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AN ALTERNATIVE PROOF FOR THE MINIMALITY OF STRONGLY QUASI-POSITIVE FIBERED KNOTS IN THE RIBBON CONCORDANCE POSET

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Abstract. Baker proved that any strongly quasi-positive fibered knot is minimal with respect to the ribbon concordance among fibered knots in the three-sphere. By applying Rapaport's conjecture, which has been solved by Kochloukova, we can check that any strongly quasi-positive fibered knot is minimal with respect to the ribbon concordance among all knots in the three-sphere. In this short note, we give an alternative proof for the fact by utilizing the knot Floer homology.

1. Introduction and result

Let $\mathcal{K}$ be the set of oriented knots in $\mathbf{S}^3$. For $K_0, K_1 \in \mathcal{K}$, an annulus $C \subset \mathbf{S}^3 \times [0,1]$ is a *concordance* from K_1 to K_0 if $\partial(\mathbf{S}^3 \times [0,1], C) = (\mathbf{S}^3 \times \{1\}, K_1) \cup (-\mathbf{S}^3 \times \{0\}, -K_0)$. We write $K_1 \sim K_0$ if there is a concordance from K_1 to K_0. A concordance is *ribbon* if it has no local maxima with respect to the projection $\mathbf{S}^3 \times [0,1] \to [0,1]$. We denote $K_1 \geq K_0$ or $K_0 \leq K_1$ if there is a ribbon concordance from K_1 to K_0.

Gordon [5] conjectured that the ribbon concordance forms a partial order. Agol [2] solved Gordon's conjecture affirmatively. We denote the partially ordered set by $(\mathcal{K}, \geq)$.

Baker [3, Lemma 2] showed that any strongly quasi-positive fibered knot is minimal in the subposet $(\mathcal{F}, \geq) \subset (\mathcal{K}, \geq)$, where $\mathcal{F}$ is the set of fibered knots in $\mathbf{S}^3$. Moreover, Silver [17] observed that the solution of Rapaport's conjecture, which has been solved affirmatively in [8], implies that if $K_1 \geq K_0$ and K_1 is fibered, then K_0 is also fibered (see also [13, Proposition 5]). Hence, we obtain the following corollary.

Corollary 1.1. *Any strongly quasi-positive fibered knot is minimal in* $(\mathcal{K}, \geq)$.

In this short note, we introduce an alternative proof for Corollary 1.1.

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An alternative proof for Corollary 1.1. [1] Let K be a strongly quasi-positive fibered knot. Suppose that $K' \leq K$. As mentioned above, K is minimal in $(\mathcal{F}, \geq)$ ([3, Lemma 2]). Hence, if we can prove that K' is fibered, we see $K' = K$ and K is minimal.

Let us prove that K' is also fibered. Since K is strongly quasi-positive, we have $2g_3(K) = s(K)$, where $g_3(K)$ is the genus of K and s is the Rasmussen invariant (for example see [16]). Now $K' \leq K$, in particular $K' \sim K$. Since it is known that s is a concordance invariant and satisfies $s \leq 2g_3$, we obtain

$$2g_3(K) = s(K) = s(K') \leq 2g_3(K').$$

Let C be a ribbon concordance from K to K'. Zemke [19, Theorem 1.1] proved that the grading preserving map F_C on knot Floer homologies derived from a ribbon concordance $K' \leq K$, which is constructed by Juhász and Marengon [7], induces an injection

$$F_C \colon \widehat{HFK}_i(K', a) \to \widehat{HFK}_i(K, a) \tag{1}$$

for all $i, a \in \mathbf{Z}$, where $\widehat{HFK}_i(K, a)$ is the hat version of the knot Floer homology of K (with $\mathbf{F}_2$-coefficient). The subscription i is the Maslov (homological) grading and a is the Alexander grading. Denote $\bigoplus_{i \in \mathbf{Z}} \widehat{HFK}_i(K, a)$ by $\widehat{HFK}(K, a)$. Then, recall the following well-known results:

(a) $g_3(K) = \max\{a \in \mathbf{Z} \mid \widehat{HFK}(K, a) \neq 0\}$ ([15]),

(b) $\mathrm{rank}_{\mathbf{F}_2} \widehat{HFK}(K, g_3(K)) = 1$ if and only if K is fibered ([14]).

We remark that although the coefficient ring of the knot Floer homology appearing in [14,15] is $\mathbf{Z}$, the same statements are true for $\mathbf{F}_2$ due to the Universal Coefficient Theorem and the fact that the differential on knot Floer homology preserves the Alexander grading (for example see [10]). Hence, the injection (1) and (a) induce $g_3(K') \leq g_3(K)$, that is, $g_3(K') = g_3(K)$. Moreover, we have

$$\begin{aligned} 0 \neq \mathrm{rank}_{\mathbf{F}_2} \widehat{HFK}(K', g_3(K')) &= \mathrm{rank}_{\mathbf{F}_2} \widehat{HFK}(K', g_3(K)) \\ &\leq \mathrm{rank}_{\mathbf{F}_2} \widehat{HFK}(K, g_3(K)) = 1, \end{aligned}$$

where the last equality follows from the fiberedness of K. Again applying (b), we see K' is fibered. □

1.1. Minimality of 3_1

The minimality of 3_1 can be also proved by utilizing the trefoil and the unknot detections of the knot Floer homology. The following proof is inspired by [4].

Proposition 1.2. *The trefoil knot 3_1 is minimal in $(\mathcal{K}, \geq)$.*

[1] This proof was firstly given in the preprint arXiv:2210.04044 (the part written by the author), which has been withdrawn from arXiv by the corresponding author of the preprint.

Proof. Suppose that $K \leq 3_1$. Let us prove that $K = 3_1$. Let C be a ribbon concordance from 3_1 to K. By utilizing the same argument as (1), there is a grading preserving injection

$$F_C \colon \widehat{HFK}(K, a) \to \widehat{HFK}(3_1, a).$$

Here, the total dimension of $\bigoplus_{a \in \mathbf{Z}} \widehat{HFK}(3_1, a)$ is 3. Hence the total dimension of $\bigoplus_{a \in \mathbf{Z}} \widehat{HFK}(K, a)$ is 1 or 3. Remark that the total dimension of the knot Floer homology of a knot is odd because it is known that

$$\sum_{i, a \in \mathbf{Z}} (-1)^i \mathrm{rank}_{\mathbf{F}_2} \widehat{HFK}_i(K, a) t^a = \Delta_K(t),$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial (for example see [15]) and $\Delta_K(1) = 1$. If the total dimension is 1, then K is the unknot because of (a). This is a contradiction to $K \sim 3_1$. If the total dimension is 3, then $K = 3_1$ ([6, Corollary 8]). This is the desired case. □

2. Table of minimal knots with up to 9 crossings

Gordon [5, Lemma 3.4] proved that if a transfinitely nilpotent knot K_1 satisfies $K_1 \geq K_0$ and $\deg \Delta_{K_1}(t) = \deg \Delta_{K_0}(t)$, then $K_1 = K_0$. This result is helpful to determine the minimality of a transfinitely nilpotent knot in $(\mathcal{K}, \geq)$.

It is known that the following knots are transfinitely nilpotent (see also [18]):

- fibered knots,
- two-bridge knots [11] and
- pseudo-alternating knots (including alternating knots and positive knots) for which the leading coefficient of Alexander polynomial is a power of a prime [12].

For example, by using Gordon's method, the author [18] proved that 8_{15} is minimal. For completeness, we recall the proof. Suppose that $8_{15} \geq K$. Then, because of the sliceness of $8_{15}\#(-\overline{K})$, the Alexander polynomial $\Delta_{8_{15}\#(-\overline{K})}(t)$ satisfies

$$\Delta_{8_{15}}(t)\Delta_{-\overline{K}}(t) \doteq \Delta_{8_{15}\#(-\overline{K})}(t) \doteq f(t)f(t^{-1})$$

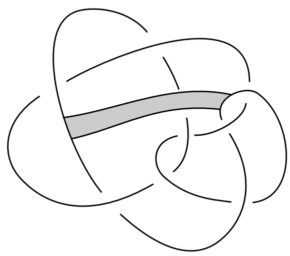
for some $f(t) \in \mathbf{Z}[t]$, where "$\doteq$" means "$=$" up to multiplication by a unit in $\mathbf{Z}[t, t^{-1}]$. Now, the Alexander polynomial $\Delta_{8_{15}}(t)$ has the irreducible decomposition $(t^{-1} - 1 + t)(3t^{-1} - 5 + 3t)$. Moreover, [5, Lemma 3.4] implies $\deg \Delta_K(t) \leq \deg \Delta_{8_{15}}(t)$. Hence, we see that $\Delta_K(t) = \Delta_{8_{15}}(t)$. As mentioned above, by applying [12, Theorem B], we can check that the commutator subgroup of the knot group of 8_{15} is transfinitely nilpotent. Then by utilizing [5, Lemma 3.4], we see that $K = 8_{15}$ and 8_{15} is minimal in $(\mathcal{K}, \geq)$. By applying similar discussions, we can check that all knots up to 9 crossings are minimal except for 6_1, 8_8, 8_9, 8_{10}, 8_{11}, 8_{18}, 9_{24}, 9_{27}, 9_{37}, 9_{40}, 9_{41}, and 9_{46}.

The knots 6_1, 8_8, 8_9, 9_{27}, 9_{41} and 9_{46} are ribbon and not minimal. We can directly check that $8_{10} \geq 3_1$, $9_{24} \geq 4_1$ and $9_{37} \geq 4_1$ (see Figures 1 and 2). In [1, Table 1], we have proved $8_{11} \geq 3_1$.

The knots 8_{18} and 9_{40} are minimal. In fact they are fibered and of genus 3, and Livingston [9] proved that their concordance genera are 3. The concordance genus of a knot K is the minimum of genera of knots concordant to K. Hence, if $8_{18} \geq K$, the knot K is a fibered knot of genus 3 (for checking the fiberedness see Section 1). This means $\deg \Delta_K(t) = 6 = \deg \Delta_{8_{18}}(t)$. By applying Gordon's method, we see that $K = 8_{18}$ and 8_{18} is minimal. Similarly, 9_{40} is also minimal. As a summary we obtain Table 1.

TABLE 1. Minimal prime knots in $(\mathcal{K}, \geq)$ up to 9 crossings. The check-mark ✓ means that the knot is minimal in $(\mathcal{K}, \geq)$.

U	✓	8_3	✓	8_{20}	$\geq U$	9_{16}	✓	9_{33}	✓
3_1	✓	8_4	✓	8_{21}	✓	9_{17}	✓	9_{34}	✓
4_1	✓	8_5	✓	9_1	✓	9_{18}	✓	9_{35}	✓
5_1	✓	8_6	✓	9_2	✓	9_{19}	✓	9_{36}	✓
5_2	✓	8_7	✓	9_3	✓	9_{20}	✓	9_{37}	$\geq 4_1$
6_1	$\geq U$	8_8	$\geq U$	9_4	✓	9_{21}	✓	9_{38}	✓
6_2	✓	8_9	$\geq U$	9_5	✓	9_{22}	✓	9_{39}	✓
6_3	✓	8_{10}	$\geq 3_1$	9_6	✓	9_{23}	✓	9_{40}	✓
7_1	✓	8_{11}	$\geq 3_1$	9_7	✓	9_{24}	$\geq 4_1$	9_{41}	$\geq U$
7_2	✓	8_{12}	✓	9_8	✓	9_{25}	✓	9_{42}	✓
7_3	✓	8_{13}	✓	9_9	✓	9_{26}	✓	9_{43}	✓
7_4	✓	8_{14}	✓	9_{10}	✓	9_{27}	$\geq U$	9_{44}	✓
7_5	✓	8_{15}	✓	9_{11}	✓	9_{28}	✓	9_{45}	✓
7_6	✓	8_{16}	✓	9_{12}	✓	9_{29}	✓	9_{46}	$\geq U$
7_7	✓	8_{17}	✓	9_{13}	✓	9_{30}	✓	9_{47}	✓
8_1	✓	8_{18}	✓	9_{14}	✓	9_{31}	✓	9_{48}	✓
8_2	✓	8_{19}	✓	9_{15}	✓	9_{32}	✓	9_{49}	✓



8_{10}

FIGURE 1. The band surgery along the gray band yields $3_1 \cup U$.

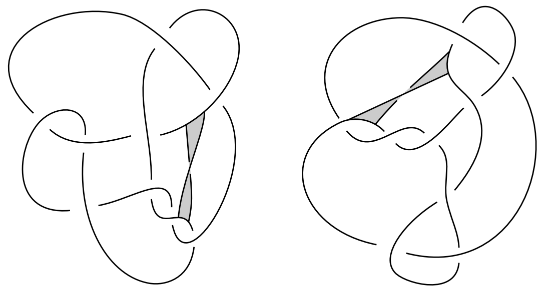


FIGURE 2. The band surgeries along gray bands yield $4_1 \cup U$.

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