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DELTA-SHOCK FOR THE NONHOMOGENEOUS PRESSURELESS EULER SYSTEM

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Abstract. We study the Riemann problem for the pressureless Euler system with the source term depending on the time. By means of the variable substitution, two kinds of Riemann solutions including delta-shock and vacuum are constructed. The generalized Rankine-Hugoniot relation and entropy condition of the delta-shock are clarified. Because of the source term, the Riemann solutions are non-self-similar. Moreover, we propose a time-dependent viscous system to show all of the existence, uniqueness and stability of solutions involving the delta-shock by the vanishing viscosity method.

1. Introduction

Consider the pressureless Euler system with the source term

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = -s(t)\rho u, \end{cases} \tag{1.1}$$

where ρ and u denote the density and velocity of the gas, respectively, the source term $-s(t)\rho u$ with $s(t) \in C([0,+\infty);[0,+\infty))$ may represent the time-dependent damping.

If $s(t) = 0$, then the system (1.1) reduces to the pressureless Euler system, which may be used to model the motion of free particles sticking under collision [2, 31] and the formation of large-scale structures in the universe [20]. Since 1994, it has been extensively studied. For instance, the existence of measure solutions to the Riemann problem was proved by Bouchut [1]. Weinan et al. [31] proved the existence of global weak solution and the behavior of such global solution with random initial data. Sheng and Zhang [24] solved the Riemann problems with the use of the characteristic analysis and vanishing viscosity method. It has been shown that delta-shocks and vacuum states appear in Riemann solutions. As regards the delta-shocks, they are an important kind of nonclassical waves for systems of conservation laws. Mathematically, they are

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characterized by the delta functions appearing in the state variables. Physically, they represent the process of concentration of the mass and formation of the universe. For interested readers, see [4,7,10–12,14,16,18,19,21–24,27–29,33,34].

When $s(t) = \alpha > 0$, the system (1.1) becomes the pressureless Euler system with constant damping. The Riemann solutions include two kinds: delta-shock solutions and vacuum solutions. But the viscous stability of the non-self-similar solutions involving delta-shock has not been studied. Moreover, it should be noted that the damping is usually time-dependent. For example, one very good example [8, 15, 25] is $s(t) = \mu/(1 + t)$, where μ is a positive number to describe the scale of the damping. However, there are few works on the stability of Riemann solutions to the system (1.1) with $s(t) = \mu/(1 + t)$ by vanishing viscosity method. Solving both problems is interesting and exciting, but it is also just one of the objectives of the present paper. To this end, we first discuss the Riemann problem for the (1.1) with initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \tag{1.2}$$

By introducing a transformation $u(x, t) = v(x, t)e^{-\int_0^t s(r)dr}$, the (1.1) can be transformed into a homogeneous system, which is a non-strictly hyperbolic system. With the help of the characteristic analysis method, we solve the Riemann problem for the modified system with the same Riemann initial data. There are two kinds of solutions: the one involving vacuum and the other including delta-shock. The generalized Rankine-Hugoniot relation and entropy condition of the delta-shock are clarified. Further, the position, strength and propagation speed of the delta-shock are given explicitly. By using the change of state variables $(\rho, u)(x, t) = (\rho, ve^{-\int_0^t s(r)dr})(x, t)$, the solutions to the original system (1.1) with (1.2) are constructed by contact discontinuities, vacuum or delta-shock connecting two non-constant states $(\rho_\pm, u_\pm e^{-\int_0^t s(r)dr})$. Under the impact of the source term, the Riemann solutions are not self-similar anymore. The contact discontinuities and delta-shock are monotonic curves whose shapes depend on $s(t)$.

In order to explore the stability of the non-self-similar solutions to (1.1)-(1.2), we propose the following time-dependent viscous system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \varepsilon e^{-\int_0^t s(r)dr} \big(\int_0^t e^{-\int_0^y s(r)dr} dy \big) u_{xx} - s(t)\rho u, \end{cases} \tag{1.3}$$

where ε is the coefficient of viscosity. The motivation for the viscous system (1.3) comes from scalar conservation law with time-dependent viscosity

$$u_t + (F(u))_x = G(t)u_{xx}, \quad G(t) > 0.$$

Tupciev [30] and Dafermos [3] independently proposed the systems of hyperbolic conservation laws with time-dependent viscosity $G(t) = \varepsilon t$ such that the systems possess smooth solutions depending only upon the single variable $\xi = x/t$. Tan, Zhang and Zheng [29] initially used the method to study the delta-shocks for a nonstrictly hyperbolic conservation laws. Subsequently, many researchers studied the delta-shocks for various systems by adopting the vanishing viscosity method, see papers [9, 13, 24, 26, 32]. For the triangular systems of conservation laws, De la cruz [5] considered Riemann problem for a 2×2 hyperbolic system with linear damping when $G(t)$ is nonlinear. De la cruz and Juajibioy [6] studied vanishing viscosity limit for Riemann solutions to a 2×2 hyperbolic system with linear damping.

Obviously, we have $e^{-\int_0^t s(r)dr}\big(\int_0^t e^{-\int_0^y s(r)dr}dy\big) = t$ when $s(r) = 0$. By making use of the transformation $u(x,t) = v(x,t)e^{-\int_0^t s(r)dr}$, the (1.3) becomes

$$(1.4) \qquad \begin{cases} \rho_t + (\rho v e^{-\int_0^t s(r)dr})_x = 0, \\ (\rho v)_t + (\rho v^2 e^{-\int_0^t s(r)dr})_x = \varepsilon e^{-\int_0^t s(r)dr}\big(\int_0^t e^{-\int_0^y s(r)dr}dy\big)v_{xx}. \end{cases}$$

It is not hard to observe that if $(\rho^\varepsilon, v^\varepsilon)$ solves the problem (1.4) and (2.2), then $(\rho^\varepsilon, u^\varepsilon)$ given by $(\rho^\varepsilon, u^\varepsilon) = (\rho^\varepsilon, v^\varepsilon e^{-\int_0^t s(r)dr})$ solves the problem (1.3) and (1.2). Though the Riemann solutions to (1.1)-(1.2) are non-self-similar, the solutions of (2.1)-(2.2) still remain similar structure if the initial data belong to a bounded total variation space, except that the weight of the delta-shock is a function determined by $s(t)$. It is natural to hope that these solutions are the limits of corresponding similar solutions of viscous system as $\varepsilon \to 0^+$. Inspired by [5, 6, 17], we introduce the similarity variable $\xi = \frac{x}{\int_0^t e^{-\int_0^y s(r)dr}dy}$ to obtain a two-point boundary value problem of high-order ordinary differential equations with the boundary value in the infinity. Adopting the method in [24], we can show that, the boundary value problem (4.1)-(4.2) has a weak solution $(\rho, v) \in L^1(-\infty, +\infty) \times C^2(-\infty, +\infty)$. Further, it is proved that when $u_- > u_+$, the similarity solution of the system (1.4) with (2.2) converges weakly star to the delta-shock solution of the modified homogeneous system (2.1) with (2.2) as $\varepsilon \to 0^+$. As a consequence, the delta-shock solution of (2.1)-(2.2) is stable under viscous perturbation. The previous analysis, plus the fact that ε is independent of t, show that the delta-shock solution of the system (1.1) with (1.2) is also stable under viscous perturbation.

The remaining part of this paper is organized as follows. Section 2 investigates the Riemann problem for a modified homogeneous system. Section 3 constructs the Riemann solutions of (1.1)-(1.2). Section 4 proves the existence of solutions to the modified viscous system (1.4) with (2.2), and analyzes the limiting behavior of solutions as viscosity term vanishes, that is, $\varepsilon \to 0^+$.

2. Riemann solutions of a modified homogeneous system

In this section we construct the solutions of Riemann problem associated with a modified homogeneous system (2.1). Under the transformation $u(x,t) = v(x,t)e^{-\int_0^t s(r)dr} = v(x,t)e^{M(t)}$, the system (1.1) is transformed into

$$(2.1) \qquad \begin{cases} \rho_t + (\rho v e^{M(t)})_x = 0, \\ (\rho v)_t + (\rho v^2 e^{M(t)})_x = 0, \end{cases}$$

and the initial data (1.2) changes to

$$(\rho, v)(x,0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \tag{2.2}$$

The (2.1) has a double characteristic root $\lambda = ve^{M(t)}$. The right characteristic vector is $\overrightarrow{r} = (1,0)^T$ satisfying $\nabla\lambda\cdot\overrightarrow{r} = 0$. As a result, the (2.1) is non-strictly hyperbolic and λ is linearly degenerate.

A bounded discontinuity at $x = x(t)$ satisfies the Rankine-Hugoniot relation

$$\begin{cases} -x'(t)[\rho] + [\rho v e^{M(t)}] = 0, \\ -x'(t)[\rho v] + [\rho v^2 e^{M(t)}] = 0, \end{cases} \tag{2.3}$$

where $[\rho] = \rho_l - \rho_r$ with $\rho_l = \rho(x(t)-0,t)$, $\rho_r = \rho(x(t)+0,t)$, in which $[\rho]$ denotes the jump of ρ across the discontinuity. Eliminating $x'(t)$ in (2.3), we get

$$(\rho_l v_l - \rho_r v_r)(\rho_l v_l e^{M(t)} - \rho_r v_r e^{M(t)}) = (\rho_l - \rho_r)(\rho_l v_l^2 e^{M(t)} - \rho_r v_r^2 e^{M(t)}), \tag{2.4}$$

which yields

$$\rho_l \rho_r (v_l - v_r)^2 = 0. \tag{2.5}$$

Obviously, when $v_l = v_r$, the two states (ρ_l, v_l) and (ρ_r, v_r) are connected only by a contact discontinuity J with the propagation speed

$$x'(t) = v_l e^{M(t)} = v_r e^{M(t)}. \tag{2.6}$$

Let us now construct the Riemann solutions in two cases. For the case $u_- < u_+$, we obtain the Riemann solution which consists of two contact discontinuities J_1 and J_2 and a vacuum state (denoted by Vac). The solution can be represented as

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x_1(t), \\ \text{Vac}, & x_1(t) < x < x_2(t), \\ (\rho_+, u_+), & x > x_2(t), \end{cases} \tag{2.7}$$

in which the locations of the J_1 and J_2 are

$$x_1(t) = u_- \int_0^t e^{M(s)} ds, \qquad x_2(t) = u_+ \int_0^t e^{M(s)} ds \tag{2.8}$$

and the propagation speeds of J_1 and J_2 are $x_1'(t) = u_- e^{M(t)}$ and $x_2'(t) = u_+ e^{M(t)}$, respectively.

When $u_- > u_+$, the solution can not be constructed by the contact discontinuities. Because the characteristic lines from the initial data overlap in $\Omega = \{(x,t) \,|\, u_+ \int_0^t e^{M(s)} ds \le x(t) \le u_- \int_0^t e^{M(s)} ds\}$, the singularity of solutions must develop in this domain. Therefore there is no solution in bounded variation space. Now we set out to show that even when the initial data is smooth, ρ and v_x must blow up simultaneously at a finite time.

Consider (2.1) with smooth initial value $(\rho(x,0),v(x,0)) = (\rho_0(x),v_0(x))$ satisfying $v_0'(x)<0$. The characteristic equations of system (2.1) are found to be

$$\frac{dx}{dt}=ve^{M(t)},\qquad \frac{dv}{dt}=0,\qquad \frac{d\rho}{dt}=-\rho v_x e^{M(t)}. \tag{2.9}$$

So the characteristic curve passing through any given point $(0,a)$ on the x-axis is expressed as

$$x=a+v_0(a)\int_0^t e^{M(s)}ds \tag{2.10}$$

on which v takes the constant value $v_0(a)$. Combining (2.1) with (2.9)-(2.10), we arrive at

$$v_x=\frac{v_0'(a)}{1+(\int_0^t e^{M(s)}ds)v_0'(a)},\qquad \rho=\frac{\rho_0(a)}{1+(\int_0^t e^{M(s)}ds)v_0'(a)}. \tag{2.11}$$

In virtue of $v_0'(a)<0$, it follows from (2.11) that

$$\lim_{t\to t^*}(\rho,v_x)=(\infty,\infty), \tag{2.12}$$

where t^* is uniquely determined by $\int_0^{t^*}e^{M(s)}ds=-\frac{1}{v_0'(a)}$ (let $v_0'(a)$ be sufficiently small if necessary). The formula (2.12) shows that ρ and v_x must blow up simultaneously at a finite time.

In such circumstances, motivated by [24, 32], we will seek the solution containing delta-shock. To this end, the following definitions of two-dimensional weighted delta function and delta-shock solution are introduced.

Definition 2.1. A two-dimensional weighted delta function $\omega(s)\delta_S$ supported on a smooth curve $S=\{(x(s),t(s)):c\le s\le d\}$ is defined by

$$\langle\omega(s)\delta_S,\phi(x,t)\rangle=\int_c^d\omega(s)\phi(x(s),t(s))ds \tag{2.13}$$

for all test functions $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$.

Definition 2.2. A pair (ρ,v) is called a delta-shock type solution to the system (2.1) in the sense of distributions if there exist a smooth curve $S=\{(x(t),t): 0\le t\le\infty\}$ and a weight $\omega\in C^1(S)$ such that ρ and v are represented in the following form

$$\rho(x,t)=\rho_0(x,t)+\omega(t)\delta_S,\quad v(x,t)=v_0(x,t),\quad v(x,t)|_S=v_\delta(t), \tag{2.14}$$

where $\rho_0(x,t)=\rho_l(x,t)-[\rho]H(x-x(t))$, $v_0(x,t)=v_l(x,t)-[v]H(x-x(t))$, in which $(\rho_l,v_l)(x,t)$ and $(\rho_r,v_r)(x,t)$ are piecewise smooth solutions to the system (2.1), $H(x)$ is the Heaviside function whose value is zero for negative argument and one for positive argument, and it satisfies

$$\langle\rho,\phi_t\rangle+\langle\rho ve^{M(t)},\phi_x\rangle=0,\quad \langle\rho v,\phi_t\rangle+\langle\rho v^2e^{M(t)},\phi_x\rangle=0 \tag{2.15}$$

for all test functions $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$, in which

$$\langle \rho, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 \phi dxdt + \langle \omega(t)\delta_S, \phi \rangle,$$
$$\langle \rho v, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 v_0 \phi dxdt + \langle \omega(t) v_\delta(t) \delta_S, \phi \rangle.$$

Based on these definitions, the delta-shock solution to the system (2.1) can be written in the following form

$$(\rho, v)(x,t) = \begin{cases} (\rho_l, v_l)(x,t), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_r, v_r)(x,t), & x > x(t), \end{cases} \tag{2.16}$$

where $\delta(\cdot)$ is the standard Dirac measure.

Theorem 2.3. *A pair (ρ, v) of the form* (2.16) *is a solution to the system* (2.1) *in the sense of distributions if the generalized Rankine-Hugoniot relation*

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t) = v_\delta(t) e^{M(t)}, \\ \frac{d\omega(t)}{dt} = -[\rho]\sigma(t) + [\rho v e^{M(t)}], \\ \frac{d(\omega(t) v_\delta(t))}{dt} = -[\rho v]\sigma(t) + [\rho v^2 e^{M(t)}] \end{cases} \tag{2.17}$$

holds.

Proof. For an arbitrary test function $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$, with the use of the Green's formula and integration by parts, we calculate

$$\begin{aligned} &\langle \rho v, \phi_t \rangle + \langle \rho v^2 e^{M(t)}, \phi_x \rangle \\ &= \int_0^{+\infty} \int_{-\infty}^{x(t)} (\rho_l v_l \phi_t + \rho_l v_l^2 e^{M(t)} \phi_x) dxdt \\ &\quad + \int_0^{+\infty} \int_{x(t)}^{+\infty} (\rho_r v_r \phi_t + \rho_r v_r^2 e^{M(t)} \phi_x) dxdt \\ &\quad + \int_0^{+\infty} \omega(t) v_\delta(t) (\phi_t + v_\delta(t) e^{M(t)} \phi_x) dt \\ &= \int_0^{+\infty} \int_{-\infty}^{x(t)} \{(\rho_l v_l \phi)_t + (\rho_l v_l^2 e^{M(t)} \phi)_x\} dxdt \\ &\quad + \int_0^{+\infty} \int_{x(t)}^{+\infty} \{(\rho_r v_r \phi)_t + (\rho_r v_r^2 e^{M(t)} \phi)_x\} dxdt + \int_0^{+\infty} \omega(t) v_\delta(t) \frac{d\phi}{dt} dt \\ &= \oint \{-\rho_l v_l \phi dx + \rho_l v_l^2 e^{M(t)} \phi\} dt + \oint \{-\rho_r v_r \phi dx + \rho_r v_r^2 e^{M(t)} \phi\} dt \\ &\quad - \int_0^{+\infty} \frac{d(\omega(t) v_\delta(t))}{dt} \phi dt \end{aligned} \tag{2.18}$$

$$= \int_0^{+\infty} \Big(-\rho_l v_l \frac{dx}{dt} + \rho_l v_l^2 e^{M(t)} + \rho_r v_r \frac{dx}{dt} - \rho_r v_r^2 e^{M(t)} \Big) \phi dt$$
$$- \int_0^{+\infty} \frac{d(\omega(t) v_\delta(t))}{dt} \phi dt$$
$$= \int_0^{+\infty} \Big\{ -[\rho v]\sigma(t) + [\rho v^2 e^{M(t)}] - \frac{d(\omega(t) v_\delta(t))}{dt} \Big\} \phi dt.$$

A completely similar argument leads to

$$\langle \rho, \phi_t \rangle + \langle \rho v e^{M(t)}, \phi_x \rangle = \int_0^{+\infty} \Big\{ -[\rho]\sigma(t) + [\rho v e^{M(t)}] - \frac{d\omega(t)}{dt} \Big\} \phi dt. \tag{2.19}$$

This completes the proof. □

The generalized Rankine-Hugoniot relation (2.17) reflects the exact relationship among the location, weight and propagation speed of the discontinuity. And so beyond that, in order that such a discontinuity is unique, we supplement the entropy condition

$$\lambda(\rho_r, v_r) < \frac{dx(t)}{dt} < \lambda(\rho_l, v_l). \tag{2.20}$$

The equation (2.20) means that all characteristics on both sides of the discontinuity are incoming. A discontinuity is known as a delta-shock (symbolized by δ) of the system (2.1) if it satisfies generalized Rankine-Hugoniot relation (2.17) and entropy condition (2.20).

In what follows, we continue to solve the Riemann problem (2.1) and (2.2) for the case $u_- > u_+$. At this moment, the Riemann solution is a delta-shock of the form, besides two constant states,

$$(\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t). \end{cases} \tag{2.21}$$

The generalized Rankine-Hugoniot relation (2.17) with

$$t = 0: \ x(0) = 0, \ \omega(0) = 0, \tag{2.22}$$

and the entropy condition

$$u_+ < v_\delta(t) < u_- \tag{2.23}$$

enable us to determine $x(t)$, $\omega(t)$ and $v_\delta(t)$ uniquely. Integrating (2.17) we obtain

$$\begin{cases} \omega(t) = -(\rho_- - \rho_+)x(t) + (\rho_- u_- - \rho_+ u_+) \int_0^t e^{M(s)} ds, \\ \omega(t) v_\delta(t) = -(\rho_- u_- - \rho_+ u_+)x(t) + (\rho_- u_-^2 - \rho_+ u_+^2) \int_0^t e^{M(s)} ds, \end{cases} \tag{2.24}$$

which gives

$$(2.25)\qquad \begin{aligned}&(\rho_- - \rho_+)v_\delta(t)x(t) - (\rho_- u_- - \rho_+ u_+)\Big(v_\delta(t)\int_0^t e^{M(s)}ds + x(t)\Big)\\ &+ \big(\rho_- u_-^2 - \rho_+ u_+^2\big)\int_0^t e^{M(s)}ds = 0.\end{aligned}$$

The equation (2.25) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}(\rho_- - \rho_+)x^2(t) - (\rho_- u_- - \rho_+ u_+)x(t)\int_0^t e^{M(s)}ds\\ &+ \frac{1}{2}\big(\rho_- u_-^2 - \rho_+ u_+^2\big)\Big(\int_0^t e^{M(s)}ds\Big)^2\Big) = 0,\end{aligned}$$

which means that

$$(2.26)\qquad \begin{aligned}&\frac{1}{2}(\rho_- - \rho_+)x^2(t) - (\rho_- u_- - \rho_+ u_+)x(t)\int_0^t e^{M(s)}ds\\ &+ \frac{1}{2}\big(\rho_- u_-^2 - \rho_+ u_+^2\big)\Big(\int_0^t e^{M(s)}ds\Big)^2 = 0.\end{aligned}$$

When $\rho_- \neq \rho_+$, solving the equation (2.26) gives

$$(2.27)\qquad x(t) = \frac{(\rho_- u_- - \rho_+ u_+) \pm \sqrt{\rho_- \rho_+ (u_- - u_+)^2}}{\rho_- - \rho_+}\int_0^t e^{M(s)}ds.$$

Integrating the first equation in (2.17) yields $x(t) = \int_0^t v_\delta(s)e^{M(s)}ds$. And so $v_\delta(t)$ is a constant in virtue of (2.27). Then there is a lengthly but straightforward computation to show that

$$(2.28)\qquad \begin{aligned}v_\delta &= \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}},\\ \omega(t) &= \sqrt{\rho_- \rho_+}(u_- - u_+)\int_0^t e^{M(s)}ds,\\ x(t) &= v_\delta \int_0^t e^{M(s)}ds.\end{aligned}$$

If $\rho_- = \rho_+$, it follows from (2.24) and (2.26) that

$$(2.29)\qquad v_\delta = \frac{u_- + u_+}{2},\ \omega(t) = \rho_-(u_- - u_+)\int_0^t e^{M(s)}ds,\ x(t) = v_\delta \int_0^t e^{M(s)}ds.$$

The result can now be summarised as follows:

Theorem 2.4. *When $u_- > u_+$, the Riemann problem* (2.1) *and* (2.2) *admits a unique entropy solution in the sense of distributions, which can be written in the form*

$$(2.30)\qquad (\rho, v)(x,t) = \begin{cases} (\rho_-, u_-), & x < v_\delta \int_0^t e^{M(s)}ds,\\ \big(\omega(t)\delta\big(x - v_\delta \int_0^t e^{M(s)}ds\big), v_\delta\big), & x = v_\delta \int_0^t e^{M(s)}ds,\\ (\rho_+, u_+), & x > v_\delta \int_0^t e^{M(s)}ds,\end{cases}$$

where v_δ and $\omega(t)$ are shown in (2.28).

3. Construction of Riemann solutions to (1.1)-(1.2)

According to the transformation of state variables

$$(\rho, u)(x,t) = (\rho, v e^{M(t)})(x,t),$$

we will, in this section, construct the Riemann solutions of the original system (1.1) with (1.2).

If $u_- < u_+$, the Riemann solutions of (1.1)-(1.2) can be written as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- e^{M(t)}), & x < x_1(t), \\ \text{Vac}, & x_1(t) < x < x_2(t), \\ (\rho_+, u_+ e^{M(t)}), & x > x_2(t), \end{cases} \tag{3.1}$$

where $x_1(t)$ and $x_2(t)$ are given by (2.8).

On differentiation $x_1(t)$ and $x_2(t)$ with respect to t, we have

$$x_1'(t) = u_- e^{M(t)},\ x_1''(t) = -u_- e^{M(t)} s(t),\ x_2'(t) = u_+ e^{M(t)},\ x_2''(t) = -u_+ e^{M(t)} s(t),$$

which means that the $x_i(t)$ $(i = 1, 2)$ must be monotonic. Since the sign of $s(t)$ is assumed to be unchanging, the curves are concave or convex.

Definition 3.1. A pair (ρ, u) is called a delta-shock solution to the system (1.1) in the sense of distributions if there exist a smooth curve $S = \{(x(t), t) : 0 \leq t \leq \infty\}$ and a weight $\omega \in C^1(S)$ such that ρ and u are represented in the following form

$$\rho(x,t) = \rho_0(x,t) + \omega(t)\delta_S, \quad u(x,t) = u_0(x,t), \quad u(x,t)|_S = u_\delta(t), \tag{3.2}$$

where $\rho_0(x,t) = \rho_l(x,t) - [\rho]H(x - x(t))$, $u_0(x,t) = u_l(x,t) - [u]H(x - x(t))$, in which $(\rho_l, u_l)(x,t)$ and $(\rho_r, u_r)(x,t)$ are piecewise smooth solutions to the system (1.1), and it satisfies

$$\langle \rho, \phi_t \rangle + \langle \rho u, \phi_x \rangle = 0, \quad \langle \rho u, \phi_t \rangle + \langle \rho u^2, \phi_x \rangle = \langle s(t) \rho u, \phi \rangle \tag{3.3}$$

for all test functions $\phi \in C_0^\infty(R \times R^+)$, where

$$\langle \rho, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 \phi dx dt + \langle \omega(t)\delta_S, \phi \rangle,$$
$$\langle \rho u, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0 u_0 \phi dx dt + \langle \omega(t) u_\delta(t) \delta_S, \phi \rangle.$$

According to the above definition, if $u_- > u_+$, we construct the delta-shock solution of (1.1) and (1.2) in the form

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_- e^{M(t)}), & x < x(t), \\ (\omega(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+ e^{M(t)}), & x > x(t), \end{cases} \tag{3.4}$$

which should satisfy the following generalized Rankine-Hugoniot relation

$$\begin{cases} \frac{dx(t)}{dt} = u_\delta(t), \\ \frac{d\omega(t)}{dt} = -[\rho]u_\delta(t) + [\rho u], \\ \frac{d(\omega(t)u_\delta(t))}{dt} = -[\rho u]u_\delta(t) + [\rho u^2] - s(t)\omega(t)u_\delta(t), \end{cases} \tag{3.5}$$

where $u_\delta(t)e^{-M(t)}$ is a constant, the jumps across the discontinuity are

$$[\rho u] = \rho_- u_- e^{M(t)} - \rho_+ u_+ e^{M(t)}, \quad [\rho u^2] = \rho_-(u_- e^{M(t)})^2 - \rho_+(u_+ e^{M(t)})^2. \tag{3.6}$$

In addition, the over-compressive entropy condition for the delta-shock

$$u_+ e^{M(t)} < u_\delta(t) < u_- e^{M(t)} \tag{3.7}$$

should be imposed in order to ensure the uniqueness.

Theorem 3.2. *When $u_- > u_+$, the Riemann problem* (1.1)-(1.2) *has a delta-shock solution which can be expressed as the formula* (3.4)*, where*

$$u_\delta(t) = v_\delta e^{M(t)},\ \omega(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)\int_0^t e^{M(s)}ds,\ x(t) = v_\delta \int_0^t e^{M(s)}ds, \tag{3.8}$$

in which v_δ is shown in (2.28)*.*

Proof. The second equation in (3.5) gives

$$\frac{d\omega(t)}{dt} = -u_\delta(t)(\rho_- - \rho_+) + (\rho_- u_- e^{M(t)} - \rho_+ u_+ e^{M(t)}). \tag{3.9}$$

Remember that $u_\delta(t)e^{-M(t)}$ is a constant, the third equality of (3.5) can be reduced to

$$\begin{aligned} \frac{d\omega(t)}{dt}u_\delta(t) = &\ - u_\delta(t)(\rho_- u_- e^{M(t)} - \rho_+ u_+ e^{M(t)}) \\ &+ (\rho_-(u_- e^{M(t)})^2 - \rho_+(u_+ e^{M(t)})^2). \end{aligned} \tag{3.10}$$

Together the equation (3.9) with the equation (3.10) we have

$$(\rho_- - \rho_+)(u_\delta(t)e^{-M(t)})^2 - 2(\rho_- u_- - \rho_+ u_+)(u_\delta(t)e^{-M(t)}) + (\rho_- u_-^2 - \rho_+ u_+^2) = 0. \tag{3.11}$$

Therefore, $u_\delta(t) = v_\delta e^{M(t)}$ can be obtained by virtue of the entropy condition (3.7). Then it follows from (3.5) that (3.8). This completes the proof. □

4. Stability of solutions to the systems (1.1) and (2.1)

This section establishes the stability of Riemann solutions to the systems (1.1) and (2.1). First, we show the existence of solutions to the modified viscous system (1.4) with (2.2). Then we analyse the limiting behavior of the solutions as $\varepsilon \to 0^+$. Because the delta-shock is what we are interested in, we mainly discuss the case $u_- > u_+$. A similar analysis can be done for the case $u_- < u_+$.

We look for the solutions which depend on the variable $\xi = \frac{x}{\int_0^t e^{-\int_0^y s(r)dr}dy}$. Then the (1.4) and initial data (2.2) become

$$\left\{\begin{array}{l} -\xi\rho_\xi + (\rho v)_\xi = 0, \\ -\xi(\rho v)_\xi + (\rho v^2)_\xi = \varepsilon v_{\xi\xi} \end{array}\right. \tag{4.1}$$

and

$$(\rho, v)(\pm\infty) = (\rho_\pm, u_\pm). \tag{4.2}$$

By the same arguments as used in [24], we can arrive at the following theorem.

Theorem 4.1. *There exists a weak solution* $(\rho, v) \in L^1(-\infty, +\infty) \times C^2(-\infty, +\infty)$ *for the boundary value problem* (4.1) *with* (4.2).

In what follows, we analyze the limiting behavior of the solutions depending on the variable $\xi = \frac{x}{\int_0^t e^{-\int_0^y s(r)dr}dy}$ of (1.4) with (2.2) as $\varepsilon \to 0^+$.

Lemma 4.2. *Assume that* $u_- > u_+$ *and* $(\rho^\varepsilon(\xi), v^\varepsilon(\xi))$ *is a solution of* (4.1) *and* (4.2) *for fixed* ε. *Let* ξ_σ^ε *be the unique point satisfying* $\xi_\sigma^\varepsilon = v^\varepsilon(\xi_\sigma^\varepsilon)$, $\xi_\sigma = \lim_{\varepsilon\to 0^+} \xi_\sigma^\varepsilon$ *(pass to a subsequence if necessary). Then for each* $\eta > 0$,

$$\lim_{\varepsilon\to 0^+} v_\xi^\varepsilon(\xi) = 0 \quad \textit{for} \quad |\xi - \xi_\sigma| \geq \eta,$$

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(\xi) = \left\{\begin{array}{ll} u_- & \textit{for} \ \ \xi \leq \xi_\sigma - \eta, \\ u_+ & \textit{for} \ \ \xi \geq \xi_\sigma + \eta \end{array}\right.$$

uniformly in the above intervals.

Proof. Take $\xi_3 = \xi_\sigma - \eta/2$, and let ε be small such that $\xi_\sigma^\varepsilon > \xi_3 + \eta/4$. When $\xi < \xi_3$, we have

$$\rho^\varepsilon(\xi) = \rho_- \exp\Big(\int_{-\infty}^{\xi} \frac{-(v^\varepsilon(s))'}{v^\varepsilon(s) - s} ds\Big) \leq \rho_- \frac{u_- - \xi}{v^\varepsilon(\xi) - \xi},$$

which gives $\rho^\varepsilon(\xi)(v^\varepsilon(\xi) - \xi) \leq \rho_-(u_- - \xi)$. Integrating the second equation of (4.1) twice on $[\xi, \xi_3]$ yields

$$v^\varepsilon(\xi_3) - v^\varepsilon(\xi) \leq (v^\varepsilon(\xi_3))' \int_{\xi-\xi_3}^{0} \exp\frac{\rho_-}{\varepsilon}\Big((u_- - \xi_3)r - \frac{1}{2}r^2\Big)dr.$$

From this it follows that in the limit $\xi \to -\infty$,

$$\begin{aligned} u_- - u_+ &\geq -(v^\varepsilon(\xi_3))' \int_{-\infty}^{0} \exp\frac{\rho_-}{2\varepsilon}\big(2(u_- - \xi_3)r - r^2\big)dr \\ &\geq -(v^\varepsilon(\xi_3))' \int_{-\infty}^{0} \exp\frac{\rho_-}{2\varepsilon}\big(2\big(u_- - \xi_3\big)r - r^2\big)dr \\ &\geq -(v^\varepsilon(\xi_3))'\varepsilon M_1, \end{aligned}$$

where M_1 is a positive constant independent of ε. Consequently, we have

$$|(v^\varepsilon(\xi_3))'| \le \frac{u_- - u_+}{\varepsilon M_1}$$

and

$$|(v^\varepsilon(\xi))'| \le \frac{u_- - u_+}{\varepsilon M_1}\exp\Big(\int_\xi^{\xi_3} -\frac{\rho^\varepsilon(v^\varepsilon - s)}{\varepsilon}ds\Big).$$

When $\xi < \xi_3$, one has

$$\rho^\varepsilon(\xi) = \rho_-\exp\Big(\int_{-\infty}^{\xi} \frac{-(v^\varepsilon(s))'}{v^\varepsilon(s)-s}ds\Big) \ge \rho_-\frac{v^\varepsilon(\xi_3)-\xi}{v^\varepsilon(\xi)-\xi},$$

which leads to

$$\rho^\varepsilon(\xi)(v^\varepsilon(\xi)-\xi) \ge \rho_-(v^\varepsilon(\xi_3)-\xi) \quad \text{for } \ \xi < \xi_3.$$

Hence, we arrive at

$$|(v^\varepsilon(\xi))'| \le \frac{u_- - u_+}{\varepsilon M_1}\exp\Big(-\frac{\rho_-}{\varepsilon}\int_\xi^{\xi_3}(v^\varepsilon(\xi_3)-s)ds\Big),$$

which shows that

$$\lim_{\varepsilon\to 0^+} v_\xi^\varepsilon(\xi) = 0 \quad \text{uniformly for } \ \xi \le \xi_\sigma - \eta.$$

For $\xi < \xi_4 \le \xi_\sigma - \eta$, noticing that

$$\begin{aligned}\rho^\varepsilon(\xi)(v^\varepsilon(\xi)-\xi) &\ge \rho_-(v^\varepsilon(\xi_4)-\xi)\\ &> \rho_-(\xi_\sigma^\varepsilon - \xi_4) \to \rho_-(\xi_\sigma-\xi_4) > 0 \ \text{ as } \ \varepsilon\to 0^+,\end{aligned}$$

we have

$$\begin{aligned}|v^\varepsilon(\xi_4) - v^\varepsilon(\xi)| &= |(v^\varepsilon(\xi_4))'|\int_\xi^{\xi_4}\exp\Big(\int_r^{\xi_4} -\frac{\rho^\varepsilon(v^\varepsilon-s)}{\varepsilon}ds\Big)dr\\ &\le |(v^\varepsilon(\xi_4))'|\int_\xi^{\xi_4}\exp\Big(\frac{-M_2(\xi_4-r)}{\varepsilon}\Big)dr\\ &\le \frac{\varepsilon}{M_2}|(v^\varepsilon(\xi_4))'|\Big(1-\exp\Big(\frac{-M_2(\xi_4-\xi)}{\varepsilon}\Big)\Big),\end{aligned}$$

where $M_2 = \frac{\rho_-(\xi_\sigma-\xi_4)}{2}$. Let ξ approach negative infinity, one obtains

$$|v^\varepsilon(\xi_4) - u_-| \le \frac{\varepsilon}{M_2}|(v^\varepsilon(\xi_4))'|.$$

Thus we have

$$\lim_{\varepsilon\to 0^+} v^\varepsilon(\xi) = u_- \quad \text{uniformly for } \ \xi \le \xi_\sigma - \eta.$$

The similar proof works for $\xi \ge \xi_\sigma + \eta$. The proof is completed. □

Lemma 4.3. *For any* $\eta > 0$,

$$\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = \begin{cases} \rho_- & \text{for } \xi < \xi_\sigma - \eta, \\ \rho_+ & \text{for } \xi > \xi_\sigma + \eta \end{cases}$$

uniformly.

Proof. For $\xi < \xi_5 < \xi_\sigma - \eta$, we have

$$\rho_- \frac{v^\varepsilon(\xi_5) - \xi}{v^\varepsilon(\xi) - \xi} \le \rho^\varepsilon(\xi) \le \rho_- \frac{u_- - \xi}{v^\varepsilon(\xi) - \xi}.$$

By taking use of Lemma 4.2, we have $\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = \rho_-$ for $\xi < \xi_\sigma - \eta$. Similarly, the conclusion for $\xi > \xi_\sigma + \eta$ can be obtained. The proof is finished. □

Next we investigate the limiting behavior of ρ^ε in the neighborhood of $\xi = \sigma$ as $\varepsilon \to 0^+$. Setting

$$\sigma = \xi_\sigma = \lim_{\varepsilon\to 0^+} \xi_\sigma^\varepsilon = \lim_{\varepsilon\to 0^+} v^\varepsilon(\xi_\sigma^\varepsilon) = v(\sigma) = v_\delta, \tag{4.3}$$

we arrive at

$$u_+ < \sigma < u_-. \tag{4.4}$$

We choose $\varphi \in C_0^\infty[\xi_1, \xi_2]$ with $\xi_1 < \sigma < \xi_2$ satisfying $\varphi(\xi) \equiv \varphi(\sigma)$ for ξ in a neighborhood Ω of $\xi = \sigma$. When $\xi_\sigma^\varepsilon \in \Omega \subset (\xi_1, \xi_2)$, it follows from (4.1) that

$$-\int_{\xi_1}^{\xi_2} \rho^\varepsilon(v^\varepsilon - \xi)\varphi' d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon \varphi d\xi = 0 \tag{4.5}$$

and

$$-\int_{\xi_1}^{\xi_2} \rho^\varepsilon v^\varepsilon(v^\varepsilon - \xi)\varphi' d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon v^\varepsilon \varphi d\xi = \varepsilon \int_{\xi_1}^{\xi_2} v^\varepsilon \varphi'' d\xi. \tag{4.6}$$

For $\alpha_1, \alpha_2 \in \Omega$ such that $\alpha_1 < \sigma < \alpha_2$, in view of Lemmas 4.2-4.3, we calculate

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_{\xi_1}^{\xi_2} \rho^\varepsilon v^\varepsilon (v^\varepsilon - \xi)\varphi' d\xi \\ &= \int_{\xi_1}^{\alpha_1} \rho_- u_-(u_- - \xi)\varphi' d\xi + \int_{\alpha_2}^{\xi_2} \rho_+ u_+(u_+ - \xi)\varphi' d\xi \\ &= (\rho_- u_-^2 - \rho_+ u_+^2 - \rho_- u_- \alpha_1 + \rho_+ u_+ \alpha_2)\varphi(\sigma) \\ &\quad + \int_{\xi_1}^{\alpha_1} \rho_- u_- \varphi(\xi) d\xi + \int_{\alpha_2}^{\xi_2} \rho_+ u_+ \varphi(\xi) d\xi. \end{aligned}$$

From this it follows that in the limit $\alpha_1 \to \sigma^-$ and $\alpha_2 \to \sigma^+$,

$$\lim_{\varepsilon\to 0^+} \int_{\xi_1}^{\xi_2} \rho^\varepsilon v^\varepsilon(v^\varepsilon - \xi)\varphi' d\xi = (-\sigma[\rho v] + [\rho v^2])\varphi(\sigma) + \int_{\xi_1}^{\xi_2} H_0(\xi - \sigma)\varphi(\xi) d\xi,$$

where $[\rho v]=\rho_- u_- - \rho_+ u_+$, $[\rho v^2]=\rho_- u_-^2 - \rho_+ u_+^2$, $H_0(x)$ is a step function: $H_0(x)=\rho_- u_-$ for $x<0$ and $H_0(x)=\rho_+ u_+$ for $x>0$. Combining with (4.6), we get

$$\lim_{\varepsilon\to 0^+}\int_{\xi_1}^{\xi_2}(\rho^\varepsilon v^\varepsilon - H_0(\xi-\sigma))\varphi(\xi)d\xi = (-\sigma[\rho v]+[\rho v^2])\varphi(\sigma) \tag{4.7}$$

for all sloping test functions $\varphi\in C_0^\infty[\xi_1,\xi_2]$. By the approximation process, we obtain that (4.7) holds for all $\varphi\in C_0^\infty[\xi_1,\xi_2]$. In consequence, $\rho^\varepsilon v^\varepsilon$ converges to a sum of a step function $H_0(\xi-\sigma)$ and a weighted Dirac delta function with the strength $(-\sigma[\rho v]+[\rho v^2])$ in the weak star topology of $C_0^\infty(\mathbb{R})$.

In a similar way, it follows from (4.5) that

$$\lim_{\varepsilon\to 0^+}\int_{\xi_1}^{\xi_2}(\rho^\varepsilon - H_\rho(\xi-\sigma))\varphi(\xi)d\xi = (-\sigma[\rho]+[\rho v])\varphi(\sigma) \tag{4.8}$$

for all $\varphi\in C_0^\infty[\xi_1,\xi_2]$, where $H_\rho(x)$ is a step function: $H_\rho(x)=\rho_-$ for $x<0$ and $H_\rho(x)=\rho_+$ for $x>0$. Consequently, ρ^ε converges to a sum of a step function $H_\rho(\xi-\sigma)$ and a weighted Dirac delta function with the weight $\omega_0=-\sigma[\rho]+[\rho v]$ in the weak star topology of $C_0^\infty(\mathbb{R})$.

In addition, if one takes the test function as $\varphi/(\bar{v}^\varepsilon+\nu)$ in (4.6), where $\bar{v}^\varepsilon$ is a modified function satisfying $v^\varepsilon(\sigma)$ inside Ω and v^ε outside Ω, and lets $\nu\to 0^+$, then the following formula

$$\lim_{\varepsilon\to 0^+}\int_{\xi_1}^{\xi_2}(\rho^\varepsilon - H_\rho(\xi-\sigma))\varphi(\xi)d\xi\cdot\sigma = (-\sigma[\rho v]+[\rho v^2])\varphi(\sigma) \tag{4.9}$$

can be obtained. Combining (4.8) with (4.9), one has

$$\sigma^2[\rho]-2\sigma[\rho v]+[\rho v^2]=0. \tag{4.10}$$

On solving equation (4.10) for σ, we have $\sigma=v_\delta=\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}$ satisfying the formula (4.4). Put σ into $\omega_0=-\sigma[\rho]+[\rho v]$ to get $\omega_0=\sqrt{\rho_-\rho_+}(u_- - u_+)$.

Hence, we come to a conclusion.

Theorem 4.4. *When $u_->u_+$, let $(\rho^\varepsilon,v^\varepsilon)(x,t)$ be the solution depending on the variable $\xi=\frac{x}{\int_0^t e^{-\int_0^y s(r)dr}dy}$ of (1.4), (2.2). Then the limit functions $\rho(x,t)$ and $v(x,t)$ of $\rho^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ exist in the sense of distributions, and $(\rho,v)(x,t)$ solves (2.1) with (2.2). The solution $(\rho,v)(x,t)$ can be explicitly shown as*

$$(\rho,v)(x,t)=\begin{cases}(\rho_-,u_-), & x<v_\delta\int_0^t e^{M(s)}ds,\\ \big(\omega_0\int_0^t e^{M(s)}ds\cdot\delta\big(x-v_\delta\int_0^t e^{M(s)}ds\big),v_\delta\big), & x=v_\delta\int_0^t e^{M(s)}ds,\\ (\rho_+,u_+), & x>v_\delta\int_0^t e^{M(s)}ds,\end{cases} \tag{4.11}$$

in which

$$v_\delta=\frac{\sqrt{\rho_-}u_-+\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}},\qquad \omega_0=\sqrt{\rho_-\rho_+}(u_- - u_+). \tag{4.12}$$

It should be noted that the difference between the strength ω_0 and weight $\omega(t)$ in (2.28) is due to the result of the introduction of the similarity variable.

Theorem 4.4 shows that the delta-shock solution of (2.1) and (2.2) is stable under viscous perturbation. Moreover, it is not difficult to see that, if $(\rho^\varepsilon, v^\varepsilon)(x,t)$ solves the problem (1.4) and (2.2), then $(\rho^\varepsilon, u^\varepsilon)(x,t)$ given by $(\rho^\varepsilon, u^\varepsilon)(x,t) = (\rho^\varepsilon, v^\varepsilon e^{M(t)})(x,t)$ solves the problem (1.3) and (1.2). Note that ε is independent of t, then the limit, $\lim_{\varepsilon\to 0^+}(\rho^\varepsilon, u^\varepsilon)(x,t) = (\rho, u)(x,t) = (\rho, ve^{M(t)})(x,t)$ exists in the sense of distributions, and $(\rho, u)(x,t)$ solves (1.1) with (1.2). The $(\rho, u)(x,t)$ can be given explicitly as

$$\begin{aligned}&(\rho, u)(x,t)\\ &= \begin{cases} (\rho_-, u_- e^{M(t)}), & x < v_\delta \int_0^t e^{M(s)} ds, \\ \big(\omega_0 \int_0^t e^{M(s)} ds \cdot \delta\big(x - v_\delta \int_0^t e^{M(s)} ds\big), v_\delta e^{M(t)}\big), & x = v_\delta \int_0^t e^{M(s)} ds, \\ (\rho_+, u_+ e^{M(t)}), & x > v_\delta \int_0^t e^{M(s)} ds \end{cases}\end{aligned}$$

for $u_- > u_+$. Therefore the delta-shock solution to the system (1.1) with initial data (1.2) is also stable under viscous perturbation.

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