Bull. Korean Math. Soc. **61** (2024), No. 3, pp. 813–823
https://doi.org/10.4134/BKMS.b230431
pISSN: 1015-8634 / eISSN: 2234-3016

2-LOCAL DERIVATIONS ON C*-ALGEBRAS

Wenbo Huang and Jiankui Li

Abstract. In this paper, we prove that every 2-local derivation on several classes of C*-algebras, such as unital properly infinite, type I or residually finite-dimensional C*-algebras, is a derivation. We show that the following statements are equivalent: (1) every 2-local derivation on a C*-algebra is a derivation, (2) every 2-local derivation on a unital primitive antiliminal and no properly infinite C*-algebra is a derivation. We also show that every 2-local derivation on a group C*-algebra $C^*(\mathbb{F})$ or a unital simple infinite-dimensional quasidiagonal C*-algebra, which is stable finite antiliminal C*-algebra, is a derivation.

1. Introduction

Throughout this paper, $\mathcal{A}$ is an algebra over the complex field $\mathbb{C}$. By an ideal we always mean a two-sided ideal unless otherwise specified. Recall that $\mathcal{A}$ is *prime* if for each $a, b \in \mathcal{A}$ the identity $a\mathcal{A}b = 0$ implies that $a = 0$ or $b = 0$. $\mathcal{A}$ is said to be *semiprime* if for each a in $\mathcal{A}$, $a\mathcal{A}a = 0$ implies $a = 0$. Obviously, every C*-algebra is semiprime.

A linear mapping D on $\mathcal{A}$ is called a *Jordan derivation* if $D(a^2) = D(a)a + aD(a)$ for each a in $\mathcal{A}$. In particular, if $D(ab) = D(a)b + aD(b)$ for each a, b in $\mathcal{A}$, then D is called a *derivation.* A classical result of Herstein [13] asserts that every Jordan derivation on a 2-torsion free prime ring is a derivation. Cusack [8] generalizes the above result to 2-torsion free semiprime rings.

The Gleason-Kahane-Żelazko theorem, a fundamental contribution in the theory of Banach algebras, in modern terminology, asserts that every unital local homomorphism from a Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is a homomorphism. Kadison [17] and Larson and Sourour [21] independently introduce the concept of local homomorphisms or local derivations. A classical result of Johnson [16] shows that every local derivation from a C*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is a derivation. After the Gleason-Kahane-Żelazko theorem was

Received July 26, 2023; Revised September 17, 2023; Accepted October 5, 2023.
2020 *Mathematics Subject Classification.* 47B47, 46L05.
Key words and phrases. C*-algebra, derivation, 2-local derivation.
The first author was partially supported by National Natural Science Foundation of China (Grant No. 12026252, 12026250). The second author was partially supported by National Natural Science Foundation of China (Grant No. 11871021).

established, Kowalski and Slodkowski [20] showed that at the cost of requiring the local behavior at two points, the condition of linearity can be dropped, that is, suppose that $\mathcal{A}$ is a unital Banach algebra and if $\phi : \mathcal{A} \to \mathbb{C}$ is a mapping (no linearity is assumed) having the property that $\phi(1_{\mathcal{A}}) = 1$ and for every $a, b \in \mathcal{A}$, there exists a homomorphism $\phi_{a,b} : \mathcal{A} \to \mathbb{C}$ such that $\phi_{a,b}(a) = \phi(a)$ and $\phi_{a,b}(b) = \phi(b)$, then ϕ is a homomorphism (cf. [19, 25]).

Motivated by the above ideas, Šemrl [25] introduces the concepts of 2-local homomorphisms and 2-local derivations. Recall that a mapping $\Delta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a 2-*local derivation* if, for every $a, b \in \mathcal{A}$, there exists a derivation $D_{a,b} : \mathcal{A} \to \mathcal{A}$ such that $D_{a,b}(a) = \Delta(a)$ and $D_{a,b}(b) = \Delta(b)$.

In [25], Šemrl shows that every 2-local derivation on $\mathcal{B}(\mathcal{H})$ is a derivation for an infinite-dimensional separable Hilbert space $\mathcal{H}$, and states the same result is true when $\mathcal{H}$ is finite-dimensional by a long proof involving tedious computations. Kim and Kim [18] give a short proof of the fact that every 2-local derivation on a finite-dimensional complex matrix algebra is a derivation. Ayupov and Kudaybergenov [2] extend this result to an arbitrary von Neumann algebra. Zhang and Li [26] construct an example of a 2-local derivation which is not a derivation on the algebra of all upper triangular complex 2×2 matrices. Let $\mathcal{M}$ be a commutative von Neumann algebra and $S(\mathcal{M})$ be the algebra of all measurable operators affiliated with $\mathcal{M}$. Ayupov, Kudaybergenov and Alauadinov [3] prove that $S(\mathcal{M})$ admits a 2-local derivation which is not a derivation if and only if the lattice $P(\mathcal{M})$ of projections in $\mathcal{M}$ is not atomic. For more information about this topic, we refer to [3, 4, 14, 15, 18, 25].

In 2016, Ayupov, Kudaybergenov and Peralta [4] presented an open problem: is every 2-local derivation on a C*-algebra a derivation? At the present time, there are few results in this topic. Kim and Kim [19] show that every continuous 2-local derivation on a unital approximately finite-dimensional (AF) C*-algebra is a derivation. He et al. [12] show that every 2-local derivation on $\mathbf{M}_n(\mathcal{A})$ ($n > 2$) is a derivation, where $\mathcal{A}$ is a unital Banach algebra. In the same paper, the authors also show that every 2-local derivation on a uniformly hyperfinite (UHF) C*-algebra is a derivation. However, there is no known example of a C*-algebra that admits a 2-local derivation which is not a derivation.

The aim of the paper is to devote the above topic, to be precise, we shall prove the following main results:

1. Every 2-local derivation on a unital properly infinite C*-algebra is a derivation.

2. Every 2-local derivation on a type I C*-algebra is a derivation.

3. The following statements are equivalent:

(1) every 2-local derivation on a C*-algebra is a derivation,

(2) every 2-local derivation on a unital primitive antiliminal and no properly infinite C*-algebra is a derivation.

4. Every 2-local derivation on a group C*-algebra $C^*(\mathbb{F})$ or a unital simple infinite-dimensional quasidiagonal C*-algebra, which is stable finite antiliminal C*-algebra, is a derivation.

2. Preliminaries

2.1. 2-local derivations and their properties

Lemma 2.1. *Let $\mathcal{I}$ be an ideal of an algebra $\mathcal{A}$ and $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local derivation. If $D(\mathcal{I}) \subset \mathcal{I}$ for every derivation D on $\mathcal{A}$, then $\Delta(\mathcal{I}) \subset \mathcal{I}$.*

Proof. By assumption, $\Delta(a) = D_{a,a}(a) \in \mathcal{I}$ for each $a \in \mathcal{I}$. □

Let $\mathcal{I}$ be a closed ideal of a C*-algebra $\mathcal{A}$. For every derivation D on $\mathcal{A}$, it is well known that $D(\mathcal{I}) \subset \mathcal{I}$, so $\Delta(\mathcal{I}) \subset \mathcal{I}$ for every 2-local derivation on $\mathcal{A}$.

Let $\mathcal{I}_0$ be a nonzero ideal of $\mathcal{A}$. Suppose that $D(\mathcal{I}_0) \subset \mathcal{I}_0$ for every derivation D on $\mathcal{A}$. Denote by $a \mapsto \langle a \rangle$ the canonical map of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}_0$. Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local derivation. By Lemma 2.1, then $\Delta(\mathcal{I}_0) \subset \mathcal{I}_0$. In this case, setting

$$\Delta_0(\langle a \rangle) = \langle \Delta(a) \rangle, \ \langle a \rangle \in \mathcal{A}/\mathcal{I}_0. \tag{1}$$

If $\langle a \rangle = \langle b \rangle$, then $a - b \in \mathcal{I}_0$, by definition of 2-local derivations, there is a derivation $D_{a,b} : \mathcal{A} \to \mathcal{A}$ such that $D_{a,b}(a) = \Delta(a)$ and $D_{a,b}(b) = \Delta(b)$, then $\Delta(a) - \Delta(b) = D_{a,b}(a) - D_{a,b}(b) = D_{a,b}(a - b) \in \mathcal{I}_0$, this means that $\langle \Delta(a) \rangle = \langle \Delta(b) \rangle$. Thus Δ_0 is well-defined. In particular, if Δ is a derivation on $\mathcal{A}$, it is straightforward to verify that Δ_0 is a derivation on $\mathcal{A}/\mathcal{I}_0$.

Lemma 2.2. *Let $\mathcal{I}_0$ be a nonzero ideal of $\mathcal{A}$ and $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local derivation. If $D(\mathcal{I}_0) \subset \mathcal{I}_0$ for every derivation D on $\mathcal{A}$, then $\Delta_0 : \mathcal{A}/\mathcal{I}_0 \to \mathcal{A}/\mathcal{I}_0$ is a 2-local derivation.*

Proof. For any $\langle x \rangle, \langle y \rangle \in \mathcal{A}/\mathcal{I}_0$, fix elements $a \in \langle x \rangle$ and $b \in \langle y \rangle$, respectively. Since $\Delta : \mathcal{A} \to \mathcal{A}$ is a 2-local derivation, there exists a derivation $D : \mathcal{A} \to \mathcal{A}$, depending on a and b, such that $D(a) = \Delta(a)$ and $D(b) = \Delta(b)$. By Eq. (1), D determines a derivation $D_0 : \mathcal{A}/\mathcal{I}_0 \to \mathcal{A}/\mathcal{I}_0$. It is easy to verify that $\Delta_0(\langle x \rangle) = D_0(\langle x \rangle)$ and $\Delta_0(\langle y \rangle) = D_0(\langle y \rangle)$. Thus $\Delta_0 : \mathcal{A}/\mathcal{I}_0 \to \mathcal{A}/\mathcal{I}_0$ is a 2-local derivation. The proof is complete. □

Theorem 2.3. *Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local derivation. Suppose that there is a family of nonzero ideals $\{\mathcal{I}_\lambda : \lambda \in \Lambda\}$ in $\mathcal{A}$ satisfying the following conditions:*

(a) $\bigcap_{\lambda \in \Lambda} \mathcal{I}_\lambda = \{0\}$,

(b) $D(\mathcal{I}_\lambda) \subseteq \mathcal{I}_\lambda$, $\lambda \in \Lambda$, *for every derivation D on $\mathcal{A}$,*

(c) *every 2-local derivation $\Delta_\lambda : \mathcal{A}/\mathcal{I}_\lambda \to \mathcal{A}/\mathcal{I}_\lambda$, $\lambda \in \Lambda$, is a derivation.*

Then $\Delta : \mathcal{A} \to \mathcal{A}$ is a Jordan derivation. Moreover, if $\mathcal{A}$ is semiprime, then Δ is a derivation.

Proof. If $\Delta : \mathcal{A} \to \mathcal{A}$ is a 2-local derivation, it easily follows that Δ is homogenous and $\Delta(a^2) = \Delta(a)a + a\Delta(a)$ for every $a \in \mathcal{A}$. This implies that every additive 2-local derivation is a Jordan derivation. Thus to prove Δ is a Jordan derivation, it suffices to show that Δ is additive.

Let $a, b \in \mathcal{A}$ for each $\lambda \in \Lambda$, by condition (c), $\Delta_\lambda : \mathcal{A}/\mathcal{I}_\lambda \to \mathcal{A}/\mathcal{I}_\lambda$ is a derivation, we have

$$\begin{aligned}\langle 0\rangle_\lambda &= \Delta_\lambda(\langle a\rangle_\lambda + \langle b\rangle_\lambda) - \Delta_\lambda(\langle a\rangle_\lambda) - \Delta_\lambda(\langle b\rangle_\lambda)\\ &= \Delta_\lambda(\langle a+b\rangle_\lambda) - \Delta_\lambda(\langle a\rangle_\lambda) - \Delta_\lambda(\langle b\rangle_\lambda)\\ &= \langle \Delta(a+b)\rangle_\lambda - \langle \Delta(a)\rangle_\lambda - \langle \Delta(b)\rangle_\lambda\\ &= \langle \Delta(a+b) - \Delta(a) - \Delta(b)\rangle_\lambda.\end{aligned}$$

Thus $\Delta(a+b) - \Delta(a) - \Delta(b) \in \mathcal{I}_\lambda$ for each $\lambda \in \Lambda$. Using condition (a), we have $\Delta(a+b)-\Delta(a)-\Delta(b) \in \bigcap \mathcal{I}_\lambda = \{0\}$. This implies that $\Delta(a+b) = \Delta(a)+\Delta(b)$. Therefore Δ is a Jordan derivation. In particular, if $\mathcal{A}$ is semiprime, it follows from [8, Corollary 5] that Δ is a derivation. The proof is complete. □

2.2. C*-algebras

We denote by $\sim$ the usual Murray-von Neumann equivalence relation on the set of projections in a C*-algebra. A nonzero projection p is *finite* if $p \sim q \leq p$ implies that $q = p$, p is *properly infinite* if there are mutually orthogonal subprojections p_1, p_2 of p such that $p_1 \sim p \sim p_2$. A unital C*-algebra $\mathcal{A}$ is finite (properly infinite) if $1_{\mathcal{A}}$ is finite (properly infinite). The Calkin algebra and every Cuntz algebra are properly infinite.

A unital C*-algebra $\mathcal{A}$ is *stably finite* if $\mathbf{M}_n(\mathcal{A})$ is finite for all n. If $\mathcal{A}$ is nonunital, it is called stably finite if its unitization $\widetilde{\mathcal{A}}$ is stably finite. Obviously, every C*-algebra with a separating family of tracial states is stably finite. Every AF algebra is stably finite.

A C*-algebra $\mathcal{A}$ is *residually finite-dimensional* (RFD) if it has a separating family of finite-dimensional representations. Every RFD C*-algebra is stably finite.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bound linear operators of $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ the set of all compact operators in $\mathcal{B}(\mathcal{H})$. A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be *irreducible* if $\pi(\mathcal{A})$ has no nontrivial invariant subspace. A C*-algebra is called *primitive* if it has a faithful irreducible representation. It is easy to verify that every primitive C*-algebra is prime, and for separable algebras the converse is also true (cf. [22]).

A C*-algebra $\mathcal{A}$ is *quasidiagonal* (QD) if there exists a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{A})$ is a quasidiagonal C*-algebra of operators. AF algebras, irrational rotation algebras and RFD C*-algebras are QD. Every QD C*-algebra is stably finite. For more information about this topic, we refer to [6].

Suppose that a C*-algebra $\mathcal{A}$ has a minimal left ideal $\mathcal{J}$, or equivalently, there is a minimal projection $p \in \mathcal{A}$ such that $\mathcal{J} = \mathcal{A}p$. Then the sum of all minimal left ideals is called the *socle* of $\mathcal{A}$, which we denote by soc($\mathcal{A}$). If $\mathcal{A}$ does not has minimal left ideal, we define soc($\mathcal{A}$) = 0. It is well known that soc($\mathcal{A}$) is an ideal of $\mathcal{A}$. For example, the socle of $\mathcal{B}(\mathcal{H})$ is the ideal of all finite rank operators (cf. [1, 9]).

A C*-algebra $\mathcal{A}$ is called *exact* if $\mathcal{A}$ has the property that minimal tensor product with $\mathcal{A}$ is an exact functor. Every nuclear C*-algebra is exact, but the converse is in general not true.

Recall that a C*-algebra $\mathcal{A}$ is said to be *type* I if $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) \neq \{0\}$ for every irreducible representation π of $\mathcal{A}$. This implies that soc$(\pi(\mathcal{A})) \neq 0$ for every irreducible representation π of $\mathcal{A}$. A type I C*-algebra needs not be finite, the Toeplitz algebra is a counterexample. Any type I C*-algebra cannot be properly infinite. Every type I C*-algebra is nuclear.

Recall that a C*-algebra $\mathcal{A}$ is said to be *antiliminal* if no nonzero positive element in $\mathcal{A}$ generates an abelian hereditary C*-subalgebra. Equivalently, the largest postliminal ideal in $\mathcal{A}$ is zero. It is well known that the Calkin algebra is antiliminal.

For more general information we refer to [5, 11, 24].

2.3. Antiliminal C*-algebras

Proposition 2.4. *A prime* C**-algebra $\mathcal{A}$ is antiliminal if and only if soc*$(\mathcal{A}) = 0$.

Proof. The conclusion follows from [23, Proposition 2.3]. □

Let $\mathbb{F}_2$ be a free group on two generators and π be the universal unitary representation of $\mathbb{F}_2$ on a Hilbert space $\mathcal{H}$. We denote by $C^*(\mathbb{F}_2)$ the full group C*-algebra in $\mathcal{B}(\mathcal{H})$ generated by the set $\{\pi(g) : g \in \mathbb{F}_2\}$. Choi [7] shows that $C^*(\mathbb{F}_2)$ is a primitive RFD C*-algebra without nontrivial projection. In addition, all properties mentioned above about $C^*(\mathbb{F}_2)$ can be extended to $C^*(\mathbb{F})$, where $\mathbb{F}$ is any free group.

Proposition 2.5. $C^*(\mathbb{F})$ *is an antiliminal* C**-algebra.*

Proof. By [7, Theorem 1], $C^*(\mathbb{F})$ has no nontrivial projection. Thus soc$(C^*(\mathbb{F}))$ $= 0$. It follows from Proposition 2.4 that $C^*(\mathbb{F})$ is antiliminal. □

Proposition 2.6. *A primitive* C**-algebra $\mathcal{A}$ is antiliminal if and only if $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$ for any faithful irreducible representation π of $\mathcal{A}$.*

Proof. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ be a faithful irreducible representation. Suppose that $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$. If soc$(\pi(\mathcal{A})) \neq 0$, then there is a minimal projection $p \in soc(\pi(\mathcal{A}))$, this means that $p \in \mathcal{K}(\mathcal{H}_\pi)$, which contradicts $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$. Thus soc$(\pi(\mathcal{A})) = 0$. It follows from Proposition 2.4 that $\pi(\mathcal{A})$ is antiliminal.

Conversely, if $\pi(\mathcal{A})$ is antiliminal, we assume that $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) \neq \{0\}$. By [24, Theorem 6.1.5] or [11, Corollary 4.1.10], $\pi(\mathcal{A}) \supset \mathcal{K}(\mathcal{H}_\pi)$, this means that soc$(\pi(\mathcal{A})) \neq 0$, which contradicts Proposition 2.4. Thus $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$. Therefore $\pi(\mathcal{A})$ is antiliminal if and only if $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}_\pi) = \{0\}$.

Since π is an isomorphism, this implies that p is a minimal projection in $\mathcal{A}$ if and only if $\pi(p)$ is a minimal projection in $\pi(\mathcal{A})$. This is to say, $soc(\mathcal{A}) = 0$ if and only if $soc(\pi(\mathcal{A})) = 0$. It follows from Proposition 2.4 that $\mathcal{A}$ is antiliminal if and only if $\pi(\mathcal{A})$ is antiliminal. By the above claim, the conclusion is proved. □

Pedersen shows that every UHF algebra is antiliminal ([24, Theorem 6.5.7]). We have the following corollary.

Corollary 2.7. *Every unital simple infinite-dimensional* C*-*algebra is antiliminal.*

Proof. Let $\mathcal{A}$ be a unital simple infinite dimensional C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a nontrivial irreducible representation. By assumption, π is faithful and $\mathcal{H}$ is infinite-dimensional. Suppose that $\mathcal{A}$ is not antiliminal. Proposition 2.6 implies that $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$. By [24, Theorem 6.1.5], $\pi(\mathcal{A}) \supset \mathcal{K}(\mathcal{H})$. However, as $\pi(\mathcal{A})$ is simple, this means that $\pi(\mathcal{A}) = \mathcal{K}(\mathcal{H})$, which contradicts that $\pi(\mathcal{A})$ is unital. Thus $\mathcal{A}$ is antiliminal. The proof is complete. □

3. Main results

Let $\mathcal{A}$ be a C*-algebra without unit and $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ be the unitization of $\mathcal{A}$.

Lemma 3.1. *The following statements are equivalent*:

(a) *every* 2-*local derivation on* $\mathcal{A}$ *is a derivation,*

(b) *every* 2-*local derivation on* $\widetilde{\mathcal{A}}$ *is a derivation.*

Proof. (a)⟹(b): Let $\Delta : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ be a 2-local derivation. Then $\Delta(\lambda) = \lambda\Delta(1) = 0$ for each $\lambda \in \mathbb{C}$. Thus

$$\Delta(a+\lambda) = D_{a+\lambda,a}(a+\lambda) = D_{a+\lambda,a}(a) + D_{a+\lambda,a}(\lambda) = D_{a+\lambda,a}(a) = \Delta(a),$$

i.e.,

$$\Delta(a+\lambda) = \Delta(a).$$

This means that the restriction $\Delta|_{\mathcal{A}}$ is a 2-local derivation. By assumption, $\Delta|_{\mathcal{A}}$ is a derivation. By the above equation, it is easily verified that Δ is a derivation on $\widetilde{\mathcal{A}}$.

(b)⟹(a): Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local derivation. We define $\widetilde{\Delta}$ on $\widetilde{\mathcal{A}}$ by $\widetilde{\Delta}(a+\lambda) = \Delta(a)$ for any $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. In particular, if D is a derivation on $\mathcal{A}$, then it is straightforward to check that $\widetilde{D}$ is a derivation on $\widetilde{\mathcal{A}}$.

Firstly, we show that $\widetilde{\Delta}$ is a 2-local derivation. For any $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$, as $\widetilde{\Delta}(a+\lambda) = \Delta(a)$ and $\widetilde{\Delta}(b+\mu) = \Delta(b)$, there exists a derivation $D_{a,b}$ on $\mathcal{A}$ such that $\Delta(a) = D_{a,b}(a)$ and $\Delta(b) = D_{a,b}(b)$. It follows that $\widetilde{\Delta}(a+\lambda) = \Delta(a) = D_{a,b}(a) = \widetilde{D}_{a,b}(a+\lambda)$ and $\widetilde{\Delta}(b+\mu) = \Delta(b) = D_{a,b}(b) = \widetilde{D}_{a,b}(b+\mu)$. This implies that $\widetilde{\Delta}$ is a 2-local derivation on $\widetilde{\mathcal{A}}$. By assumption, $\widetilde{\Delta}$ is a derivation. By the definition of $\widetilde{\Delta}$, we have $\Delta = \widetilde{\Delta}|_{\mathcal{A}}$, therefore $\Delta : \mathcal{A} \to \mathcal{A}$ is a derivation. □

Theorem 3.2. *Let $\mathcal{A}$ be a* C*-*algebra. Then every* 2*-local derivation* $\Delta : \mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathcal{A})$, $n \geq 3$, *is a derivation. In particular, if $\mathcal{A}$ is commutative, then every* 2*-local derivation* $\Delta : \mathbf{M}_n(\mathcal{A}) \to \mathbf{M}_n(\mathcal{A})$, $n \geq 2$, *is a derivation.*

Proof. The conclusion follows from Lemma 3.1, [12, Corollary 2.17] and [15, Corollary 3.7]. □

Theorem 3.3. *Let $\mathcal{A}$ be a unital properly infinite* C*-*algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. Take mutually orthogonal projections p_1, p_2 in $\mathcal{A}$ such that $p_1 \sim 1_{\mathcal{A}} \sim p_2$. Since p_1 is properly infinite too, there exist mutually orthogonal subprojections p_3, p_4 of p_1 such that $p_3 \sim p_1 \sim p_4$. Put $p = p_2 + p_3 + p_4$, then $1_{\mathcal{A}} \sim p$ and p_2, p_3, p_4 are mutually orthogonal equivalent subprojections of p. This implies that

$$\mathcal{A} \cong p\mathcal{A}p \cong \mathbf{M}_3(p_2\mathcal{A}p_2).$$

The result follows from Theorem 3.2. □

As a direct application of Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let $\mathcal{A}$ be a Calkin algebra or Cuntz algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Theorem 3.5. *Let $\mathcal{A}$ be a* C*-*algebra and soc($\mathcal{A}$) be an essential ideal of $\mathcal{A}$. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. The conclusion follows from [14, Theorem 3.6]. □

Theorem 3.6. *Let $\mathcal{A}$ be a prime and not antiliminal* C*-*algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. By assumption, it follows from Proposition 2.4 that soc($\mathcal{A}$) $\neq 0$. In addition, $\mathcal{A}$ is prime implies that soc($\mathcal{A}$) is essential. The conclusion follows from Theorem 3.5. □

Theorem 3.7. *Let $\mathcal{A}$ be a* C*-*algebra with a separating family of irreducible representations* $\{\pi_\lambda\}$. *For each* λ, *if* $\pi_\lambda(\mathcal{A})$ ($\pi_\lambda(\widetilde{\mathcal{A}})$, *if $\mathcal{A}$ is nonunital*) *is unital properly infinite or not antiliminal, then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. By Lemma 3.1, it is sufficient to consider the case when $\mathcal{A}$ is a unital C^*-algebra. Let $\Delta : \pi_\lambda(\mathcal{A}) \to \pi_\lambda(\mathcal{A})$ be a 2-local derivation. By Theorem 3.3 or Theorem 3.6, Δ is a derivation on $\pi_\lambda(\mathcal{A})$ for every λ. By Theorem 2.3, the result holds. □

As a direct application of Theorem 3.7, we have the following corollary.

Corollary 3.8. *Suppose that $\mathcal{A}$ is a* C*-*algebra of type* I. *Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Theorem 3.9. *The following statements are equivalent*:

(a) *every* 2*-local derivation on any* C*-*algebra is a derivation,*
(b) *every* 2*-local derivation on any unital* C*-*algebra is a derivation,*
(c) *every* 2*-local derivation on any unital primitive antiliminal no properly infinite* C*-*algebra is a derivation.*

Proof. By Lemma 3.1, (a)$\Longleftrightarrow$(b). Obviously, (b)$\Longrightarrow$(c).

(c)$\Longrightarrow$(b): Let $\mathcal{A}$ be a unital C*-algebra. We denote by $\hat{\mathcal{A}}$ the set of all nontrivial irreducible representations of $\mathcal{A}$. We denote $E = \{\pi(\mathcal{A}) : \pi \in \hat{\mathcal{A}}\}$, $F = \{\pi(\mathcal{A}) \text{ is not antiliminal} : \pi \in \hat{\mathcal{A}}\}$, $G = \{\pi(\mathcal{A}) \text{ is properly infinite} : \pi \in \hat{\mathcal{A}}\}$ and $H = \{\pi(\mathcal{A}) \text{ is antiliminal and is not properly infinite} : \pi \in \hat{\mathcal{A}}\}$. Then $E = F \cup G \cup H$. If $\pi(\mathcal{A}) \in F$, by Theorem 3.6, every 2-local derivation on $\pi(\mathcal{A})$ is a derivation. If $\pi(\mathcal{A}) \in G$, it follows from Theorem 3.3 that every 2-local derivation on $\pi(\mathcal{A})$ is a derivation. If $\pi(\mathcal{A}) \in H$, by assumption, every 2-local derivation on $\pi(\mathcal{A})$ is a derivation. By Theorem 2.3, every 2-local derivation on $\mathcal{A}$ is a derivation. The proof is complete. □

Theorem 3.10. *Let $\mathcal{A}$ be an* RFD C*-*algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. If π is a finite-dimensional representation of $\mathcal{A}$, then $\pi(\mathcal{A})$ is a finite-dimensional C^*-algebra. Thus $soc(\pi(\mathcal{A})) = \pi(\mathcal{A})$. By Theorem 3.5, every 2-local derivation on $\pi(\mathcal{A})$ is a derivation. By Theorems 2.3, every 2-local derivation on $\mathcal{A}$ is a derivation. □

Corollary 3.11. *Let $\mathbb{F}$ be any free group. Then every* 2*-local derivation on the group* C*-*algebra $C^*(\mathbb{F})$ is a derivation.*

Proof. Since $C^*(\mathbb{F})$ is an RFD C^*-algebra, the conclusion follows from Theorem 3.10. □

Theorem 3.12. *Let $\mathcal{A}$ be a unital* C*-*algebra with a separating family of tracial states $\{\tau_\lambda\}$. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. Let τ be a tracial state on $\mathcal{A}$. Then there are a Hilbert space $\mathcal{H}$, a unit vector $\xi \in \mathcal{H}$, and a representation $\pi_\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi_\tau(\mathcal{A})\xi$ is dense in $\mathcal{H}$ and $\tau(a) = (\pi_\tau(a)\xi, \xi)$ for every $a \in \mathcal{A}$. We define

$$\hat{\tau} : \pi_\tau(\mathcal{A})'' \to \mathbb{C} \text{ by } \hat{\tau}(a) = (a\xi, \xi).$$

Then $\hat{\tau}$ is a faithful normal trace on $\pi_\tau(\mathcal{A})''$.

Let $\Delta : \pi_\tau(\mathcal{A}) \to \pi_\tau(\mathcal{A})$ be a 2-local derivation. For any $a, b \in \pi_\tau(\mathcal{A})$, there exists a derivation $D_{a,b}$ on $\pi_\tau(\mathcal{A})$ such that $\Delta(a) = D_{a,b}(a)$ and $\Delta(b) = D_{a,b}(b)$. By [10, Theorem 10.6], $D_{a,b}$ is weak*-continuous. This implies that there is an element $m \in \pi_\tau(\mathcal{A})''$ such that $D_{a,b}(ab) = mab - abm$. Thus $\hat{\tau}(D_{a,b}(ab)) = 0$. We have $0 = \hat{\tau}(D_{a,b}(ab)) = \hat{\tau}(D_{a,b}(a)b + aD_{a,b}(b)) = \hat{\tau}(\Delta(a)b + a\Delta(b))$, i.e.,

$$\hat{\tau}(\Delta(a)b) = -\hat{\tau}(a\Delta(b)).$$

Using the above equation, we obtain $\hat{\tau}((\Delta(a+b)-\Delta(a)-\Delta(b))c)=0$ for any $c\in\pi_\tau(\mathcal{A})$. Put $c=(\Delta(a+b)-\Delta(a)-\Delta(b))^*$. Since $\hat{\tau}$ is faithful, we have

$$\Delta(a+b)-\Delta(a)-\Delta(b)=0.$$

So Δ is additive. Therefore Δ is a derivation on $\pi_\tau(\mathcal{A})$. By assumption, $\bigcap ker\pi_{\tau_\lambda}=\{0\}$. It follows from Theorem 2.3 that every 2-local derivation on $\mathcal{A}$ is a derivation. The proof is complete. □

Corollary 3.13. *Let $\mathcal{A}$ be a unital simple stably finite exact* C*-*algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. [5, Corollary V. 2.1.16] implies that $\mathcal{A}$ has a faithful tracial state. The result follows from Theorem 3.12. □

Corollary 3.14. *Let $\mathcal{A}$ be a unital simple* QD C*-*algebra. Then every* 2*-local derivation on $\mathcal{A}$ is a derivation.*

Proof. By [5, Proposition V. 4.2.7], $\mathcal{A}$ has a faithful tracial state. The result follows from Theorem 3.12. □

Remark 3.15. There is a separable simple unital C*-algebra which is QD but not exact (even nuclear). However, there is no known example of a stably finite nuclear C*-algebra which is not QD (cf. [5, pp. 460–463] or [6]).

References

[1] B. H. Aupetit and H. T. Mouton, *Trace and determinant in Banach algebras*, Studia Math. **121** (1996), no. 2, 115–136.

[2] S. Ayupov and K. Kudaybergenov, 2*-local derivations on von Neumann algebras*, Positivity **19** (2015), no. 3, 445–455. https://doi.org/10.1007/s11117-014-0307-3

[3] S. Ayupov, K. Kudaybergenov, and A. Alauadinov, 2*-local derivations on matrix algebras over commutative regular algebras*, Linear Algebra Appl. **439** (2013), no. 5, 1294–1311. https://doi.org/10.1016/j.laa.2013.04.013

[4] S. Ayupov, K. Kudaybergenov, and A. M. Pereira, *A survey on local and* 2*-local derivations on* C*- *and von Neumann algebras*, in Topics in functional analysis and algebra, 73–126, Contemp. Math., 672, Amer. Math. Soc., Providence, RI, 2016. https://doi.org/10.1090/conm/672/13462

[5] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, 122, Springer, Berlin, 2006. https://doi.org/10.1007/3-540-28517-2

[6] N. P. Brown, *On quasidiagonal C*-algebras*, in Operator algebras and applications, 19–64, Adv. Stud. Pure Math., 38, Math. Soc. Japan, Tokyo, 2004. https://doi.org/10.2969/aspm/03810019

[7] M. D. Choi, *The full C*-algebra of the free group on two generators*, Pacific J. Math. **87** (1980), no. 1, 41–48. http://projecteuclid.org/euclid.pjm/1102780313

[8] J. M. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 321–324. https://doi.org/10.2307/2040004

[9] L. Dalla, S. Giotopoulos, and N. Katseli, *The socle and finite-dimensionality of a semiprime Banach algebra*, Studia Math. **92** (1989), no. 2, 201–204. https://doi.org/10.4064/sm-92-2-201-204

[10] K. R. Davidson, *Nest algebras*, Pitman Research Notes in Mathematics Series, 191, Longman Sci. Tech., Harlow, 1988.

[11] J. Dixmier, C^*-*algebras*, translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15, North-Holland, Amsterdam, 1977.

[12] J. He, J. Li, G. An, and W. Huang, *Characterizations of* 2*-local derivations and local Lie derivations on some algebras*, Sib. Math. J. **59** (2018), no. 4, 721–730; translated from Sibirsk. Mat. Zh. **59** (2018), no. 4, 912–926. https://doi.org/10.1134/s0037446618040146

[13] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110. https://doi.org/10.2307/2032688

[14] W. Huang and J. K. Li, 2*-local derivations on semisimple Banach algebras with minimal left ideals*, J. Aust. Math. Soc. **110** (2021), no. 3, 321–332. https://doi.org/10.1017/S144678872000021X

[15] W. Huang, J. K. Li, and W. H. Qian, *Derivations and* 2*-local derivations on matrix algebras and algebras of locally measurable operators*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 1, 227–240. https://doi.org/10.1007/s40840-018-0675-0

[16] B. E. Johnson, *Local derivations on* C^**-algebras are derivations*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 313–325. https://doi.org/10.1090/S0002-9947-00-02688-X

[17] R. V. Kadison, *Local derivations*, J. Algebra **130** (1990), no. 2, 494–509. https://doi.org/10.1016/0021-8693(90)90095-6

[18] S. O. Kim and J. S. Kim, *Local automorphisms and derivations on* $\mathbb{M}_n$, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1389–1392. https://doi.org/10.1090/S0002-9939-03-07171-5

[19] S. O. Kim and J. S. Kim, *Local automorphisms and derivations on certain* C^**-algebras*, Proc. Amer. Math. Soc. **133** (2005), no. 11, 3303–3307. https://doi.org/10.1090/S0002-9939-05-08059-7

[20] S. Kowalski and Z. Slodkowski, *A characterization of multiplicative linear functionals in Banach algebras*, Studia Math. **67** (1980), no. 3, 215–223. https://doi.org/10.4064/sm-67-3-215-223

[21] D. R. Larson and A. R. Sourour, *Local derivations and local automorphisms of* $\mathcal{B}(X)$, in Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), 187–194, Proc. Sympos. Pure Math., 51, Part 2, Amer. Math. Soc., Providence, RI, 1990. https://doi.org/10.1090/pspum/051.2/1077437

[22] B. Magajna, *The Haagerup norm on the tensor product of operator modules*, J. Funct. Anal. **129** (1995), no. 2, 325–348. https://doi.org/10.1006/jfan.1995.1053

[23] M. Mathieu, *Elementary operators on prime* C^**-algebras. II*, Glasgow Math. J. **30** (1988), no. 3, 275–284. https://doi.org/10.1017/S0017089500007369

[24] G. K. Pedersen, C^**-algebras and their automorphism groups*, second edition, Academic Press London, 2018.

[25] P. Šemrl, *Local automorphisms and derivations on* $\mathcal{B}(H)$, Proc. Amer. Math. Soc. **125** (1997), no. 9, 2677–2680. https://doi.org/10.1090/S0002-9939-97-04073-2

[26] J. H. Zhang and H. X. Li, 2*-local derivations on digraph algebras*, Acta Math. Sinica (Chinese Ser.) **49** (2006), no. 6, 1411–1416.

Wenbo Huang
School of Mathematics and Physics
Jiangsu University of Technology
Changzhou 213001, P. R. China
and
School of Mathematics
East China University of Science and Technology
Shanghai 200237, P. R. China
Email address: huangwenbo2015@126.com

Jiankui Li
School of Mathematics
East China University of Science and Technology
Shanghai 200237, P. R. China
Email address: jkli@ecust.edu.cn