Bull. Korean Math. Soc. **61** (2024), No. 3, pp. 763–777
https://doi.org/10.4134/BKMS.b230348
pISSN: 1015-8634 / eISSN: 2234-3016

RINGS WHOSE ASSOCIATED EXTENDED ZERO-DIVISOR GRAPHS ARE COMPLEMENTED

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Abstract. Let R be a commutative ring with identity $1 \neq 0$. In this paper, we continue the study started in [10] to further investigate when the extended zero-divisor graph of R, denoted as $\overline{\Gamma}(R)$, is complemented. We also study when $\overline{\Gamma}(R)$ is uniquely complemented. We give a complete characterization of when $\overline{\Gamma}(R)$ of a finite ring R is complemented. Various examples are given using the direct product of rings and idealizations of modules.

1. Introduction

Throughout the paper, R will be a commutative ring with identity and $Z(R)$ will be its set of zero-divisors. Let x be an element of R. The annihilator of x is defined as $\mathrm{Ann}_R(x) := \{y \in R \mid xy = 0\}$ and if there is no confusion, we denote it simply by $\mathrm{Ann}(x)$. For an ideal I of R, $\sqrt{I}$ means the radical of I. An element x of R is called nilpotent if $x^n = 0$ for some positive integer n and we denote n_x its index of nilpotency; that is, the smallest integer n such that $x^n = 0$. The set of all nilpotent elements is denoted by $\mathrm{Nil}(R) := \sqrt{0}$. The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo an integer n will be denoted by $\mathbb{Z}_n$. For a subset X of R, we denote by X^* the set $X \setminus \{0\}$.

Recall that the zero-divisor graph, denoted by $\Gamma(R)$, is the simple graph whose vertex set is the set of nonzero zero-divisors, $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The extended zero-divisor graph, denoted by $\overline{\Gamma}(R)$, is the simple graph which has the same vertex set like $\Gamma(R)$ and two distinct vertices x and y are adjacent if and only if $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$ for some integers $n, m \in \mathbb{N}^*$. We assume the reader has a basic familiarity with the zero-divisor graph theory. For general background on the zero-divisor graph theory, we refer the reader to [1, 3–5, 7–10].

This paper deals with complementedness and uniquely complementedness notions of graphs. A graph $G = (V, E)$ is said to be complemented if every

Received June 12, 2023; Revised September 12, 2023; Accepted October 5, 2023.
2020 *Mathematics Subject Classification.* 13A70, 05C25, 05E40, 13B30.
Key words and phrases. Commutative ring, zero-divisor graph, extended zero-divisor graph, complemented, uniquely complemented, zero-dimensional ring.

vertex v has an orthogonal; that is, an adjacent vertex u to v such that the edge $v - u$ is not a part of a triangle, we write $v \perp u$. The graph G is said to be uniquely complemented if it is complemented and, for any three vertices $u, v, w \in V$, if v is orthogonal to both u and w, then $u \sim w$, where $\sim$ is the equivalence relation on G given by $u \sim w$ if their open neighborhoods coincide. In [2, Theorem 3.5], these notions were used, for the classical zero-divisor graph, to characterize when the total quotient ring of a reduced ring R is von Neumann regular. Also, [10, Proposition 4.8] gives a similar result. Namely, it was shown that, when $\overline{\Gamma}(R) \neq \Gamma(R)$, $\overline{\Gamma}(R)$ is complemented is a sufficient condition so that the total quotient ring of R is zero-dimensional. But, it seems that the proof holds true only when $girth(\overline{\Gamma}(R)) = 4$. In this paper, using a new treatment, we prove that [10, Proposition 4.8] still holds true without any further assumption (see Theorem 4.2). Namely, in this paper, we continue the investigation begun in [10] to further study when $\overline{\Gamma}(R)$ is complemented and when it is uniquely complemented.

This article is organized as follows: In Section 2, we study when the extended zero-divisor graph of a commutative ring is complemented. We start by showing that, if $\overline{\Gamma}(R)$ is complemented such that $|Z(R)| \geq 4$, then the ring R has at most one nonzero nilpotent element (see Theorem 2.4 and Example 2.5). When R is finite, we get the converse of Theorem 2.4 (see Corollary 2.8). In fact, this is a consequence of the characterization of finite rings with complemented extended zero-divisor graphs (see Theorem 2.6). In Section 3, we show as a main result that, when $\Gamma(R) \neq \overline{\Gamma}(R)$, the complementedness and the uniquely complementedness notions coincide (see Theorem 3.2). In Section 4, we show that, when $\Gamma(R) \neq \overline{\Gamma}(R)$, the total quotient ring $T(R)$ of R is zero-dimensional once $\Gamma(R)$ is complemented (see Theorem 4.2). The proof of this result leads us to show that when $\overline{\Gamma}(R)$ is complemented, every non nilpotent element has an orthogonal which is not nilpotent (see Theorem 4.4). Also, if $\overline{\Gamma}(R)$ is complemented such that $\overline{\Gamma}(R) \neq \Gamma(R)$, then orthogonals to the unique nonzero nilpotent element cannot be an end (see Corollary 4.5). At the end of this section we prove that, for any ring R such that $|\mathrm{Nil}(R)| = 2$, R is not local or $\overline{\Gamma}(R)$ is not complemented (see Proposition 4.6). Finally, Section 5 is devoted to the study of when the extended zero-divisor graph of a finite direct product of rings as well as the one of an idealization of an R-module are complemented (see Theorems 5.1, 5.2 and 5.3, and Proposition 5.4).

2. When the extended zero-divisor graph of a commutative ring is complemented?

In this section we study when the extended zero-divisor graph of a commutative ring is complemented. We start by showing that the ring R will have at most one nonzero nilpotent element if $\overline{\Gamma}(R)$ is complemented and $|Z(R)| \geq 4$. But first, we need the following lemmas which will be very useful throughout this paper.

Lemma 2.1. *Let R be a non reduced ring. If $\overline{\Gamma}(R)$ is complemented, then every nonzero nilpotent element has index* 2.

Proof. Assume that $\mathrm{Nil}(R) \neq \{0\}$. Let $x \in \mathrm{Nil}(R)$ such that $n_x \geq 3$. Let $z \in Z(R)$ such that z is adjacent to x. If $x^{n_x-1} \neq z$, then x^{n_x-1} is adjacent to both z and x. Otherwise, we can easily see that $x^{n_x-1}+x$ is adjacent to both x^{n_x-1} and x. Hence, $\overline{\Gamma}(R)$ is not complemented. □

Notice that the converse of this lemma does not hold in general since, for instance, the extended zero-divisor graph $\overline{\Gamma}(\mathbb{Z}_{18})$, illustrated in Figure 1, is not complemented (since, for example, $\overline{6}$ has not an orthogonal element) even if the index of nilpotency of every nilpotent element is 2.

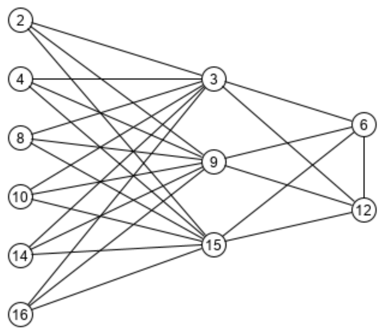

FIGURE 1. $\overline{\Gamma}(\mathbb{Z}_{18})$

Example 2.2. (1) Let p be a prime number and n be a positive integer. Then, $\overline{\Gamma}(\mathbb{Z}_{p^n})$ is complemented if and only if $n = 2$ and $p = 3$ (since K_2 is the only complete graph that is complemented).

(2) Consider the ring $\mathbb{R}[X,Y]/(X^3, XY^3)$. The index of nilpotency of $\overline{X}$ is 3, so the graph $\overline{\Gamma}(\mathbb{R}[X,Y]/(X^3, XY^3))$ is not complemented.

Lemma 2.3. *Let R be a ring such that $|Z(R)^*| \geq 3$. If $\overline{\Gamma}(R)$ is complemented, then the following assertions hold*:

(1) *For every $\alpha \in \mathrm{Nil}(R)^*$, $2\alpha = 0$.*

(2) *For every $\alpha \in \mathrm{Nil}(R)^*$, if $\beta \in Z(R)^*$ such that $\beta \perp \alpha$, then $\beta \notin \mathrm{Nil}(R)$.*

Proof. (1) Assume that there exists $\alpha \in \mathrm{Nil}(R)^*$ such that $2\alpha \neq 0$. Then, α is adjacent to $(-\alpha)$. On the other hand, $|Z(R)^*| \geq 3$ and since $\overline{\Gamma}(R)$ is connected, there exists $z \in Z(R)^* \setminus \{\alpha, -\alpha\}$ which is adjacent to α. But, such an element is adjacent to $(-\alpha)$. Namely, this means that α has not an orthogonal, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented.

(2) Let $\alpha \in \mathrm{Nil}(R)^*$ and consider $\beta \in Z(R)^*$ such that $\alpha \perp \beta$. If $\beta \in \mathrm{Nil}(R)^*$, then $\alpha + \beta \neq 0$, otherwise $\alpha = -\beta$ and with the fact that $2\alpha = 0$, $\alpha = \beta$, a

contradiction since $\alpha \perp \beta$. Thus, $\alpha + \beta$ is adjacent to both α and β (since α and β are adjacent, and by Lemma 2.1, $\beta^2 = \alpha^2 = 0$). So, α and β are not orthogonal, a contradiction. □

Now, we are in position to show that when $\overline{\Gamma}(R)$ is complemented and $|Z(R)| \geq 4$, the ring R has at most one nonzero nilpotent element.

Notice that, if $|Z(R)| = 2$, which means that R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, $\overline{\Gamma}(R)$ is not complemented. If $|Z(R)| = 3$, then $\overline{\Gamma}(R)$ is complemented. Explicitly, R is either isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ (and in this case $\mathrm{Nil}(R) = \{0, a, -a\} = Z(R)$ for some $0 \neq a \in R$), or R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (and in this case $\mathrm{Nil}(R) = \{0\}$).

Theorem 2.4. *Let R be a ring such that $|Z(R)| \geq 4$. If $\overline{\Gamma}(R)$ is complemented, then $|\mathrm{Nil}(R)| \leq 2$.*

Proof. Assume that there exist $a, b \in \mathrm{Nil}(R)^*$ such that $a \neq b$. Then, $a + b \in \mathrm{Nil}(R)^*$ by Lemma 2.3. Let $x, y, z \in Z(R) \setminus \mathrm{Nil}(R)$ such that $x \perp a$, $y \perp b$ and $z \perp a + b$. Let n be a positive integer such that $z^n(a + b) = 0$. We have the two following cases:
Case $ab \neq 0$: Since $z^n(a + b) = 0$, $z^n ab = -z^n b^2 = 0$ by Lemma 2.1. Thus, ab is adjacent to both z and $a + b$ ($ab \neq z$ since $ab \in \mathrm{Nil}(R)^*$ and also $ab \neq a + b$), a contradiction.
Case $ab = 0$: If $z^n a = 0$, then a is adjacent to both z and $a + b$, a contradiction. Then, $z^n a \neq 0$. If $z^n a \neq a$, then $z^n a$ is adjacent to both a and x, a contradiction. Otherwise, since $z^n(a + b) = 0$ and $b \in \mathrm{Nil}(R)^*$, $z^n a = -z^n b = z^n b$. Then, $z^n a = a = z^n b$ is adjacent to both b and y, a contradiction. □

Example 2.5. (1) Consider the ring $R = D \times \mathbb{Z}_2[X]/(X^2)$, where D is an integral domain. Then, $\mathrm{Nil}(R) = \{(0, \bar{0}), (0, \bar{X})\}$ and its extended zero-divisor graph is illustrated in Figure 2. Namely, $\overline{\Gamma}(R)$ is a complete bipartite graph and hence it is complemented.

(2) For the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, we have $\mathrm{Nil}(R) = \{(\bar{0}, \bar{0}, \bar{0}), (\bar{0}, \bar{0}, \bar{2})\}$. The extended zero-divisor graph of this ring is illustrated in Figure 3. We can easily show that $\overline{\Gamma}(R)$ is complemented.

(3) For the ring $\mathbb{Z}_2[X, Y]/(X^3, XY)$, we have $\mathrm{Nil}(\mathbb{Z}_2[X, Y]/(X^3, XY)) = \{\bar{0}, \bar{X}, \bar{X}^2, \bar{X} + \bar{X}^2\}$. The extended zero-divisor of this ring is illustrated in Figure 4.

Since $\bar{X} + \bar{Y}$ has not an orthogonal element, $\overline{\Gamma}(\mathbb{Z}_2[X, Y]/(X^3, XY))$ is not complemented.

When R is finite, the converse of Theorem 2.4 holds as shown in Corollary 2.8 which is a consequence of the following one.

Theorem 2.6. *Let R be a finite ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. Then, $\overline{\Gamma}(R)$ is complemented if and only if $R \cong B \times A_1 \times \cdots \times A_n$ such that $B \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and $A_1, \ldots, A_n$ are finite fields.*

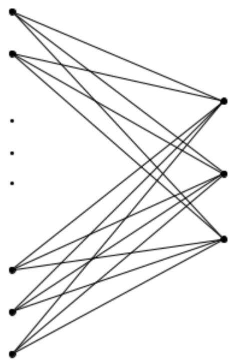

FIGURE 2. $\overline{\Gamma}(D \times \mathbb{Z}_2[X]/(X^2))$

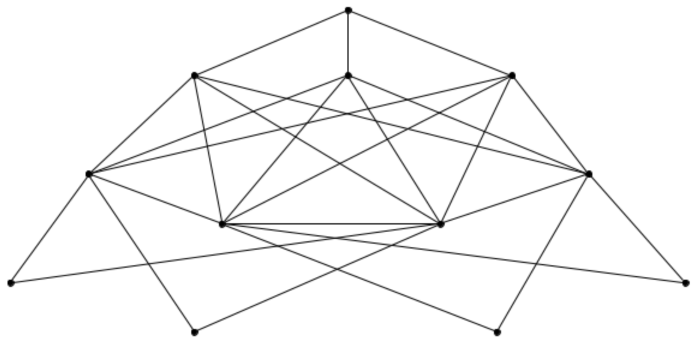

FIGURE 3. $\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$

Proof. ($\Leftarrow$) This follows by induction using Theorems 5.1 and 5.2 given in Section 5.

($\Rightarrow$) Since R is a finite ring, $R \cong A_1 \times \cdots \times A_n$ such that A_i is a finite local ring for all $i \in \{1, \ldots, n\}$, by [6, Theorem 87]. Then, for all $i \in \{1, \ldots, n\}$, $Z(A_i) = \mathrm{Nil}(A_i)$. By Theorem 2.4, $|\mathrm{Nil}(R)| \leq 2$, and since $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)| = 2$. So, one of the A_i's is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and the other rings are finite fields. Notice that $\overline{\Gamma}(\mathbb{Z}_4)$ and $\overline{\Gamma}(\mathbb{Z}_2[X]/(X^2))$ are not complemented which guarantee the existence of the fields. $\square$

Corollary 2.7. *Let $n \in \mathbb{N}^*$ such that $\Gamma(\mathbb{Z}_n) \neq \overline{\Gamma}(\mathbb{Z}_n)$. Then, $\overline{\Gamma}(\mathbb{Z}_n)$ is complemented if and only if $n = 2^2 p_1 \cdots p_r$ with $p_1, \ldots, p_r$ are distinct prime numbers and $r \geq 1$ is a positive integer.*

Now, let us prove the converse of Theorem 2.4 in the case of a finite ring.

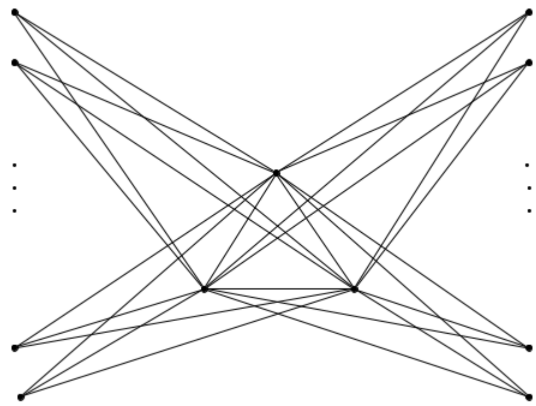

FIGURE 4. $\overline{\Gamma}(\mathbb{Z}_2[X,Y]/(X^3,XY))$

Corollary 2.8. *Let R be a finite ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. If $\text{Nil}(R) = \{0, a\}$ for some $a \in R^*$, then $\overline{\Gamma}(R)$ is complemented.*

Proof. Since R is a finite ring, by [6, Theorem 87], $R \cong A_1 \times \cdots \times A_n$ such that A_i is a finite local ring for all $i \in \{1, \ldots, n\}$. If R is indecomposable, then using the fact that $|\text{Nil}(R)| = 2 = |Z(R)|$, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Then, this contradicts the fact that $\Gamma(R) \neq \overline{\Gamma}(R)$. Thus, $R \cong A_1 \times \cdots \times A_n$ such that $Z(A_i) = \text{Nil}(A_i)$ for every $i \in \{1, \ldots, n\}$ and $n \geq 2$. Since $|\text{Nil}(R)| = 2$, one of the A_i's is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and the other rings are integral domains. Then, by Theorem 2.6, $\overline{\Gamma}(R)$ is complemented. □

The authors are not able to prove the equivalence of Theorem 2.6 for infinite rings. We let it then as an open important question.

3. Complementedness and uniquely complementedness notions coincide for the extended zero-divisor graphs

In [2, Theorem 3.5], it was shown that, when R is reduced, $\Gamma(R)(= \overline{\Gamma}(R))$ is uniquely complemented if and only if $\Gamma(R)$ is complemented if and only if $T(R)$ is von Neumann regular. The main result of this section generalizes [2, Theorem 3.5]. Namely, it shows that, when R is not reduced, the complementedness and the uniquely complementedness notions coincide. To show this, we first prove the following lemma.

Lemma 3.1. *Let R be a ring and $a, b, c \in Z(R) \setminus \text{Nil}(R)$. If $a \perp b$ and $a \perp c$ in $\overline{\Gamma}(R)$, then $b \sim c$.*

Proof. We have $a^{n_1}b^{m_1} = a^{n_2}c^{m_2} = 0$ for some $n_1, m_1, n_2, m_2 \in \mathbb{N}^*$. We first show that b and c are not adjacent; that is, $b^\alpha c^\beta \neq 0$ for every $\alpha, \beta \in \mathbb{N}^*$. If $b^\alpha c^\beta = 0$ for some $\alpha, \beta \in \mathbb{N}$, then, $b = c$ or $a = c$ (since $a \perp b$ and $a \perp c$).

Thus, $b \in \mathrm{Nil}(R)$ or $a \in \mathrm{Nil}(R)$, a contradiction. Then, b and c are not adjacent. Now, let us prove that $N(b) = N(c)$. Let $d \in N(b)$. Then $d^n b^m = 0$ with $d^n \neq 0$ for some $n, m \in \mathbb{N}$. Thus, $(d^n c^{m_2}) a^{n_2} = d^n (c^{m_2} a^{n_2}) = 0$ and $(d^n c^{m_2}) b^m = (d^n b^m) c^{m_2} = 0$. Then, $d^n c^{m_2} = 0$, otherwise $d^n c^{m_2}$ is adjacent to both a and b (and $d^n c^{m_2} \neq a$ and $d^n c^{m_2} \neq b$ since $a, b \notin \mathrm{Nil}(R)$) which contradicts the fact that $a \perp b$. This shows that $N(b) \subseteq N(c)$. Similarly, we show the other inclusion and then $b \sim c$. □

Now, we are ready to prove the main result of this section.

Theorem 3.2. *Let R be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. Then, $\overline{\Gamma}(R)$ is uniquely complemented if and only if $\overline{\Gamma}(R)$ is complemented.*

Proof. ($\Rightarrow$) By definition of uniquely complemented.

($\Leftarrow$) Suppose that $\overline{\Gamma}(R)$ is complemented. Then, by Theorem 2.4, $\mathrm{Nil}(R) = \{0, \alpha\}$ for some $0 \neq \alpha \in R$. So, by Lemma 3.1, we have just to prove that, for every $b, c \in Z(R)^*$, if $\alpha \perp b$ and $\alpha \perp c$, then $b \sim c$, and if $\alpha \perp c$ and $b \perp c$, then $\alpha \sim b$. Let us prove the first implication. So, suppose by contradiction that there exist $b, c \in Z(R)^*$ such that $\alpha \perp b$ and $\alpha \perp c$ but $b \nsim c$. Then, there exists $x \in N(c) \setminus N(b)$; that is, $x^{n_1} c^{m_1} = 0$ for some $n_1, m_1 \in \mathbb{N}^*$ and $x^n b^m \neq 0$ for every $n, m \in \mathbb{N}^*$. Assume that $xb \neq c$. Then, $(xb)^{n_1} c^{m_1} = 0$ and so xb and c are adjacent. On the other hand, α and b are adjacent. Then, $\alpha b^t = 0$ for some $t \in \mathbb{N}^*$. Thus, $\alpha (xb)^t = 0$ which shows that xb is adjacent to both α and c, a contradiction since $\alpha \perp c$. Then, $xb = c$, and with $x^{n_1} c^{m_1} = 0$ we get $x^{n_1 + m_1} b^{m_1} = x^{n_1} (xb)^{m_1} = 0$, a contradiction.

Now, we prove the second implication. Assume that $\alpha \perp c$ and $b \perp c$. Then, $\alpha c^{m_1} = b^{n_1} c^{m_2} = 0$ for some $n_1, m_1, m_2 \in \mathbb{N}^*$. Thus, α is not adjacent to b, otherwise b is adjacent to both α and c, a contradiction since $\alpha \perp c$. Let $d \in N(\alpha)$. Then $d^n \alpha = 0$ for some $n \in \mathbb{N}^*$ and so $(d^n b^{n_1}) c^{m_2} = d^n (b^{n_1} c^{m_2}) = 0$ and $(d^n b^{n_1}) \alpha = b^{n_1} d^n \alpha = 0$. If $d^n b^{n_1} \in Z(R) \setminus \mathrm{Nil}(R)$, then $d^n b^{n_1} \neq \alpha$ and $d^n b^{n_1} \neq c$. Thus, $d^n b^{n_1}$ is adjacent to both c and α, a contradiction (since $\alpha \perp c$). Then, $d^n b^{n_1} \in \mathrm{Nil}(R)$. If $d^n b^{n_1} = 0$, then d is adjacent to b ($d \neq b$ since $d, b \in Z(R) \setminus \mathrm{Nil}(R)$). Thus, $d \in N(b)$. If $d^n b^{n_1} = \alpha$, then $d^{2n} b^{2n_1} = \alpha^2 = 0$ ($d^{2n} \neq 0$, $b^{2n_1} \neq 0$ and $d \neq b$ since $b, d \in Z(R) \setminus \mathrm{Nil}(R)$). Thus, $d \in N(b)$. This shows that $\alpha \sim b$. Therefore, $\overline{\Gamma}(R)$ is uniquely complemented. □

Corollary 3.3. *Let R be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$ and $\overline{\Gamma}(R)$ is complemented. Then, for every orthogonal $b \in Z(R)^*$ to the nonzero nilpotent element α, we have $b \sim \alpha + b$.*

Proof. Assume that $\Gamma(R) \neq \overline{\Gamma}(R)$ and $\overline{\Gamma}(R)$ is complemented. Then, by Theorem 2.3, $\mathrm{Nil}(R) = \{0, \alpha\}$ for some $0 \neq \alpha \in R$. Let $b \in Z(R)^* \setminus \{\alpha\}$ such that $\alpha \perp b$; that is, $\alpha b^n = 0$ for some positive integer n and there is no vertex adjacent to both α and b. Let us prove that $\alpha \perp (\alpha + b)$. We have $\alpha(\alpha + b)^n = \alpha(b^n + n\alpha b^{n-1} + \cdots + \alpha^n) = \alpha b^n = 0$. Since $\alpha + b \neq \alpha$ and $(\alpha + b)^n \neq 0$ (because $b \notin \mathrm{Nil}(R)$), α and $\alpha + b$ are adjacent. Now, assume that there exists c which is adjacent to both α and $\alpha+b$. Then, $c^{n_1}\alpha = 0 = c^{n_1}(\alpha+b)^{m_1} = m_1c^{n_1}\alpha b^{m_1-1} + c^{n_1}b^{m_1} = 0 + c^{n_1}b^{m_1}$. So, c is adjacent to b, a contradiction since $\alpha \perp b$. Therefore, $\alpha \perp (\alpha+b)$, which shows using Theorem 3.2 that $b \sim \alpha + b$. □

4. Complemented extended zero-divisor graphs and zero-dimensional rings

If $|Z(R)| = 2$, then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ and so $T(R)$ is zero-dimensional. If $|Z(R)| = 3$, then R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ and so, $\overline{\Gamma}(R)$ is complemented and $T(R)$ is zero-dimensional. In this section, we show that, when $\Gamma(R) \neq \overline{\Gamma}(R)$ (in particular $|Z(R)| \geq 4$), $T(R)$ is zero-dimensional once $\Gamma(R)$ is complemented. In fact, this result was already given in [10, Proposition 4.8]. But, in the third line of the proof, [10, Corollary 3.4] is used to show that an element z_0 is not nilpotent. This means that we have supposed that the girth of $\overline{\Gamma}(R)$ is not 3. But, there are $\overline{\Gamma}(R)$ which are complemented with girth equal to 3. For this consider $\overline{\Gamma}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ (see Figure 3). Now, using a new way, we show that [10, Proposition 4.8] holds true. To show that, we need the following lemma.

Lemma 4.1. *Let R be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. If $\Gamma(R)$ is uniquely complemented, then $\overline{\Gamma}(R)$ is not complemented.*

Proof. The result holds because once $\Gamma(R)$ is uniquely complemented it will be a star graph by [2, Theorem 3.9]. In this case $\overline{\Gamma}(R)$ is not complemented. □

Using the previous lemma, we get the main result of this section.

Theorem 4.2. *Let R be a ring such that $\Gamma(R) \neq \overline{\Gamma}(R)$. If $\overline{\Gamma}(R)$ is complemented, then $T(R)$ is zero-dimensional.*

Proof. There are two cases to discuss:
Case 1. For every $x \in Z(R)^*$, $x^{\perp} \cap (Z(R) \setminus \mathrm{Nil}(R)) \neq \emptyset$. In this case, we show that for every $\frac{x_1}{x_2}$ in $T(R)$, there exists $\frac{m_1}{m_2} \in T(R)$ such that $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is a unit and $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent. This shows that $T(R)$ is π-regular and so zero-dimensional (see [11, Theorems 3.1 and 3.2]). Then, let $\frac{x_1}{x_2}$ in $T(R)$. We distinguish three sub-cases:
Sub-case 1. Assume that $x_1 \in R \setminus Z(R)$. Since $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)| = 2$. We denote by α the nonzero nilpotent element of R. Using Lemma 2.3, we have $\alpha^2 = 2\alpha = 0$. It is clear that $\frac{x_1}{x_2}\frac{\alpha}{x_2}$ is nilpotent and also $\frac{x_1}{x_2} + \frac{\alpha}{x_2}$ is a unit since $(x_1+\alpha)^2 = {x_1}^2 \notin Z(R)$.
Sub-case 2. Assume that $x_1 = \alpha$. We have $\frac{x_1}{x_2}\frac{1}{x_2}$ is nilpotent and also $\frac{x_1}{x_2} + \frac{1}{x_2}$ is a unit since $(x_1+1)^2 = 1 \notin Z(R)$.
Sub-case 3. Assume that $x_1 \in Z(R) \setminus \mathrm{Nil}(R)$. Then, there exists $m_1 \in {x_1}^{\perp} \cap (Z(R) \setminus \mathrm{Nil}(R))$. Since x_1 and m_1 are adjacent, $\frac{x_1}{x_2}\frac{m_1}{x_2}$ is nilpotent. So, it remains to show that $\frac{x_1}{x_2} + \frac{m_1}{x_2}$ is a unit, which means to prove that $x_1 + m_1$ does

not belong to $Z(R)$. Otherwise, there exists $z \in R^*$ such that $z(x_1 + m_1) = 0$. We have x_1m_1 is nilpotent (since x_1 and m_1 are adjacent), then there are the two following sub-subcases to discuss:

Sub-subcase 1. Suppose that $x_1m_1 = 0$. We have $z(x_1 + m_1) = 0$, then $zx_1m_1 + zm_1{}^2 = 0$ and $zx_1{}^2 + zx_1m_1 = 0$, so $zm_1{}^2 = 0$ and $zx_1{}^2 = 0$. Then, $z \neq x_1$ and $z \neq m_1$ since x_1 and m_1 are not nilpotent. Thus, z is adjacent to both x_1 and m_1, a contradiction (since x_1 and m_1 are orthogonal).

Sub-subcase 2. Suppose that $x_1m_1 = \alpha$. We have $zx_1{}^2 + zx_1m_1 = 0$ and $zx_1m_1+zm_1{}^2 = 0$, then $zx_1{}^2+z\alpha = 0$ and $z\alpha+zm_1{}^2 = 0$. Thus, $z\alpha x_1{}^2 = 0$ and $z\alpha m_1{}^2 = 0$. Then, $z\alpha \neq x_1$ and $z\alpha \neq m_1$ since x_1 and m_1 are not nilpotent. If $z\alpha \neq 0$, then it is adjacent to both x_1 and m_1, a contradiction (since x_1 and m_1 are orthogonal). Then, $z\alpha = 0$ which implies that $zx_1^2 = 0 = zm_1^2$. Thus, $z \neq x_1$ and $z \neq m_1$ since x_1 and m_1 are not nilpotent. Then, z is adjacent to both x_1 and m_1, a contradiction (since x_1 and m_1 are orthogonal).

Case 2. There exists $x \in Z(R)^*$, $x^\perp = \{\alpha\} \subset \mathrm{Nil}(R) = \{0, \alpha\}$. In this case, one can show that $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Otherwise, there exist s and t in $N_{\overline{\Gamma}(R)}(x)$ such that s and t are adjacent. Since $x^\perp = \{\alpha\}$, $s\alpha = \alpha$. Also, since s and t are adjacent, $st = \alpha$ or $st = 0$. Then, $s\alpha t = 0$, which implies that $\alpha t = 0$. Thus, t and α are adjacent, which is a contradiction with the fact that $x^\perp = \{\alpha\}$. Thus, $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. On the other hand, $\Gamma(R)$ is not uniquely complemented (using Lemma 4.1). Then, in this case, there are two sub-cases to discuss:

Sub-case 1. $\Gamma(R)$ is complemented. Since $\Gamma(R)$ is not uniquely complemented, by [2, Theorem 3.14], $R \cong \mathbb{Z}_4 \times D$ or $R \cong \mathbb{Z}_2[X]/(X^2) \times D$ such that D is an integral domain, but this contradicts the fact that $|x^\perp| = 1$ (since for every $y \in Z(R)^*$, $|y^\perp| > 1$).

Sub-case 2. $\Gamma(R)$ is not complemented. Then, there exists $b \in Z(R)^*$ which has an orthogonal in $\overline{\Gamma}(R)$ and not in $\Gamma(R)$ (one can see that $b \neq \alpha$ and $bx \notin \{0, \alpha\}$ since α has x as an orthogonal in $\Gamma(R)$ and $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$). Then, there exists $t \in b^\perp$ such that $bt = \alpha$. One can show that $t \neq \alpha$. Otherwise, $b\alpha = \alpha$ and so for every $n \in \mathbb{N}^*$, $b^n\alpha = \alpha \neq 0$. Then, b and α are not adjacent in $\overline{\Gamma}(R)$, which is a contradiction with the fact that $t = \alpha$ and b are orthogonal. There are two sub-subcases to discuss:

Sub-subcase 1. $b\alpha \neq 0$. Then, $b\alpha = \alpha$. Since for every $z \in N_{\overline{\Gamma}(R)}(b)$, $zb = \alpha$ or $zb = 0$, $z(b\alpha) = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$, then $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$ (since $b\alpha = \alpha$). In this case, we show that $(bx)^\perp = \emptyset$, therefore, we determine firstly $N_{\overline{\Gamma}(R)}(bx)$. Let $h \in N_{\overline{\Gamma}(R)}(bx)$. Then there exist n, m in $\mathbb{N}^*$ such that $(bx)^n h^m = 0$ with $(bx)^n \neq 0$ and $h^m \neq 0$. Then, $(bh)^n x^n = 0$ (resp., $(bh)^m x^n = 0$) if $n \geq m$ (resp., $m \geq n$). If $(bh)^n \neq 0$, then $bh = \alpha$ since $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Then, $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(b)$. If $(bh)^n = 0$, then $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(b)$. Thus, $N_{\overline{\Gamma}(R)}(bx) = N_{\overline{\Gamma}(R)}(b) \cup \{\alpha\}$. Thus, $(bx)^\perp = \emptyset$ since $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(b)$, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented.

Sub-subcase 2. $b\alpha = 0$. Then, $t\alpha \neq 0$ since $t \in b^\perp$, which implies that $t\alpha = \alpha$. As in the previous case, $z\alpha = 0$ for every $z \in N_{\overline{\Gamma}(R)}(t)$. In this case, we show that $(tx)^\perp = \emptyset$. First, let us determine $N_{\overline{\Gamma}(R)}(tx)$. Let $h \in N_{\overline{\Gamma}(R)}(tx)$. Then there exist n, m in $\mathbb{N}^*$ such that $(tx)^n h^m = 0$ with $(tx)^n \neq 0$ and $h^m \neq 0$. Then, $(th)^n x^n = 0$ (resp., $(th)^m x^n = 0$) if $n \geq m$ (resp., $m \geq n$). If $(th)^n \neq 0$, then $th = \alpha$ since $N_{\overline{\Gamma}(R)}(x) = \{\alpha\}$. Then, $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(t)$. If $(th)^n = 0$, then $h = \alpha$ or $h \in N_{\overline{\Gamma}(R)}(t)$. Thus, $N_{\overline{\Gamma}(R)}(tx) = N_{\overline{\Gamma}(R)}(t) \cup \{\alpha\}$. Then, $(tx)^\perp = \emptyset$ since for every $z \in N_{\overline{\Gamma}(R)}(t)$, $z\alpha = 0$, which is a contradiction with the fact that $\overline{\Gamma}(R)$ is complemented. □

The following example shows that $T(R)$ is zero-dimensional does not imply that $\overline{\Gamma}(R)$ is complemented.

Example 4.3. $T(\mathbb{Z}_{16})$ is zero dimensional and $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented.

Proof. Since $\mathbb{Z}_{16}$ is zero-dimensional, $T(\mathbb{Z}_{16}) \cong \mathbb{Z}_{16}$ is zero-dimensional. On the other hand, $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented ($\overline{2}$ is a nilpotent element in $\mathbb{Z}_{16}$ of index of nilpotency 4). Then, by Lemma 2.1, $\overline{\Gamma}(\mathbb{Z}_{16})$ is not complemented. □

An observation of the proof of Theorem 4.2 leads us to show that, if $\overline{\Gamma}(R)$ is complemented, then every non nilpotent element has a non nilpotent orthogonal, as shown in the following result.

Theorem 4.4. *Let R be a ring such that $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$. Then, for all $x \in Z(R) \setminus \mathrm{Nil}(R)$, $x^\perp \cap (Z(R) \setminus \mathrm{Nil}(R)) \neq \emptyset$.*

Proof. Since $\overline{\Gamma}(R)$ is complemented and $\overline{\Gamma}(R) \neq \Gamma(R)$, $|\mathrm{Nil}(R)^*| = 1$ and $n_x \leq 2$ for every nilpotent element x. We denote by α the nonzero nilpotent element of R. We suppose that there exists $x_1 \in Z(R) \setminus \mathrm{Nil}(R)$ such that ${x_1}^\perp = \{\alpha\}$. On the other hand, by Theorem 4.2, $T(R)$ is zero-dimensional. Then, using [11, Theorems 3.1 and 3.2], there exists m_1 in R such that $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent and $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is a unit for some $m_2, x_2 \in R \setminus Z(R)$. Since $\frac{x_1}{x_2}\frac{m_1}{m_2}$ is nilpotent, $x_1 m_1 \in \mathrm{Nil}(R)$. Then, there are two cases to discuss:
Case 1. Suppose that $x_1 m_1 = 0$. Then, x_1 and m_1 are adjacent ($x_1 \neq m_1$ and $m_1 \neq 0$ since $x_1 \in Z(R) \setminus \mathrm{Nil}(R)$ and $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is a unit). If m_1 is nilpotent, then $m_1(x_1 m_2 + m_1 x_2) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, m_1 and x_1 are not orthogonal, then there exists $z \in Z(R)^*$ that is adjacent to both x_1 and m_1. Then, $z^2{x_1}^2 = z^2{m_1}^2 = 0$ (since zx_1 and zm_1 are nilpotent). If $z^2 x_1 = 0$ and $z^2 m_1 = 0$, then $z^2(x_1 m_2 + x_2 m_1) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, $z^2 x_1(x_1 m_2 + x_2 m_1) = 0$ with $z^2 x_1 \neq 0$ or $z^2 m_1(x_1 m_2 + x_2 m_1) = 0$ with $z^2 x_1 \neq 0$. Then, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction.
Case 2. Suppose that $x_1 m_1 = \alpha$. If m_1 is nilpotent, then x_1 and m_1 are adjacent. Thus, there exists $n \in \mathbb{N}^*$ such that ${x_1}^n m_1 = 0$. Consider β such that ${x_1}^\beta m_1 = 0$ and ${x_1}^{(\beta-1)} m_1 \neq 0$. We have ${x_1}^{(\beta-1)} m_1(x_1 m_2 + m_1 x_2) = 0$, then

$\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. If m_1 is not nilpotent, then ${x_1}^2{m_1}^2 = 0$ with ${x_1}^2 \neq 0$ and ${m_1}^2 \neq 0$. Then, x_1 and m_1 are adjacent, but they are not orthogonal (since m_1 is not nilpotent). Then, there exists $z \in Z(R)^*$ that is adjacent to both x_1 and m_1. Thus, $z^2(x_1)^2 = 0$ and $z^2 m_1 = 0$. If $z^2 x_1 = 0$ and $z^2 m_1 = 0$, then $z^2(x_1 m_2 + m_1 x_2) = 0$. Thus, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. Otherwise, $z^2 x_1(x_1 m_2 + m_1 x_2) = 0$ if $z^2 x_1 \neq 0$, or $z^2 m_1(x_1 m_2 + m_1 x_2) = 0$ if $z^2 m_1 \neq 0$ and $z^2 x_1 = 0$. Then, $\frac{x_1}{x_2} + \frac{m_1}{m_2}$ is not a unit, a contradiction. □

It was proven in [2, Lemma 3.7] that if $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then there exists a unique nonzero nilpotent element in R and any orthogonal to such an element is an end. This is not the case for $\overline{\Gamma}(R)$ as shown in the following corollary.

Corollary 4.5. *Let R be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. If $\overline{\Gamma}(R)$ is complemented, then every orthogonal to the nonzero nilpotent element is not an end.*

In Corollary 2.8, we showed, for a finite ring R, that when $\Gamma(R) \neq \overline{\Gamma}(R)$ and $|\mathrm{Nil}(R)| = 2$, $\overline{\Gamma}(R)$ is complemented. For the infinite case we get the following result.

Proposition 4.6. *Let R be an infinite ring such that $\mathrm{Nil}(R) = \{0, \alpha\}$ for some $\alpha \in R^*$. Then, either R is not local or $\overline{\Gamma}(R)$ is not complemented.*

Proof. Assume that $\mathrm{Nil}(R) = \{0, \alpha\}$ and suppose that R is local with the maximal ideal $\mathrm{Ann}(\alpha)$ and that $\overline{\Gamma}(R)$ is complemented. Let $x \in Z(R) \setminus \{0, \alpha\}$. Then, there exists $y \in Z(R) \setminus \{0, \alpha\}$ such that $x \perp y$, by Theorem 4.4. But, since $\mathrm{Ann}(\alpha)$ is the maximal ideal of R, $x, y \in \mathrm{Ann}(\alpha)$. So, $x - y$ is a part of a triangle, a contradiction. □

5. When the graphs $\overline{\Gamma}(R_1 \times R_2)$ and $\overline{\Gamma}(R(+)M)$ are complemented?

In the first part of this section, we investigate when the extended zero-divisor graph of the product of two rings, $R_1 \times R_2$, is complemented. Namely, we treat three cases following the cardinality of $Z(R_2)$: $|Z(R_2)| = 1$, $|Z(R_2)| = 2$ and $|Z(R_2)| \geq 3$.

For the case when R_2 is an integral domain, we have the following theorem.

Theorem 5.1. *Let R_1 and R_2 be two rings such that R_2 is an integral domain. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented if and only if either $|Z(R_1)| = 2$ or ($\overline{\Gamma}(R_1)$ is complemented and $|\mathrm{Nil}(R_1)| \leq 2$).*

Proof. ($\Rightarrow$) Assume that $\overline{\Gamma}(R_1 \times R_2)$ is complemented and $|Z(R_1)| \neq 2$. If $|\mathrm{Nil}(R_1)| \geq 3$, then $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$, a contradiction (by Theorem 2.4). Now, suppose that $\overline{\Gamma}(R_1)$ is not complemented. Then there exists $z \in Z(R_1)^*$ such that x is not an orthogonal to z for every $x \in Z(R_1)^*$. We have $(z, 0) \in Z(R_1 \times R_2)$. Let $(a, b) \in Z(R_1 \times R_2)$ such that (a, b) is adjacent to $(z, 0)$. So,

$(a,b)^n(z,0)^m = (0,0)$ for some $n,m \in \mathbb{N}^*$ with $(a,b)^n \neq (0,0)$ and $(z,0)^m \neq (0,0)$, then $a^n z^m = 0$ and so we have three cases to discuss:
Case 1. If $a^n = 0$ and $b \neq 0$, then for every vertex y adjacent to z, $(y,0)$ is adjacent to both (a,b) and $(z,0)$.
Case 2. If $a^n \neq 0$ and $b \neq 0$, then a is adjacent to z and so there exists x adjacent to both a and z since z is not orthogonal to a. Thus, $(x,0)$ is adjacent to both $(z,0)$ and (a,b).
Case 3. If $a^n \neq 0$ and $b = 0$, then $(0,1)$ is adjacent to both (a,b) and $(z,0)$.

In all cases, $(z,0)$ has not an orthogonal in $\overline{\Gamma}(R_1 \times R_2)$, a contradiction.

($\Leftarrow$) If $|Z(R_1)| = 2$, then $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus, $\overline{\Gamma}(R_1 \times R_2)$ is a complete bipartite graph. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented.

Now, assume that $\overline{\Gamma}(R_1)$ is complemented and $|\mathrm{Nil}(R_1)| \leq 2$. If $|\mathrm{Nil}(R_1)| = 1$, then $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times \{0\}) \cup (Z(R_1) \times \{0\}) \cup (Z(R_1) \times R_2^*)$. If $(a,b) \in R_1 \setminus Z(R_1) \times \{0\}$, then $(a,0) \perp (0,1)$. If $(a,b) \in Z(R_1) \times \{0\}$, then $b = 0$ and $(a,0) \perp (c,1)$ with $c \in a^{\perp}$. If $(a,b) \in Z(R_1) \times R_2^*$, then $(a,b) \perp (c,0)$ with $c \in a^{\perp}$.

If $|\mathrm{Nil}(R_1)| = 2$, then $\mathrm{Nil}(R_1) = \{0, \alpha\}$ for some $0 \neq \alpha \in R_1$ and $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times \{0\}) \cup (Z(R_1) \setminus \mathrm{Nil}(R_1) \times R_2) \cup (\mathrm{Nil}(R_1) \times R_2)$. Let $(a,b) \in Z(R_1 \times R_2)^*$. If $(a,b) \in R_1 \setminus Z(R_1) \times \{0\}$, then $(a,0) \perp (0,1)$. If $(a,b) \in Z(R_1) \setminus \mathrm{Nil}(R_1) \times R_2$, then if $b = 0$, $(a,0) \perp (c,b')$ with $c \in a^{\perp}$, otherwise $(a,b) \perp (c,0)$ with $c \in a^{\perp}$. For the case where $(a,b) \in \mathrm{Nil}(R_1) \times R_2$, we distinguish three cases:

If $a = \alpha$ and $b = 0$, then $(\alpha,0) \perp (c,b')$ with $c \in \alpha^{\perp}$ and $b' \in R_2^*$.
If $a = \alpha$ and $b \neq 0$, then $(\alpha,b) \perp (c,0)$ with $c \in R_1 \setminus Z(R_1)$.
If $a = 0$ and $b \neq 0$, then $(0,b) \perp (c,0)$ with $c \in R_1 \setminus Z(R_1)$.
This completes the proof. □

Now, for the case when $|Z(R_2)| = 2$, we have the following result.

Theorem 5.2. *Let R_1 and R_2 be two rings such that $|Z(R_2)| = 2$. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented if and only if $\overline{\Gamma}(R_1)$ is complemented and R_1 is reduced.*

Proof. ($\Rightarrow$) Assume that $\overline{\Gamma}(R_1 \times R_2)$ is complemented. We have $|\mathrm{Nil}(R_2)| = |Z(R_2)| = 2$, then if R_1 is not reduced, $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$, a contradiction (by Theorem 2.4). Now, suppose that $\overline{\Gamma}(R_1)$ is not complemented. Then there exists $z \in Z(R_1)^*$ which has not an orthogonal. We have $(z,0) \in Z(R_1 \times R_2)$. Suppose that there exists $(a,b) \in Z(R_1 \times R_2)$ such that $(z,0) \perp (a,b)$. Then, $(a,b)^n(z,0)^m = (0,0)$ for some $n,m \in \mathbb{N}^*$ and so $a^n z^m = 0$. Thus, we have two cases to discuss:
Case 1. If $a^n = 0$, then $b^n \neq 0$. So, consider $y \in Z(R_1)^*$ which is adjacent to z. Then, $(y,0)$ is adjacent to both (a,b) and $(z,0)$, a contradiction.
Case 2. If $a^n \neq 0$, then a is adjacent to z and so there exists $x \in Z(R_1)^*$ such that x is adjacent to both z and a. Thus, $(x,0)$ is adjacent to both (a,b) and $(z,0)$, a contradiction.

Hence, $(z,0)$ has not an orthogonal in $\overline{\Gamma}(R_1 \times R_2)$, a contradiction.

($\Leftarrow$) We have $Z(R_1 \times R_2) = (R_1 \setminus Z(R_1) \times Z(R_2)) \cup (Z(R_1) \times R_2 \setminus Z(R_2)) \cup (Z(R_1) \times Z(R_2))$. Let $(a,b) \in Z(R_1 \times R_2)$. If $(a,b) \in R_1 \setminus Z(R_1) \times Z(R_2)$, then $(a,b) \perp (0,1)$. If $(a,b) \in Z(R_1) \times R_2 \setminus Z(R_2)$, then $(a,b) \perp (c,0)$ with $c \in a^{\perp}$. If $(a,b) \in Z(R_1) \times Z(R_2)$, then $(a,b) \perp (c,1)$ with $c \in a^{\perp}$. □

For the case where R_2 is a non reduced ring such that $|Z(R_2)| \geq 3$, we give the following theorem.

Theorem 5.3. *Let R_1 be a ring and R_2 be a non reduced ring such that $|Z(R_2)| \geq 3$. Then, $\overline{\Gamma}(R_1 \times R_2)$ is complemented if and only if $\overline{\Gamma}(R_2)$ and $\overline{\Gamma}(R_1)$ are both complemented and R_1 is reduced.*

Proof. ($\Rightarrow$) If R_1 is not reduced, then $|\mathrm{Nil}(R_1 \times R_2)| \geq 3$ since R_2 is not reduced. Then, $\overline{\Gamma}(R_1 \times R_2)$ is not complemented, by Theorem 2.4, since $|Z(R_1 \times R_2)| \geq 4$, a contradiction.

Now, assume that $\overline{\Gamma}(R_2)$ is not complemented. Then, there exists $z \in Z(R_2)^*$ which has not an orthogonal. Let $(a,b) \in Z(R_1 \times R_2)^*$ such that (a,b) is adjacent to $(0,z)$. Then, $(a,b)^n(0,z)^m = (0,0)$ for some $n,m \in \mathbb{N}^*$ with $(a,b)^n \neq (0,0)$ and $(0,z)^m \neq (0,0)$. Thus, $b^n z^m = 0$. Then, we have two cases to discuss:

Case 1. If $b^n \neq 0$, then b is adjacent to z. So, there exists a vertex x adjacent to both z and b. Thus, $(0,x)$ is adjacent to both $(0,z)$ and (a,b).

Case 2. If $b^n = 0$, then $a^n \neq 0$. So, consider $y \in Z(R_2)^*$ which is adjacent to z. Then, $(0,y)$ is adjacent to both (a,b) and $(0,z)$, a contradiction (since $\overline{\Gamma}(R_1 \times R_2)$ is complemented). Similarly, we can prove that $\overline{\Gamma}(R_1)$ is complemented (because, if $\overline{\Gamma}(R_1)$ is not complemented, then $|Z(R_1)| \geq 3$).

($\Leftarrow$) We have $Z(R_1 \times R_2) = (Z(R_1) \times Z(R_2)) \cup (R_1 \setminus Z(R_1) \times Z(R_2)) \cup (Z(R_1) \times R_2 \setminus Z(R_2))$. Let $(a,b) \in Z(R_1 \times R_2)^*$. If $(a,b) \in R_1 \setminus Z(R_1) \times Z(R_2) \setminus \mathrm{Nil}(R_2)$, then $(a,b) \perp (0,c)$ with $c \in b^{\perp}$. If $(a,b) \in Z(R_1)^* \times R_2 \setminus Z(R_2)$, then $(a,b) \perp (c,0)$ with $c \in a^{\perp}$. If $(a,b) \in Z(R_1)^* \times Z(R_2) \setminus \mathrm{Nil}(R_2)$, then $(a,b) \perp (c_1,c_2)$ with $c_1 \in a^{\perp}$ and $c_2 \in b^{\perp}$. If $(a,b) \in R_1 \setminus Z(R_1) \times \mathrm{Nil}(R_1)^*$, then $(a,b) \perp (0,c)$ with $c \in R_2 \setminus Z(R_2)$. If $(a,b) \in Z(R_1)^* \times \mathrm{Nil}(R_2)^*$, then $(a,b) \perp (c_1,c_2)$ with $c_1 \in a^{\perp}$ and $c_2 \in R_2 \setminus Z(R_2)$. If $(a,b) \in \{0\} \times \mathrm{Nil}(R_2)^*$, then $(a,b) \perp (c_1,c_2)$ with $c_1 \in R_1 \setminus Z(R_1)$ and $c_2 \in b^{\perp}$. □

Recall that the idealization of an R-module M, denoted by $R(+)M$, is the commutative ring $R \times M$ with the following addition and multiplication: $(a,n)+(b,m) = (a+b, n+m)$ and $(a,n)(b,m) = (ab, am+bn)$ for every $(a,n),(b,m) \in R(+)M$, [11]. In the following result we study when $\overline{\Gamma}(R(+)M)$ is complemented. Notice that, if $|M| \geq 4$, then $|Z(R(+)M)| \geq 4$ and $|\mathrm{Nil}(R(+)M)| \geq 3$ and so $\overline{\Gamma}(R(+)M)$ is not complemented, by Theorem 2.4. If $M \cong \mathbb{Z}_3$, then $|\mathrm{Nil}(R(+)\mathbb{Z}_3)| \geq 3$ and so $\overline{\Gamma}(R(+)\mathbb{Z}_3)$ is complemented if and only if R is an integral domain and $\mathbb{Z}_3$ is a torsion free R-module (in particular, $\overline{\Gamma}(R(+)\mathbb{Z}_3)$ is an edge). Then, only the case where $M \cong \mathbb{Z}_2$ is of interest.

Proposition 5.4. *Let R be a non-integral domain such that, for every $x \in Z(R) \cap Z(\mathbb{Z}_2)$, $x^{\perp} \setminus Z(\mathbb{Z}_2) \neq \emptyset$. Then, $\overline{\Gamma}(R(+)\mathbb{Z}_2)$ is complemented if and only if R is reduced and $\overline{\Gamma}(R)$ is complemented.*

Proof. ($\Leftarrow$) We have $Z(R(+)\mathbb{Z}_2) = Z(R) \cup Z(\mathbb{Z}_2)(+)\mathbb{Z}_2 = \{(a, \bar{0}), (a, \bar{1}) \mid a \in Z(R) \cup Z(\mathbb{Z}_2)\}$. Let $a \in Z(R) \cup Z(\mathbb{Z}_2)$. Then we have the following three cases:
Case 1. Suppose that $a \in Z(\mathbb{Z}_2) \setminus Z(R)$. Then, $(a, \bar{0}) \perp (0, \bar{1})$ and $(a, \bar{1}) \perp (0, \bar{1})$.
Case 2. Suppose that $a \in Z(R) \setminus Z(\mathbb{Z}_2)$. Then, since $\overline{\Gamma}(R)$ is complemented, $(a, \bar{0}) \perp (x, \bar{0})$ with $x \in a^{\perp}$, and either $(a, \bar{1}) \perp (y, \bar{0})$ with $y \in a^{\perp} \cap Z(\mathbb{Z}_2)$, or $(a, \bar{1}) \perp (y, \bar{1})$ with $y \in a^{\perp} \setminus Z(\mathbb{Z}_2)$.
Case 3. Suppose that $a \in Z(R) \cap Z(\mathbb{Z}_2)$. Then, there exists $x \in a^{\perp} \setminus Z(\mathbb{Z}_2)$ such that $(a, \bar{0}) \perp (x, \bar{0})$ and $(a, \bar{1}) \perp (x, \bar{0})$.

($\Rightarrow$) Assume that $\overline{\Gamma}(R(+)\mathbb{Z}_2)$ is complemented. Then, $|\mathrm{Nil}(R(+)\mathbb{Z}_2)| = 2$ (by Theorem 2.4). In particular R is reduced. Now, let us prove that $\overline{\Gamma}(R)$ is complemented. Let $a \in Z(R)^*$. If $a \in Z(R) \cap Z(\mathbb{Z}_2)$, then a has an orthogonal, by the hypotheses. If $a \in Z(R) \setminus Z(\mathbb{Z}_2)$, then $(a, \bar{0}) \in Z(R(+)\mathbb{Z}_2)^*$ and so $(a, \bar{0})$ has an orthogonal in $\overline{\Gamma}(R(+)\mathbb{Z}_2)$. Since the vertices adjacent to $(a, \bar{0})$ are of the form $(b, \bar{1})$ or $(b, \bar{0})$ with $b \in Z(R)^*$, $(a, \bar{0}) \perp (c, \bar{0})$ or $(a, \bar{0}) \perp (c, \bar{1})$ for some $c \in a^{\perp}$. Hence, a has c as an orthogonal. □

Acknowledgment. The authors would like to express their gratitude to the referee for the meticulous review of the paper and for providing valuable suggestions that have enhanced the quality of this paper.

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