_x^2(\mathbf{S}_t(u_0) - u_0)\|_{L^\infty} \le Ct\left(\|u_0\|_{L^\infty}\|\partial_x^3 u_0\|_{L^\infty}+\|\partial_x^2 u_0\|_{L^\infty}\right). \tag{13}$$

Proof. By Lemma 2.1, we know that the solution map $\mathbf{S}_t(u_0) \in \mathcal{C}([0,T];C^1)$ and has common lifespan $T \approx 1$. Moreover, there holds

$$\|\mathbf{S}_t(u_0)\|_{L^\infty_T(L^\infty)} \lesssim \|u_0\|_{L^\infty} \quad \text{and} \quad \|\partial_x \mathbf{S}_t(u_0)\|_{L^\infty_T(L^\infty)} \lesssim 1.$$

By the fundamental theorem of calculus in the time variable, we have

$$\begin{aligned}
\|\mathbf{S}_t(u_0) - u_0\|_{L^\infty} &\le \int_0^t \|\partial_\tau \mathbf{S}_\tau(u_0)\|_{L^\infty} \mathrm{d}\tau \\
&\le \int_0^t \|\mathbf{S}_\tau(u_0)\partial_x \mathbf{S}_\tau(u_0)\|_{L^\infty} \mathrm{d}\tau \\
&\quad + \int_0^t \|\partial_x(1-\partial_x^2)^{-1}(\mathbf{S}_\tau(u_0))\|_{L^\infty} \mathrm{d}\tau \\
&\lesssim \int_0^t \|\mathbf{S}_\tau(u_0)\|_{L^\infty}\left(\|\partial_x \mathbf{S}_\tau(u_0)\|_{L^\infty}+1\right) \mathrm{d}\tau \\
&\lesssim t\|u_0\|_{L^\infty}\left(1+\|\partial_x u_0\|_{L^\infty}\right) \lesssim t\|u_0\|_{L^\infty}.
\end{aligned}$$

Setting $v = \partial_x(\mathbf{S}_t(u_0) - u_0)$ and $u(t) = \mathbf{S}_t(u_0)$, then from (7), we deduce

$$\begin{cases} \partial_t v + \frac{3}{2}u\partial_x v = -\frac{3}{2}u\partial_x^2 u_0 - \frac{3}{2}(\partial_x u)^2 + \partial_x^2(1-\partial_x^2)^{-1}u, \\ v(x,t=0) = 0. \end{cases} \tag{14}$$

By identical reasoning to (9), we have

$$\begin{aligned}
\|v(t)\|_{L^\infty} &\le \int_0^t \left\|\frac{3}{2}u\partial_x^2 u_0 + \frac{3}{2}(\partial_x u)^2 - \partial_x^2(1-\partial_x^2)^{-1}u\right\|_{L^\infty} \mathrm{d}\tau \\
&\le Ct\left(1+\|u_0\|_{L^\infty}\|\partial_x^2 u_0\|_{L^\infty}\right).
\end{aligned}$$

Setting $w = \partial_x^2(\mathbf{S}_t(u_0) - u_0)$, then from (14), we deduce

$$\begin{cases} \partial_t w + \frac{3}{2}u\partial_x w = -\frac{3}{2}u\partial_x^3 u_0 - \frac{9}{2}\partial_x u\partial_x^2 u + \partial_x^3(1-\partial_x^2)^{-1}u, \\ w(x,t=0) = 0. \end{cases} \tag{15}$$

By identical reasoning to (9), we have

$$\begin{aligned}\|w(t)\|_{L^\infty} &\le \int_0^t \left\| \frac{3}{2}u\partial_x^3 u_0 + \frac{9}{2}\partial_x u \partial_x^2 u - \partial_x^3(1-\partial_x^2)^{-1}u \right\|_{L^\infty} \mathrm{d}\tau \\ &\le Ct\left(\|u_0\|_{L^\infty}\|\partial_x^3 u_0\|_{L^\infty} + \|\partial_x^2 u_0\|_{L^\infty}\right),\end{aligned}$$

which implies (13). This completes the proof of Lemma 3.1. $\square$

3. Proof of Theorem 1.1

We need to introduce smooth, radial cut-off functions to localize the frequency region. Precisely, let $\widehat{\phi} \in \mathcal{C}_0^\infty(\mathbb{R})$ be an even, real-valued and non-negative function on $\mathbb{R}$ and satisfy

$$\widehat{\phi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le \frac{1}{4}, \\ 0, & \text{if } |\xi| \ge \frac{1}{2}. \end{cases}$$

By the Fourier version formula and the Fubini theorem, we see that

$$\|\phi\|_{L^\infty} = \sup_{x\in\mathbb{R}} \frac{1}{2\pi}\left|\int_{\mathbb{R}} \widehat{\phi}(\xi)\cos(x\xi)\mathrm{d}\xi\right| \le \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\xi)\mathrm{d}\xi$$

and

$$\phi(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\xi)\mathrm{d}\xi > 0.$$

We establish the following crucial lemmas which will be used later on.

Lemma 3.1. *Define the high-low frequency functions f_n and g_n by*

$$f_n = 2^{-n}\phi(x)\cos(2^n x) \quad \text{and} \quad g_n = 2^{-n}\phi(x), \quad n \gg 1.$$

Then there exists a positive constant $C = C(\phi)$ such that

$$\|f_n\|_{L^\infty} + \|g_n\|_{L^\infty} + \|\partial_x g_n\|_{L^\infty} + \|\partial_x^2 g_n\|_{L^\infty} \le C2^{-n}, \tag{16}$$

$$\|\partial_x f_n\|_{L^\infty} \le C, \quad \|\partial_x^2 f_n\|_{L^\infty} \le C2^n, \tag{17}$$

$$\liminf_{n\to\infty} \|g_n \partial_x f_n\|_{C^1} \ge C. \tag{18}$$

Proof. It is easy to obtain that

$$\begin{aligned}\partial_x f_n &= -\phi(x)\sin(2^n x) + 2^{-n}\phi'(x)\cos(2^n x), \\ \partial_x^2 f_n &= -2^n\phi(x)\cos(2^n x) - 2\phi'(x)\sin(2^n x) + 2^{-n}\phi''(x)\cos(2^n x).\end{aligned}$$

Thus (16) and (17) are obvious. Also,

$$\begin{aligned}g_n\partial_x^2 f_n = &-\underbrace{\phi^2(x)\cos(2^n x)}_{=:I_1} - \underbrace{2^{1-n}\phi(x)\phi'(x)\sin(2^n x)}_{=:I_2} \\ &+ \underbrace{2^{-2n}\phi(x)\phi''(x)\cos(2^n x)}_{=:I_3}.\end{aligned}$$

Then

$$\|I_1\|_{L^\infty} \ge \phi^2(0),$$

$$\|I_2\|_{L^\infty} + \|I_3\|_{L^\infty} \le C2^{-n},$$

which implies that

$$\|g_n \partial_x^2 f_n\|_{L^\infty} \ge \phi^2(0) - C2^{-n}. \tag{19}$$

Obviously,

$$\|g_n \partial_x f_n\|_{L^\infty} + \|\partial_x g_n \partial_x f_n\|_{L^\infty} \le C2^{-n}.$$

Using the above estimates and the inverse triangle inequality, we obtain the desired (18) and finish the proof of Lemma 3.1. □

To obtain the non-uniformly continuous dependence property for the FW equation, we need to prove the following crucial proposition.

Proposition 3.2. *Assume that $u_0 \in \mathcal{S}$ and $\|\partial_x u_0\|_{L^\infty} \lesssim 1$. Then the data-to-solution map $u_0 \mapsto \mathbf{S}_t(u_0)$ of the Cauchy problem* (2) *satisfies that for $t \in (0, 1]$*

$$\left\|\mathbf{S}_t(u_0) - u_0 + \frac{3}{2} t u_0 \partial_x u_0\right\|_{C^1} \lesssim \mathbf{E}(u_0) t^2 + \|u_0\|_{L^\infty},$$

where we denote

$$\begin{aligned}\mathbf{E}(u_0) = &\|u_0\|_{L^\infty}\left(1 + \|u_0\|_{L^\infty}\|\partial_x^3 u_0\|_{L^\infty} + \|\partial_x^2 u_0\|_{L^\infty}\right) \\ &+ (1 + \|u_0\|_{L^\infty})\left(1 + \|u_0\|_{L^\infty}\|\partial_x^2 u_0\|_{L^\infty}\right).\end{aligned}$$

Proof. By the fundamental theorem of calculus in the time variable, from (2), we have

$$\begin{aligned}\left\|\mathbf{S}_t(u_0) - u_0 + \frac{3}{2} t u_0 \partial_x u_0\right\|_{C^1} \le &\int_0^t \left\|\partial_\tau \mathbf{S}_\tau(u_0) + \frac{3}{2} u_0 \partial_x u_0\right\|_{C^1} \mathrm{d}\tau \\ \le &\frac{3}{2}\int_0^t \|\mathbf{S}_\tau(u_0)\partial_x \mathbf{S}_\tau(u_0) - u_0\partial_x u_0\|_{C^1} \mathrm{d}\tau \\ &+ \int_0^t \left\|\partial_x\left(1 - \partial_x^2\right)^{-1}\mathbf{S}_\tau(u_0)\right\|_{C^1} \mathrm{d}\tau.\end{aligned} \tag{20}$$

Using the following estimates

$$\begin{aligned}&\|\mathbf{S}_\tau(u_0)\partial_x \mathbf{S}_\tau(u_0) - u_0\partial_x u_0\|_{C^1} \\ \lesssim &\|(\mathbf{S}_\tau(u_0) - u_0)\partial_x \mathbf{S}_\tau(u_0)\|_{C^1} + \|u_0 \partial_x(\mathbf{S}_\tau(u_0) - u_0)\|_{C^1} \\ \lesssim &\|\mathbf{S}_\tau(u_0) - u_0\|_{L^\infty}(\|\partial_x u_0\|_{L^\infty} + \|\partial_x^2 u_0\|_{L^\infty}) \\ &+ (\|u_0\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty})\|\partial_x(\mathbf{S}_\tau(u_0) - u_0)\|_{L^\infty} \\ &+ \|u_0\|_{L^\infty}\|\partial_x^2(\mathbf{S}_\tau(u_0) - u_0)\|_{L^\infty} \lesssim \tau \mathbf{E}(u_0)\end{aligned}$$

and

$$\left\|\partial_x\left(1 - \partial_x^2\right)^{-1}\mathbf{S}_\tau(u_0)\right\|_{C^1} \lesssim \|\partial_x G * \mathbf{S}_\tau(u_0)\|_{L^\infty} \lesssim \|\mathbf{S}_\tau(u_0)\|_{L^\infty} \lesssim \|u_0\|_{L^\infty},$$

then combining (20), we complete the proof of Proposition 3.2. □

With Proposition 3.2 in hand, we can prove Theorem 1.1.

We set $u_0^n = f_n + g_n$ and compare the solution $\mathbf{S}_t(u_0^n)$ and $\mathbf{S}_t(f_n)$. Obviously, we have

$$\|u_0^n - f_n\|_{C^1} = \|g_n\|_{C^1} \le C2^{-n} \quad \Rightarrow \quad \lim_{n\to\infty} \|u_0^n - f_n\|_{C^1} = 0.$$

Notice that

$$\begin{aligned}
&\mathbf{S}_t(\underbrace{f_n + g_n}_{=u_0^n}) = \underbrace{\mathbf{S}_t(u_0^n) - u_0^n + tu_0^n\partial_x u_0^n}_{=\mathbf{I}_1(u_0^n)} + f_n + g_n - tu_0^n\partial_x u_0^n,\\
&\mathbf{S}_t(f_n) = \underbrace{\mathbf{S}_t(f_n) - f_n + tf_n\partial_x f_n}_{=\mathbf{I}_2(f_n)} + f_n - tf_n\partial_x f_n \quad \text{and}\\
&u_0^n\partial_x u_0^n - f_n\partial_x f_n = g_n\partial_x f_n + u_0^n\partial_x g_n,
\end{aligned}$$

using the triangle inequality and Proposition 3.2, we deduce that

$$\begin{aligned}
&\|\mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n)\|_{C^1}\\
&= \|\mathbf{I}_1(u_0^n) - \mathbf{I}_2(f_n) + g_n - t\big(g_n\partial_x f_n + u_0^n\partial_x g_n\big)\|_{C^1}\\
&\ge t\|g_n\partial_x f_n\|_{C^1} - t\|u_0^n\partial_x g_n\|_{C^1} - \|\mathbf{I}_1(u_0^n)\|_{C^1} - \|\mathbf{I}_2(f_n)\|_{C^1} - \|g_n\|_{C^1}\\
&\ge t\|g_n\partial_x f_n\|_{C^1} - C2^{-n} - C\|\mathbf{I}_1(u_0^n)\|_{C^1} - C\|\mathbf{I}_2(f_n)\|_{C^1}, \qquad (21)
\end{aligned}$$

where we have used

$$\|u_0^n\partial_x g_n\|_{C^1} \lesssim \|u_0^n\|_{C^1}\|\partial_x g_n\|_{C^1} \lesssim 2^{-n}.$$

Using Proposition 3.2 with $u_0 = f_n$ and $u_0 = f_n + g_n$, respectively, and by Lemma 3.1, we obtain

$$\|\mathbf{I}_1(f_n + g_n)\|_{C^1} + \|\mathbf{I}_2(f_n)\|_{C^1} \lesssim t^2 + 2^{-n}. \qquad (22)$$

Then (21) reduces to

$$\liminf_{n\to\infty} \|\mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n)\|_{C^1} \ge t \liminf_{n\to\infty} \|g_n\partial_x f_n\|_{C^1} - Ct^2. \qquad (23)$$

Hence, it follows from (23) and Lemma 3.1 that

$$\liminf_{n\to\infty} \|\mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n)\|_{C^1} \gtrsim t \quad \text{for } t \text{ small enough.}$$

This completes the proof of Theorem 1.1. □

Finally, we present another proposition which can be directly lead to the non-uniformly continuous dependence property for the FW equation. We should mention that, the norm of solution in L^∞ can be bounded by the norm of initial data in L^∞ is the key ingredient of our analysis in Propositions 3.2 and 3.3.

Proposition 3.3. *Let f_n and g_n be given in Lemma 3.1. Then the difference between the data-to-solution maps $f_n + g_n \mapsto \mathbf{S}_t(f_n + g_n)$ and $f_n \mapsto \mathbf{S}_t(f_n)$ of the Cauchy problem (2) satisfies that for $n \gg 1$ and $0 < t \ll 1$*

$$\|\mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n)\|_{C^1} \approx t.$$

Proof. From now on, we set $\mathcal{X}_t := \mathbf{S}_t(f_n+g_n)-\mathbf{S}_t(f_n)-g_n$. By the fundamental theorem of calculus in the time variable, from (2), we have

$$
\begin{aligned}
\mathcal{X}_t &= \int_0^t \partial_\tau(\mathbf{S}_\tau(f_n+g_n)-\mathbf{S}_\tau(f_n))\mathrm{d}\tau \\
&= -\frac{3}{2}\int_0^t \mathbf{S}_\tau(f_n+g_n)\partial_x\mathbf{S}_\tau(f_n+g_n)-\mathbf{S}_\tau(f_n)\partial_x\mathbf{S}_\tau(f_n)\mathrm{d}\tau \\
&\quad + \int_0^t \mathbf{P}(\mathbf{S}_\tau(f_n+g_n)-\mathbf{S}_\tau(f_n))\mathrm{d}\tau \\
&= -\frac{3}{2}\int_0^t \mathcal{X}_\tau\partial_x\mathbf{S}_\tau(f_n+g_n)+\mathbf{S}_\tau(f_n)\partial_x\mathcal{X}_\tau\mathrm{d}\tau \\
&\quad - \frac{3}{2}\int_0^t g_n\partial_x\mathbf{S}_\tau(f_n+g_n)+\mathbf{S}_\tau(f_n)\partial_x g_n\mathrm{d}\tau \\
&\quad + \int_0^t \mathbf{P}(\mathbf{S}_\tau(f_n+g_n)-\mathbf{S}_\tau(f_n))\mathrm{d}\tau \qquad (24)\\
&= -\frac{3}{2}\int_0^t \mathcal{X}_\tau\partial_x\mathbf{S}_\tau(f_n+g_n)+\mathbf{S}_\tau(f_n)\partial_x\mathcal{X}_\tau\mathrm{d}\tau \\
&\quad - \frac{3}{2}g_n\int_0^t \partial_x(\mathbf{S}_\tau(f_n+g_n)-(f_n+g_n))\mathrm{d}\tau \\
&\quad - \frac{3}{2}\partial_x g_n\int_0^t \mathbf{S}_\tau(f_n)\mathrm{d}\tau - \frac{3}{2}tg_n\partial_x(f_n+g_n) \\
&\quad + \int_0^t \mathbf{P}(\mathbf{S}_\tau(f_n+g_n)-\mathbf{S}_\tau(f_n))\mathrm{d}\tau. \qquad (25)
\end{aligned}
$$

From (24), using the triangle inequality, one has

$$
\begin{aligned}
&2^n\|\mathcal{X}_t\|_{L^\infty}+\|\partial_x\mathcal{X}_t\|_{L^\infty}+2^{-n}\left\|\partial_x^2\mathcal{X}_t\right\|_{L^\infty} \\
&\lesssim \int_0^t \left(2^n\|\mathcal{X}_\tau\|_{L^\infty}+\|\partial_x\mathcal{X}_\tau\|_{L^\infty}+2^{-n}\left\|\partial_x^2\mathcal{X}_\tau\right\|_{L^\infty}\right)\mathrm{d}\tau+t,
\end{aligned}
$$

from which, we deduce that

$$
2^n\|\mathcal{X}_t\|_{L^\infty}+\|\partial_x\mathcal{X}_t\|_{L^\infty}+2^{-n}\left\|\partial_x^2\mathcal{X}_t\right\|_{L^\infty}\lesssim t. \qquad (26)
$$

From (25), using the inverse triangle inequality, one has

$$
\begin{aligned}
\|\partial_x\mathcal{X}_t\|_{L^\infty} &\geq \frac{3}{2}t\|\partial_x(g_n\partial_x f_n)\|_{L^\infty}-C2^{-n}t \\
&\quad - \int_0^t \left(2^n\|\mathcal{X}_\tau\|_{L^\infty}+\|\partial_x\mathcal{X}_\tau\|_{L^\infty}+2^{-n}\left\|\partial_x^2\mathcal{X}_\tau\right\|_{L^\infty}\right)\mathrm{d}\tau \\
&\quad - 2^{-n}\int_0^t \|\partial_x^2(\mathbf{S}_\tau(f_n+g_n)-(f_n+g_n))\|_{L^\infty}\mathrm{d}\tau \\
&\geq \frac{3}{2}t\|g_n\partial_x^2 f_n\|_{L^\infty}-C2^{-n}t-t^2.
\end{aligned}
$$

Thus, due to (26), (19), and (13), for $n \gg 1$ and $0 < t \ll 1$, we obtain

$$\|\partial_x \mathcal{X}_t\|_{L^\infty} \gtrsim t. \tag{27}$$

Combining (26) and (27) yields the proof of Proposition 3.3. □

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