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SOME EVALUATIONS OF INFINITE SERIES INVOLVING DIRICHLET TYPE PARAMETRIC HARMONIC NUMBERS

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Abstract. In this paper, we formally introduce the notion of a general parametric digamma function $\Psi(-s;A,a)$ and we find the Laurent expansion of $\Psi(-s;A,a)$ at the integers and poles. Considering the contour integrations involving $\Psi(-s;A,a)$, we present some new identities for infinite series involving Dirichlet type parametric harmonic numbers by using the method of residue computation. Then applying these formulas obtained, we establish some explicit relations of parametric linear Euler sums and some special functions (e.g. trigonometric functions, digamma functions, Hurwitz zeta functions etc.). Moreover, some illustrative special cases as well as immediate consequences of the main results are also considered.

1. Introduction

Let $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^-$ be the sets of complex numbers, integers, positive integers and negative integers, respectively. We also denote by $\mathbb{N}_0$ the set of non-negative integers and by $\mathbb{N}_0^-$ the set of non-positive integers.

Recall that for any $a,b\in\mathbb{C}\setminus\mathbb{N}^-$, Alzer and Choi [1] defined the following four parametric linear Euler sums:

$$\begin{aligned} S^{++}_{p,q}(a,b)&:=\sum_{n=1}^\infty \frac{H^{(p)}_n(a)}{(n+b)^q}, \quad S^{+-}_{p,q}(a,b):=\sum_{n=1}^\infty \frac{H^{(p)}_n(a)}{(n+b)^q}(-1)^{n-1},\\ S^{-+}_{p,q}(a,b)&:=\sum_{n=1}^\infty \frac{\bar{H}^{(p)}_n(a)}{(n+b)^q}, \quad S^{--}_{p,q}(a,b):=\sum_{n=1}^\infty \frac{\bar{H}^{(p)}_n(a)}{(n+b)^q}(-1)^{n-1},\end{aligned} \tag{1.1}$$

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where $p,q\in\mathbb{C}$ are adjusted so that the involved series can converge. Here $H_n^{(p)}(a)$ and $\bar{H}_n^{(p)}(a)$ are the *parametric harmonic numbers* of order p and the *alternating parametric harmonic numbers* of order p, respectively, defined by

$$H_n^{(p)}(a) := \sum_{j=1}^{n} \frac{1}{(j+a)^p} \quad (n\in\mathbb{N},\ p\in\mathbb{C},\ a\in\mathbb{C}\setminus\mathbb{N}^-) \tag{1.2}$$

and

$$\bar{H}_n^{(p)}(a) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(j+a)^p} \quad (n\in\mathbb{N},\ p\in\mathbb{C},\ a\in\mathbb{C}\setminus\mathbb{N}^-). \tag{1.3}$$

For convenience, we let $H_0^{(p)}(a)=\bar{H}_0^{(p)}(a):=0$. In particular, if taking $a=0$ in (1.2) and (1.3), then we get the so-called the *generalized harmonic numbers* $H_n^{(p)}$ of order p and *generalized alternating harmonic numbers* $\bar{H}_n^{(p)}$ of order p defined by

$$H_n^{(p)} \equiv H_n^{(p)}(0) := \sum_{k=1}^{n} \frac{1}{k^p}, \quad H_n\equiv H_n^{(1)}, \quad H_0^{(p)}:=0, \tag{1.4}$$

$$\bar{H}_n^{(p)} \equiv \bar{H}_n^{(p)}(0) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^p}, \quad \bar{H}_n\equiv \bar{H}_n^{(1)}, \quad \bar{H}_0^{(p)}:=0. \tag{1.5}$$

When taking the limit $n\to\infty$ in (1.2)-(1.5) we get the so called the *Hurwitz zeta function, alternating Hurwitz zeta function, Rimann zeta function* and *alternating Riemann zeta function*, respectively:

$$\zeta(p;a+1) := \lim_{n\to\infty} H_n^{(p)}(a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^p} \quad (\mathrm{Re}(p)>1,\ a\in\mathbb{C}\setminus\mathbb{N}^-), \tag{1.6}$$

$$\bar{\zeta}(p;a+1) := \lim_{n\to\infty} \bar{H}_n^{(p)}(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+a)^p} \quad (\mathrm{Re}(p)>0,\ a\in\mathbb{C}\setminus\mathbb{N}^-), \tag{1.7}$$

$$\zeta(p)\equiv\zeta(p;1) := \lim_{n\to\infty} H_n^{(p)} = \sum_{n=1}^{\infty} \frac{1}{n^p} \quad (\mathrm{Re}(p)>1), \tag{1.8}$$

$$\bar{\zeta}(p)\equiv\bar{\zeta}(p;1) := \lim_{n\to\infty} \bar{H}_n^{(p)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad (\mathrm{Re}(p)>0). \tag{1.9}$$

Clearly, setting $a=b=0$ in (1.1) becomes the classical *linear Euler sums* defined by (see [9])

$$\begin{aligned} S_{p,q}^{++} &\equiv S_{p,q}^{++}(0,0) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \quad S_{p,q}^{+-} \equiv S_{p,q}^{+-}(0,0) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}(-1)^{n-1}, \\ S_{p,q}^{-+} &\equiv S_{p,q}^{-+}(0,0) := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}, \quad S_{p,q}^{--} \equiv S_{p,q}^{--}(0,0) := \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q}(-1)^{n-1}. \end{aligned} \tag{1.10}$$

Investigation of Euler sums has a long history. The origin of the study of linear Euler sums (or double zeta star values) $S_{p,q}$ goes back to the correspondence of Euler with Goldbach in 1742–1743; see Berndt [3, p. 253] for a discussion. Euler elaborated a method to show that the linear sums $S^{++}_{p,q}$ can be evaluated in terms of zeta values in the following cases: $p = 1$, $p = q$, $p + q$ odd, $p + q$ even but with the pair (p, q) being restricted to the set $\{(2, 4), (4, 2)\}$. For more details on linear Euler sums, the readers are referred to [2, 4, 9]. Flajolet and Salvy [9] introduced and studied the following special Dirichlet series that involve harmonic numbers

$$S_{\mathbf{p},q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_r)}}{n^q}, \tag{1.11}$$

we call them *generalized* (*non-alternating*) *Euler sums.* Moreover, if $r > 1$ in (1.11), they were called *nonlinear Euler sums.* Here $\mathbf{p} := (p_1, p_2, \ldots, p_r)$ $(r, p_i \in \mathbb{N},\ i = 1, 2, \ldots, r)$ with $p_1 \le p_2 \le \cdots \le p_r$ and $q \ge 2$. The quantity $w := p_1 + \cdots + p_r + q$ is called the weight and the quantity r is called the degree (order). They considered the contour integration involving classical digamma function and used the residue computations to establish more explicit reductions of generalized Euler sums to Euler sums with lower degree. In particular, they proved the famous theorem: a nonlinear Euler sum $S_{p_1 p_2 \cdots p_r, q}$ reduces to a combination of sums of lower orders whenever the weight $p_1 + p_2 + \cdots + p_r + q$ and the order r are of the same parity. Due to surprising applications in many branches of mathematics and theoretical physics, Euler sums and related variants have attracted a lot of attention and interest in the past three decades (for example, see the books by Srivastava-Choi [17] and Zhao [24]). Some recent results of nonlinear Euler sums and related variants may be seen in the works of [5–8, 10–16, 18, 20, 23] and the references therein.

Motivated by Alzer-Choi's paper [1] and Xu's papers [21, 22], they studied many interesting features and identities of some parametric linear Euler sums. For example, Alzer and Choi discussed the analytic continuations and mingling connections of parametric linear Euler sums $S^{++}_{p,q}(a, b)$, $S^{+-}_{p,q}(a, b)$, $S^{-+}_{p,q}(a, b)$, $S^{--}_{p,q}(a, b)$. Xu [21] defined the following two combining sums involving parametric harmonic numbers

$$S_{p,q}(a) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}(-a) - (-1)^{p+q} H_n^{(p)}(a)}{n^q},$$
$$\bar{S}_{p,q}(a) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}(-a) - (-1)^{p+q} H_n^{(p)}(a)}{n^q} (-1)^n,$$

and used the residue computations to establish two explicit formulas of $S_{p,q}(a)$ and $\bar{S}_{p,q}(a)$ via trigonometric function, (alternating) Riemann zeta values, (alternating) Hurwitz zeta function and (alternating) digamma function. For instances, for any real $0 < |a| < 1$, we have (see [21, Cor. 4.4])

$$\begin{aligned} S_{2,2}(a) = & -\frac{3}{a^4} - 3\zeta(4;a+1) + 2\frac{\psi(a+1)+\gamma}{a^3} \\ & + \zeta(2;a+1)\left(\frac{\pi^2}{\sin^2(\pi a)} - \frac{1}{a^2}\right) + 2\pi\cot(\pi a)\left(\frac{1}{a^3} + \zeta(3;a+1)\right) \\ & + \frac{\pi^2}{a^2\sin^2(\pi a)}, \\ \bar{S}_{2,2}(a) = & -\frac{3}{a^4} - 3\zeta(4;a+1) - 2\frac{\ln 2 - \bar{\zeta}(1;a+1)}{a^3} \\ & + \bar{\zeta}(2;a+1)\left(\frac{1}{a^2} - \frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\right) + \frac{2\pi}{\sin(\pi a)}\left(\frac{1}{a^3} - \bar{\zeta}(3;a+1)\right) \\ & + \frac{\pi^2\cot(\pi a)}{a^2\sin(\pi a)}. \end{aligned}$$

Let $A := \{a_n\}$, $-\infty < n < \infty$, be a sequence of complex numbers such that if $n \to \pm\infty$, $a_n = o(n^\alpha)$, $\alpha < 1$. In [21, 22], Xu defined two parametric digamma (or Psi) function $\Psi(-s,a)$ and $\Psi(-s;A)$ by

$$\Psi(-s,a)+\gamma := \frac{1}{s-a} + \sum_{k=1}^{\infty}\left(\frac{1}{k+a} - \frac{1}{k+a-s}\right) \quad (s \in \mathbb{C},\ a \in \mathbb{C}\setminus\mathbb{N}^-) \tag{1.12}$$

and

$$\Psi(-s;A) := \frac{a_0}{s} + \sum_{k=1}^{\infty}\left(\frac{a_k}{k} - \frac{a_k}{k-s}\right) \quad (s \in \mathbb{C}\setminus\mathbb{N}_0). \tag{1.13}$$

Clearly, when $a = 0$ in (1.12) then the $\Psi(-s,0)$ becomes the classical digamma function denotes by

$$\psi(-s) + \gamma := \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k-s}\right) \quad (s \in \mathbb{C}\setminus\mathbb{N}_0). \tag{1.14}$$

Moreover, Xu used these two functions and residue computations to have found a large numbers of formulas of infinite series of parametric harmonic numbers and infinite series of Dirichlet type harmonic numbers, respectively. Here γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.577215664901532860606512\ldots.$$

The primary goal of this paper is to study the explicit relations of parametric linear Euler sums $S^{++}_{p,q}(a,b)$, $S^{+-}_{p,q}(a,b)$, $S^{-+}_{p,q}(a,b)$, $S^{--}_{p,q}(a,b)$ and (alternating) Hurwitz zeta functions by using the method of contour integrations and residue computations.

For convenience, for positive integers p and $q \geq 2$, let

$$S_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{1}{(n-b)^q} \sum_{k=1}^{n} \frac{1}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{1}{(n+b)^q} \sum_{k=1}^{n} \frac{1}{(k+a)^p}$$

$$= S_{p,q}^{++}(-a,-b) - (-1)^{p+q} S_{p,q}^{++}(a,b), \tag{1.15}$$

$$\bar{S}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{1}{(n-b)^q} \sum_{k=1}^{n} \frac{(-1)^k}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{1}{(n+b)^q} \sum_{k=1}^{n} \frac{(-1)^k}{(k+a)^p}$$

$$= -S_{p,q}^{-+}(-a,-b) + (-1)^{p+q} S_{p,q}^{-+}(a,b) \tag{1.16}$$

and for positive integers p and q, let

$$L_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-b)^q} \sum_{k=1}^{n} \frac{1}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+b)^q} \sum_{k=1}^{n} \frac{1}{(k+a)^p}$$

$$= -S_{p,q}^{+-}(-a,-b) + (-1)^{p+q} S_{p,q}^{+-}(a,b), \tag{1.17}$$

$$\bar{L}_{p,q}(a,b) := \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-b)^q} \sum_{k=1}^{n} \frac{(-1)^k}{(k-a)^p} - (-1)^{p+q} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+b)^q} \sum_{k=1}^{n} \frac{(-1)^k}{(k+a)^p}$$

$$= S_{p,q}^{--}(-a,-b) - (-1)^{p+q} S_{p,q}^{--}(a,b). \tag{1.18}$$

In this paper, we extend the parametric digamma function $\Psi(-s,a)$ and $\Psi(-s;A)$ to general function $\Psi(-s;A,a)$. Define function $\Psi(-s;A,a)$ by

$$\Psi(-s;A,a) + \gamma := \frac{a_0}{s-a} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_k}{k+a-s} \right), \quad (-1 < a < 1). \tag{1.19}$$

Clearly, the function $\Psi(-s;A,a)$ is meromorphic in the entire complex plane with a simple pole at $z = n + a$ for each nonnegative integer n. Moreover, when $a = 0$ then $\Psi(-s;A,0) = \Psi(-s;A)$ and when $A = \{1^n\}$ then $\Psi(-s;\{1\},a) = \Psi(-s,a)$. We will use the method of contour integrations involving $\Psi(-s;A,a)$ and residue theorem to establish some explicit relations of infinite series involving Dirichlet type parametric harmonic numbers which are of the form

$$\sum_{k=1}^{n} \frac{a_k}{(k+a)^p} \quad (p, n \in \mathbb{N},\ a \in \mathbb{C} \setminus \mathbb{N}^-).$$

Further, this allows us to find some new explicit evaluations for $S_{p,q}(a,b)$, $\bar{S}_{p,q}(a,b)$, $L_{p,q}(a,b)$ and $\bar{L}_{p,q}(a,b)$.

2. Some preliminaries

We first state two lemmas and some theorems that will subsequently be used in our proofs of main results.

Flajolet and Salvy [9] defined a kernel function $\xi(s)$ by the two requirements:

1. $\xi(s)$ is meromorphic in the whole complex plane.

2. $\xi(s)$ satisfies $\xi(s) = o(s)$ over an infinite collection of circles $|s| = \rho_k$ with $\rho_k \to \infty$. Applying these two conditions of kernel function $\xi(s)$, Flajolet and Salvy discovered the following residue lemma.

Lemma 2.1 ([9]). *Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then*

$$\sum_{\alpha\in O}\mathrm{Res}[r(s)\,\xi(s), s=\alpha] + \sum_{\beta\in S}\mathrm{Res}[r(s)\,\xi(s), s=\beta] = 0, \tag{2.1}$$

where S is the set of poles of $r(s)$ and O is the set of poles of $\xi(s)$ that are not poles $r(s)$. Here $\mathrm{Res}[r(s), s=\alpha]$ *denotes the residue of $r(s)$ at $s = \alpha$.*

Moreover, from classical expansions and the properties of ψ function, they listed the following Laurent series of $\pi\cot(\pi s)$ and $\frac{\pi}{\sin(\pi s)}$ at an integer n.

Lemma 2.2 (cf. [9]). *For integer n,*

$$\pi\cot(\pi s) \overset{s\to n}{=} \frac{1}{s-n} - 2\sum_{k=1}^{\infty}\zeta(2k)(s-n)^{2k-1} \quad (n\in\mathbb{Z}), \tag{2.2}$$

$$\frac{\pi}{\sin(\pi s)} \overset{s\to n}{=} (-1)^n\left(\frac{1}{s-n} + 2\sum_{k=1}^{\infty}\bar{\zeta}(2k)(s-n)^{2k-1}\right) \quad (n\in\mathbb{Z}). \tag{2.3}$$

Using the similar approach of the proofs of [21, Thms. 2.1 and 2.2] and [22, Thms. 2.1 and 2.2], we can deduce the following Taylor expansion and Laurent expansion formulas.

Theorem 2.3. *For positive integers n and $a \in (-1,1)$,*

$$\begin{aligned}\Psi(-s;A,a)+\gamma \overset{s\to -n}{=} &\sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a} + \sum_{k=1}^{\infty}\frac{a_k - a_{k-n}}{k+a}\\ &+\sum_{j=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{a_{k-n}}{(k+a)^{j+1}} - \sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^{j+1}}\right)(s+n)^j.\end{aligned} \tag{2.4}$$

Proof. To prove identity (2.4), we first observe that the right-hand side of (1.19) can be rewritten as

$$\Psi(-s;A,a)+\gamma = \sum_{k=1}^{\infty}\frac{a_k}{k+a} + \frac{a_0}{s-a} - \sum_{k=n+1}^{\infty}\frac{a_{k-n}}{k+a-s-n}.$$

Hence, if setting $s \to -n$ and applying the power series expansion

$$\frac{1}{1-x} = \sum_{k=1}^{\infty}x^{k-1} \quad (x\in(-1,1)),$$

we can find

$$\Psi(-s;A,a)+\gamma \overset{s\to -n}{=} \sum_{k=1}^{\infty}\frac{a_k}{k+a} + \sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a-s-n} - \sum_{k=1}^{\infty}\frac{a_{k-n}}{k+a-s-n}$$

$$= \sum_{k=1}^{n-1} \frac{a_{k-n}}{k+a} \frac{1}{1-\frac{s+n}{k+a}} - \sum_{k=1}^{\infty} \left(\frac{a_{k-n}}{k+a} \frac{1}{1-\frac{s+n}{k+a}} - \frac{a_k}{k+a} \right)$$

$$= \sum_{k=1}^{n-1} \frac{a_{k-n}}{k+a} \sum_{j=0}^{\infty} \left(\frac{s+n}{k+a} \right)^j - \sum_{k=1}^{\infty} \left(\frac{a_{k-n}}{k+a} \sum_{j=0}^{\infty} \left(\frac{s+n}{k+a} \right)^j - \frac{a_k}{k+a} \right)$$

$$= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{a_{k-n}(s+n)^j}{(k+a)^{j+1}} - \sum_{k=1}^{\infty} \frac{a_{k-n}(s+n)^j}{(k+a)^{j+1}} \right) + \sum_{k=1}^{n-1} \frac{a_{k-n}}{k+a} + \sum_{k=1}^{\infty} \frac{a_k - a_{k-n}}{k+a}$$

$$= \sum_{k=1}^{n-1} \frac{a_{k-n}}{k+a} + \sum_{k=1}^{\infty} \frac{a_k - a_{k-n}}{k+a} + \sum_{j=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{a_{k-n}}{(k+a)^{j+1}} - \sum_{k=1}^{\infty} \frac{a_{k-n}}{(k+a)^{j+1}} \right) (s+n)^j.$$

We have now completed the proof of Theorem 2.3. □

Theorem 2.4. *For integer $n \geq 0$ and $a \in (-1, 1)$,*

$$\Psi(-s; A, a) + \gamma \overset{s \to n}{=} \frac{a_n}{s-n-a} + \sum_{k=1}^{n} \frac{a_{n-k}}{k-a} + \sum_{k=1}^{\infty} \frac{a_k - a_{k+n}}{k+a}$$

$$+ \sum_{j=1}^{\infty} \left((-1)^j \sum_{k=1}^{n} \frac{a_{n-k}}{(k-a)^{j+1}} - \sum_{k=1}^{\infty} \frac{a_{k+n}}{(k+a)^{j+1}} \right) (s-n)^j. \tag{2.5}$$

Proof. The proof of Theorem 2.4 is similar to the proof of Theorem 2.3. Applying the power series expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{and} \quad \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \quad (x \in (-1,1)),$$

we arrive at

$$\Psi(-s; A, a) + \gamma \overset{s \to n}{=} \frac{a_0}{s-a} + \frac{a_n}{n+a} - \frac{a_n}{n+a-s} + \sum_{k=1}^{n-1} \left(\frac{a_k}{k+a} - \frac{a_k}{k+a-s} \right)$$

$$+ \sum_{k=n+1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_k}{k+a-s} \right)$$

$$= \frac{a_n}{s-n-a} + \sum_{k=1}^{n} \frac{a_k}{k+a} - \sum_{k=1}^{n} \left(\frac{a_k}{k+a} - \frac{a_{n-k}}{k+s-a-n} \right)$$

$$+ \sum_{k=1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_{k+n}}{k+n+a-s} \right)$$

$$= \frac{a_n}{s-n-a} + \sum_{k=1}^{n} \frac{a_{n-k}}{k-a} \frac{1}{1+\frac{s-n}{k-a}} - \sum_{k=1}^{\infty} \left(\frac{a_{k+n}}{k+a} \frac{1}{1-\frac{s-n}{k+a}} - \frac{a_k}{k+a} \right)$$

$$= \sum_{j=0}^{\infty} \left((-1)^j \sum_{k=1}^{n} \frac{a_{n-k}(s-n)^j}{(k-a)^{j+1}} - \sum_{k=1}^{\infty} \left(\frac{a_{k+n}(s-n)^j}{(k+a)^{j+1}} - \frac{a_k}{k+a} \right) \right)$$

$$
\begin{aligned}
&+\frac{a_n}{s-n-a}\\
=\ &\frac{a_n}{s-n-a}+\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}+\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a}\\
&+\sum_{j=1}^{\infty}\left((-1)^j\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^{j+1}}-\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^{j+1}}\right)(s-n)^j.
\end{aligned}
$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *For integer $n \geq 0$ and $a \in (-1,1)$,*

$$
\begin{aligned}
\Psi(-s;A,a)+\gamma &\overset{s\to n+a}{=} \frac{a_n}{s-n-a}+\sum_{k=1}^{n}\frac{a_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_{k+n}}{k}\right)\\
&+\sum_{j=1}^{\infty}\left((-1)^j\sum_{k=1}^{n}\frac{a_{n-k}}{k^{j+1}}-\sum_{k=1}^{\infty}\frac{a_{k+n}}{k^{j+1}}\right)(s-n-a)^j. \qquad (2.6)
\end{aligned}
$$

Proof. According to the definition, it is easy to see that when $s \to n+a$ and $a \in (-1,1)$, we have

$$
\begin{aligned}
&\Psi(-s;A,a)+\gamma\\
\overset{s\to n+a}{=}\ &\frac{a_0}{s-a}+\frac{a_n}{n+a}-\frac{a_n}{n+a-s}+\sum_{k=1}^{n-1}\left(\frac{a_k}{k+a}-\frac{a_k}{k+a-s}\right)\\
&+\sum_{k=n+1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_k}{k+a-s}\right)\\
=\ &\frac{a_n}{s-n-a}+\sum_{k=1}^{n}\frac{a_k}{k+a}-\sum_{k=1}^{n}\left(\frac{a_k}{k+a}-\frac{a_{n-k}}{k+s-a-n}\right)\\
&+\sum_{k=1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_{k+n}}{k+n+a-s}\right)\\
=\ &\frac{a_n}{s-n-a}+\sum_{k=1}^{n}\frac{a_{n-k}}{k}\frac{1}{1+\frac{s-n-a}{k}}-\sum_{k=1}^{\infty}\left(\frac{a_{k+n}}{k}\frac{1}{1-\frac{s-n-a}{k}}-\frac{a_k}{k+a}\right)\\
=\ &\sum_{j=0}^{\infty}\left((-1)^j\sum_{k=1}^{n}\frac{a_{n-k}(s-n-a)^j}{k^{j+1}}-\sum_{k=1}^{\infty}\left(\frac{a_{k+n}(s-a-n)^j}{k^{j+1}}-\frac{a_k}{k+a}\right)\right)\\
&+\frac{a_n}{s-n-a}\\
=\ &\frac{a_n}{s-n-a}+\sum_{k=1}^{n}\frac{a_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_{k+n}}{k}\right)
\end{aligned}
$$

$$+\sum_{j=1}^{\infty}\left((-1)^j\sum_{k=1}^{n}\frac{a_{n-k}}{k^{j+1}}-\sum_{k=1}^{\infty}\frac{a_{k+n}}{k^{j+1}}\right)(s-n-a)^j,$$

where we used the following power series expansions

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k \quad \text{and} \quad \frac{1}{1+x}=\sum_{k=0}^{\infty}(-1)^k x^k \quad (x\in(-1,1)).$$

We have now completed the proof of Theorem 2.5. □

In particular, taking $n=0$ in (2.5) yields

$$\Psi(-s;A,a)+\gamma \overset{s\to 0}{=} \frac{a_0}{s-a}-\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j+1}}\right)s^j. \tag{2.7}$$

Differentiating identities (2.4)-(2.7) $p-1$ times with respect to s, respectively, and noting that

$$\frac{1}{(s-n-a)^p}=\frac{(-1)^p}{(s-n)^p}\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}\left(\frac{s-n}{a}\right)^{j+p-1} \quad (s\to n),$$

we can get the following corollaries.

Corollary 2.6. *For positive integers $n\geq 1$, $p>1$ and $a\in(-1,1)$,*

$$\frac{\Psi^{(p-1)}(-s;A,a)}{(p-1)!} \overset{s\to -n}{=} (-1)^p\sum_{j=0}^{\infty}\binom{j+p-1}{p-1}\left(\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^{j+p}}-\sum_{k=1}^{n-1}\frac{a_{k-n}}{(k+a)^{j+p}}\right)(s+n)^j. \tag{2.8}$$

Corollary 2.7. *For integers $n\geq 0$, $p>1$ and $0<|a|<1$,*

$$\frac{\Psi^{(p-1)}(-s;A,a)}{(p-1)!} \overset{s\to n}{=} \frac{(-1)^p}{(s-n)^p}\sum_{j=p}^{\infty}\binom{j-1}{p-1}\left(\frac{a_n}{a^j}+\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^j}+(-1)^j\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^j}\right)(s-n)^j. \tag{2.9}$$

Corollary 2.8. *For integers $n\geq 0$, $p>1$ and $a\in(-1,1)$,*

$$\begin{aligned}&\frac{\Psi^{(p-1)}(-s;A,a)}{(p-1)!}\\ &\overset{s\to n+a}{=} \frac{(-1)^p}{(s-n-a)^p}\sum_{j=p}^{\infty}\binom{j-1}{p-1}\left(\sum_{k=1}^{\infty}\frac{a_{k+n}}{k^j}+(-1)^j\sum_{k=1}^{n}\frac{a_{n-k}}{k^j}\right)(s-n-a)^j\\ &\quad+\frac{a_n}{(s-n-a)^p}.\end{aligned} \tag{2.10}$$

Corollary 2.9. *For positive integer $p>1$ and $0<|a|<1$,*

$$\frac{\Psi^{(p-1)}(-s;A,a)}{(p-1)!} \overset{s\to 0}{=} (-1)^p\sum_{j=1}^{\infty}\binom{j+p-2}{p-1}\left(\frac{a_0}{a^{j+p-1}}+\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j+p-1}}\right)s^{j-1}. \tag{2.11}$$

3. Main results

Theorem 3.1. *Let $A := \{a_n\}$ and $B := \{b_n\}$ $(-\infty < n < \infty)$ be sequences of complex numbers, $p > 1$ be an integer and $0 < |a| < 1$. Then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{a_n}{(n+a)^p} \left(\sum_{k=1}^{n} \frac{b_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{b_k}{k+a} - \frac{b_{k+n}}{k} \right) \right) \\
&+ \sum_{n=1}^{\infty} \frac{b_n}{(n+a)^p} \left(\sum_{k=1}^{n} \frac{a_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_{k+n}}{k} \right) \right) \\
&- p \sum_{n=1}^{\infty} \frac{a_n b_n}{(n+a)^{p+1}} - \frac{a_0}{a^{p-1}} \sum_{k=1}^{\infty} \frac{b_k}{k\,(k+a)} - \frac{b_0}{a^{p-1}} \sum_{k=1}^{\infty} \frac{a_k}{k\,(k+a)} \\
&+ \sum_{k=0}^{p-2} \frac{1}{a^{k+1}} \left(a_0 \sum_{n=1}^{\infty} \frac{b_n}{(n+a)^{p-k}} + b_0 \sum_{n=1}^{\infty} \frac{a_n}{(n+a)^{p-k}} \right) \\
&+ \sum_{\substack{j_1+j_2=p-1 \\ j_1,j_2 \geq 1}} \left(\sum_{k=1}^{\infty} \frac{a_k}{(k+a)^{j_1+1}} \right) \left(\sum_{k=1}^{\infty} \frac{b_k}{(k+a)^{j_2+1}} \right) = 0,
\end{aligned} \tag{3.1}$$

provided that if $n \to \pm\infty$, then $a_n = o(n^{\alpha})$, $b_n = o\left(n^{\beta}\right)$ with $\alpha, \beta < 1$.

Proof. To prove Theorem 3.1, we apply Lemma 2.1 and consider the kernel function

$$\xi_1(s) := \Psi(-s; A, a)\Psi(-s; B, a)$$

and the base function $r_1(s) = s^{-p}$. Clearly, for the function $F_1(s) := \xi_1(s)r_1(s)$, the only singularities are poles at $s = 0$ and $s = n + a$, where $n \in \mathbb{N}^-$. By Theorem 2.5, the pole at a real $s = n + a$ (n is a negative integer) has order 2 and the residue is

$$\begin{aligned}
& \mathrm{Res}[F_1(s), s = n + a] \\
= & \lim_{s \to n+a} \frac{d}{ds} (s - n - a)^2 F_1(s) \\
= & \lim_{s \to n+a} \frac{d}{ds} \left\{ \left(\frac{a_n}{s-n-a} + \sum_{k=1}^{n} \frac{a_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_{k+n}}{k} \right) \right) \right. \\
& \left. \times \left(\frac{b_n}{s-n-a} + \sum_{k=1}^{n} \frac{b_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{b_k}{k+a} - \frac{b_{k+n}}{k} \right) \right) \frac{(s-n-a)^2}{s^p} \right\} \\
= & \frac{a_n \left(\sum_{k=1}^{n} \frac{b_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{b_k}{k+a} - \frac{b_{k+n}}{k} \right) \right) (n+a)^p}{(n+a)^{2p}} \\
& + \frac{b_n \left(\sum_{k=1}^{n} \frac{a_{n-k}}{k} + \sum_{k=1}^{\infty} \left(\frac{a_k}{k+a} - \frac{a_{k+n}}{k} \right) \right) (n+a)^p}{(n+a)^{2p}}
\end{aligned}$$

$$\begin{aligned}
&-\frac{p(n+a)^{p-1}a_nb_n}{(n+a)^{2p}}\\
&=\frac{a_n}{(n+a)^p}\left(\sum_{k=1}^{n}\frac{b_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{b_k}{k+a}-\frac{b_{k+n}}{k}\right)\right)\\
&\quad+\frac{b_n}{(n+a)^p}\left(\sum_{k=1}^{n}\frac{a_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_{k+n}}{k}\right)\right)-\frac{pa_nb_n}{(n+a)^{p+1}}.
\end{aligned}$$

Similarly, using (2.7), the residue of the pole of order p at 0 is found to be

$$\begin{aligned}
&\mathrm{Res}[F_1(s),s=0]\\
&=\frac{1}{(p-1)!}\lim_{s\to 0}\frac{d^{p-1}}{ds^{p-1}}s^pF_1(s)\\
&=\frac{1}{(p-1)!}\lim_{s\to 0}\frac{d^{p-1}}{ds^{p-1}}\left\{\frac{a_0b_0}{(s-a)^2}-\frac{a_0}{s-a}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{b_k}{(k+a)^{j+1}}s^j\right.\\
&\quad-\frac{b_0}{s-a}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j+1}}s^j\\
&\quad\left.+\sum_{j_1,j_2=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j_1+1}}\sum_{k=1}^{\infty}\frac{b_k}{(k+a)^{j_2+1}}s^{j_1+j_2}\right\}\\
&=\frac{pa_0b_0}{a^{p+1}}+\sum_{k=0}^{p-2}\frac{1}{a^{k+1}}\left(a_0\sum_{n=1}^{\infty}\frac{b_n}{(n+a)^{p-k}}+b_0\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p-k}}\right)\\
&\quad+\sum_{\substack{j_1+j_2=p-1,\\ j_1,j_2\geq 1}}\left(\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j_1+1}}\sum_{k=1}^{\infty}\frac{b_k}{(k+a)^{j_2+1}}\right).
\end{aligned}$$

Then, by using Lemma 2.1, we know that

$$\sum_{n=0}^{\infty}\mathrm{Res}[F_1(s),s=n+a]+\mathrm{Res}[F_1(s),s=0]=0.$$

Summing these contributions yields the statement of the theorem. □

Corollary 3.2. *For real* $0<|a|<1$*, we have*

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^2}&=\frac{\psi(a+1)+\gamma}{a^2}+\zeta(2;a+1)\left(\psi(a+1)+\gamma-\frac{1}{a}\right)\\
&\quad+\zeta(3;a+1), \qquad (3.2)\\
\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^3}&=\frac{\psi(a+1)+\gamma}{a^3}+\frac{3}{2}\zeta(4;a+1)-\frac{1}{a^2}\zeta(2;a+1)-\frac{\zeta^2(2;a+1)}{2}
\end{aligned}$$

$$+\zeta(3;a+1)\left(\psi(a+1)+\gamma-\frac{1}{a}\right). \tag{3.3}$$

Proof. Taking $p=2,3$ and $a_k=b_k=1$ in (3.1), by straightforward calculations, we obtain the two results. □

Theorem 3.3. *For integer $p>1$ and real $0<|a|<1$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^p}\sum_{k=1}^{n}\left(\frac{a_{n-k}}{k-a}+(-1)^p\frac{a_{k-n}}{k+a}\right)\\
&=\frac{a_0}{a^{p+1}}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{a_n}{n^p}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p+1}}-\pi\cot(\pi a)\left(\frac{a_0}{a^p}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^p}\right)\\
&\quad-2a_0\sum_{k=1}^{[p/2]}\frac{\zeta(2k)}{a^{p-2k+1}}-2\sum_{\substack{2k_1+k_2=p\\k_1,k_2\ge1}}\zeta(2k_1)\left(\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{k_2+1}}\right)\\
&\quad+a_0(-1)^p\sum_{n=1}^{\infty}\frac{1}{n^p(n+a)}-\sum_{n=1}^{\infty}\frac{1}{n^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}+(-1)^p(a_k-a_{k-n})}{k+a},
\end{aligned}\tag{3.4}$$

where $n\to\pm\infty$, then $a_n=o(n^{\alpha})$ with $\alpha<1$.

Proof. The proof is similar to that of Theorem 3.1. To prove Theorem 3.3, we need to use Lemma 2.1 and consider the following function

$$F_2(s):=\frac{\pi\cot(s\pi)(\Psi(-s;A,a)+\gamma)}{s^p}\quad(p>1).$$

Clearly, the function $F_2(s)$ only has poles at $s=0,\pm n$ and $n+a-1$ for $n\in\mathbb{N}$. At a positive integer n, the pole $s=\pm n$ and $n+a-1$ is simple and by the expansions (2.2), (2.4), (2.5) and (2.6), these residues are

$$\begin{aligned}
\mathrm{Res}[F_2(s),s=-n]&=\frac{(-1)^p}{n^p}\left(\sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a}+\sum_{k=1}^{\infty}\frac{a_k-a_{k-n}}{k+a}\right),\\
\mathrm{Res}[F_2(s),s=n]&=-\frac{a_n}{an^p}+\frac{1}{n^p}\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}+\frac{1}{n^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a},\\
\mathrm{Res}[F_2(s),s=n+a-1]&=\frac{a_{n-1}\pi\cot(a\pi)}{(n+a-1)^p}.
\end{aligned}$$

Further, applying (2.2) with $n=0$ and (2.7), the residue of the pole of order p at 0 is found to be

$$\begin{aligned}
\mathrm{Res}[F_2(s),s=0]&=(-1)^p\frac{a_0}{(-a)^{p+1}}-\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p+1}}+2a_0\sum_{k=1}^{[p/2]}\frac{\zeta(2k)}{a^{p-2k+1}}\\
&\quad+2\sum_{\substack{2k_1+k_2=p\\k_1,k_2\ge1}}\zeta(2k_1)\left(\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{k_2+1}}\right).
\end{aligned}$$

Then using Lemma 2.1, we have

$$\sum_{n=1}^{\infty}\left(\operatorname{Res}[F_2(s), s=-n] + \operatorname{Res}[F_2(s), s=n] + \operatorname{Res}[F_2(s), s=n+a-1]\right) + \operatorname{Res}[F_2(s), s=0] = 0.$$

Summing these four contributions yields the statement of the theorem. □

Corollary 3.4. *For real* $0<|a|<1$, *we have*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{k}{k^2-a^2} = \frac{1}{2a^3}+\frac{\zeta(3;a+1)}{2}-\frac{\psi(a+1)+\gamma}{2a^2} - \frac{\pi\cot(\pi a)\,\zeta(2;a)}{2}, \tag{3.5}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n}\frac{1}{k^2-a^2} = \frac{1}{2a^5}-\frac{\zeta(2)\,\zeta(2;a+1)}{a}-\frac{\psi(a+1)+\gamma}{2a^4}-\frac{\zeta(2)}{2a^3} + \frac{\zeta(4;a+1)}{2a}-\frac{\pi\cot(\pi a)\,\zeta(3;a)}{2a}, \tag{3.6}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{n}\frac{k(-1)^k}{k^2-a^2} = \frac{1}{2a^3}-\frac{\psi(a+1)+\gamma}{2a^2}-\frac{\bar\zeta(3;a+1)}{2}-\frac{\pi\cot(\pi a)\,\bar\zeta(2;a)}{2} + \left(\bar\zeta(1;a+1)-\frac{1}{2a}\right)\left(\bar\zeta(2)+\zeta(2)\right). \tag{3.7}$$

Proof. This corollary follows directly from (3.4) with $p=2,3$, $a_k=1$ and $p=2$, $a_k=(-1)^k$. □

Theorem 3.5. *For integer* $p>1$ *and real* $0<|a|<1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}\sum_{k=1}^{n}\left(\frac{a_{n-k}}{k-a}+(-1)^p\frac{a_{k-n}}{k+a}\right)\\
&=\frac{a_0}{a^{p+1}}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{a_n}{n^p}(-1)^n+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p+1}}\\
&\quad-\frac{\pi}{\sin(\pi a)}\left(\frac{a_0}{a^p}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^p}(-1)^n\right)+2a_0\sum_{k=1}^{[p/2]}\frac{\bar\zeta(2k)}{a^{p-2k+1}}\\
&\quad+2\sum_{\substack{2k_1+k_2=p\\k_1,k_2\ge 1}}\bar\zeta(2k_1)\left(\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{k_2+1}}\right)+a_0(-1)^p\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p(n+a)}\\
&\quad-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}+(-1)^p(a_k-a_{k-n})}{k+a},
\end{aligned}\tag{3.8}$$

where $n\to\pm\infty$, *then* $a_n=o(n^{\alpha})$ *with* $\alpha<1$.

Proof. The proof is based on the function

$$F_3(s) := \frac{\pi(\Psi(-s;A,a)+\gamma)}{\sin(\pi s)s^p}$$

and the residue computation. Similar to $F_2(s)$, the function $F_3(s)$ also only has poles at $s=0,\pm n$ and $n+a-1$ for $n\in\mathbb{N}$. Using (2.3), (2.4), (2.5) and (2.6), by a similar argument to the proof of Theorem 3.3, we deduce that for positive integer n,

$$\begin{aligned}
&\mathrm{Res}[F_3(s),s=-n] = (-1)^{p+n}\frac{\sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a}+\sum_{k=1}^{\infty}\frac{a_k-a_{k-n}}{k+a}}{n^p},\\
&\mathrm{Res}[F_3(s),s=n] = \frac{(-1)^{n+1}a_n}{an^p}+\frac{(-1)^n}{n^p}\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}+\frac{(-1)^n}{n^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a},\\
&\mathrm{Res}[F_3(s),s=n+a-1] = (-1)^{n-1}\frac{\pi a_{n-1}}{\sin(a\pi)(n+a-1)^p}.
\end{aligned}$$

Applying (2.3) with $n=0$ and (2.7), the residue of the pole of order p at 0 is found to be

$$\begin{aligned}
\mathrm{Res}[F_3(s),s=0] = &-\frac{a_0}{a^{p+1}}-\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p+1}}-2a_0\sum_{k=1}^{[p/2]}\frac{\bar\zeta(2k)}{a^{p-2k+1}}\\
&-2\sum_{\substack{2k_1+k_2=p\\k_1,k_2\ge1}}\bar\zeta(2k_1)\left(\sum_{k=1}^{\infty}\frac{a_k}{(n+a)^{k_2+1}}\right).
\end{aligned}$$

Then, by using Lemma 2.1, we know that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\mathrm{Res}[F_3(s),s=n+a-1]+\mathrm{Res}[F_3(s),s=-n]+\mathrm{Res}[F_3(s),s=n])\\
&+\mathrm{Res}[F_3(s),s=0]=0.
\end{aligned}$$

Hence, combining these four residue results, we obtain the desired evaluation. □

Corollary 3.6. *For real* $0<|a|<1$*, we have*

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sum_{k=1}^{n}\frac{k}{k^2-a^2} = \frac{1}{2a^3}+\frac{\zeta(3;a+1)}{2}+\frac{\ln(2)-\bar\zeta(1;a+1)}{2a^2} - \frac{\pi\bar\zeta(2;a)}{2\sin(\pi a)}, \tag{3.9}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sum_{k=1}^{n}\frac{1}{k^2-a^2} = \frac{1}{2a^5}+\frac{\zeta(4;a+1)}{2a}+\frac{\ln(2)-\bar\zeta(1;a+1)}{2a^4}+\frac{\bar\zeta(2)}{2a^3} - \frac{\pi\bar\zeta(3;a)}{2a\sin(\pi a)}+\frac{\bar\zeta(2)\zeta(2;a+1)}{a}, \tag{3.10}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{k(-1)^k}{k^2-a^2}=\frac{1}{2a^3}+\frac{\ln(2)-\bar{\zeta}\left(1;a+1\right)}{2a^2}-\frac{\bar{\zeta}\left(3;a+1\right)}{2}$$
$$-\frac{\pi\zeta(2;a)}{2\sin\left(\pi a\right)}+\left(\bar{\zeta}(2)+\zeta(2)\right)\left(\frac{1}{2a}-\bar{\zeta}(1;a+1)\right). \tag{3.11}$$

Proof. Setting $p=2,3$, $a_k=1$ and $p=2$, $a_k=(-1)^k$ in (3.8) yield the three desired results with simple calculations. □

Theorem 3.7. *For integers $p,q>1$ and real $0<|a|<1$, we have*

$$\sum_{n=1}^{\infty}\frac{1}{n^q}\sum_{k=1}^{n}\left(\frac{a_{n-k}}{(k-a)^p}-(-1)^{p+q}\frac{a_{k-n}}{(k+a)^p}\right)$$
$$=(-1)^{p+1}\binom{p+q-1}{p-1}\left(\frac{a_0}{a^{p+q}}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{p+q}}\right)$$
$$+2(-1)^p\sum_{\substack{2k_1+k_2=q+1\\k_1,k_2\geq 1}}\binom{k_2+p-2}{p-1}\zeta\left(2k_1\right)\left(\frac{a_0}{a^{k_2+p-1}}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{k_2+p-1}}\right)$$
$$+(-1)^{p+q+1}a_0\sum_{n=1}^{\infty}\frac{1}{n^q(n+a)^p}-\frac{1}{(p-1)!}\sum_{n=0}^{\infty}a_n\frac{d^{p-1}}{ds^{p-1}}\left(\frac{\pi\cot\left(\pi s\right)}{s^q}\right)\Bigg|_{s=n+a}$$
$$+\frac{(-1)^{p+1}}{a^p}\sum_{n=1}^{\infty}\frac{a_n}{n^q}-(-1)^p\sum_{n=1}^{\infty}\frac{1}{n^q}\sum_{k=1}^{\infty}\left(\frac{a_{k+n}}{(k+a)^p}+(-1)^q\frac{a_{k-n}}{(k+a)^p}\right), \tag{3.12}$$

where $n\to\pm\infty$, then $a_n=o\left(n^{\alpha}\right)$ with $\alpha<1$.

Proof. Define the function $F_4(s)$ by

$$F_4(s):=\frac{\pi\cot(\pi s)\Psi^{(p-1)}(-s;A,a)}{(p-1)!s^q}.$$

It can be found that $F_4(s)$ only has poles at $s=0,\pm n$ and $n+a-1$ for $n\in\mathbb{N}$. By Eqs. (2.2), (2.8)-(2.11), for positive integer n, the corresponding residues are

$$\mathrm{Res}[F_4(s),s=-n]=(-1)^{p+q}\frac{\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^p}}{n^q}-(-1)^{p+q}\frac{\sum_{k=1}^{n}\frac{a_{k-n}}{(k+a)^p}}{n^q}$$
$$+(-1)^{p+q}\frac{a_0}{(n+a)^pn^q},$$
$$\mathrm{Res}[F_4(s),s=n]=\frac{(-1)^pa_n}{n^qa^p}+\frac{(-1)^p\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^p}}{n^q}+\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^p},$$
$$\mathrm{Res}[F_4(s),s=n+a-1]=\frac{1}{(p-1)!}\lim_{s\to n+a-1}\frac{d^{p-1}}{ds^{p-1}}\frac{a_{n-1}\pi\cot(\pi s)}{s^q},$$
$$\mathrm{Res}[F_4(s),s=0]=(-1)^p\binom{p+q-1}{p-1}\left(\frac{a_0}{a^{p+q}}+\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{p+q}}\right)$$

$$- 2(-1)^p \sum_{\substack{2k_1+k_2=q+1\\ k_1,k_2\geq 1}} \binom{k_2+p-2}{p-1}\zeta(2k_1)\left(\frac{a_0}{a^{k_2+p-1}}+\sum_{n=1}^\infty \frac{a_n}{(n+a)^{k_2+p-1}}\right).$$

Using Lemma 2.1 we have

$$\sum_{n=1}^\infty (\mathrm{Res}[F_4(s), s=n+a-1]+\mathrm{Res}[F_4(s), s=-n]+\mathrm{Res}[F_4(s), s=n])$$
$$+\mathrm{Res}[F_4(s), s=0]=0.$$

Summing these four contributions yields the statement of the theorem. □

Corollary 3.8. *For real* $0<|a|<1$*, we have*

$$\begin{aligned}\sum_{n=1}^\infty \frac{1}{n^2}\left(H_n^{(2)}(-a)-H_n^{(2)}(a)\right) = &-\frac{3}{a^4}-3\zeta\left(4;a+1\right)+2\frac{\psi\left(a+1\right)+\gamma}{a^3}\\ &+\zeta\left(2;a+1\right)\left(\frac{\pi^2}{\sin^2\left(\pi a\right)}-\frac{1}{a^2}\right)\\ &+2\pi\cot\left(\pi a\right)\zeta(3;a)+\frac{\pi^2}{a^2\sin^2\left(\pi a\right)},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\left(\bar{H}_n^{(2)}(a)-\bar{H}_n^{(2)}(-a)\right) = &\frac{1}{a^2}\left(\zeta(2)-\zeta(2;a+1)\right)-2\zeta(2)\bar{\zeta}(2;a+1)\\ &-3\bar{\zeta}(4;a)+\frac{2}{a^3}(\psi(a+1)+\gamma)\\ &+2\pi\cot(a\pi)\bar{\zeta}(3;a)+\pi^2\csc^2(a\pi)\bar{\zeta}(2;a)\\ &+\bar{\zeta}(2)\left(\frac{1}{a^2}-2\bar{\zeta}(2;a+1)\right).\end{aligned} \tag{3.14}$$

Proof. Taking $p=q=2$, $a_k=1$ and $p=q=2$, $a_k=(-1)^k$ in (3.12), by direct calculations, we arrive at the two results. □

Theorem 3.9. *For integers* $p, q>1$ *and real* $0<|a|<1$*,*

$$\begin{aligned}&\sum_{n=1}^\infty \frac{(-1)^n}{n^q}\sum_{k=1}^n\left(\frac{a_{n-k}}{(k-a)^p}-(-1)^{p+q}\frac{a_{k-n}}{(k+a)^p}\right)\\ &=(-1)^{p+1}\binom{p+q-1}{p-1}\left(\frac{a_0}{a^{p+q}}+\sum_{n=1}^\infty\frac{a_n}{(n+a)^{p+q}}\right)\\ &\quad+2(-1)^{p+1}\sum_{\substack{2k_1+k_2=q+1\\ k_1,k_2\geq 1}}\binom{k_2+p-2}{p-1}\bar{\zeta}\left(2k_1\right)\left(\frac{a_0}{a^{k_2+p-1}}+\sum_{n=1}^\infty\frac{a_n}{(n+a)^{k_2+p-1}}\right)\\ &\quad+(-1)^{p+q+1}a_0\sum_{n=1}^\infty\frac{(-1)^n}{n^q(n+a)^p}-\frac{1}{(p-1)!}\sum_{n=0}^\infty a_n\frac{d^{p-1}}{ds^{p-1}}\left(\frac{\pi}{\sin\left(\pi s\right)s^q}\right)\bigg|_{s=n+a}\end{aligned}$$

$$+\frac{(-1)^{p+1}}{a^p}\sum_{n=1}^{\infty}\frac{a_n}{n^q}(-1)^n-(-1)^p\sum_{n=1}^{\infty}\frac{(-1)^n}{n^q}\sum_{k=1}^{\infty}\left(\frac{a_{k+n}}{(k+a)^p}+(-1)^q\frac{a_{k-n}}{(k+a)^p}\right), \tag{3.15}$$

where $n\to\pm\infty$, *then* $a_n=o\left(n^{\alpha}\right)$ *with* $\alpha<1$.

Proof. Similarly, we consider the function

$$F_5(s):=\frac{\pi\Psi^{(p-1)}(-s;A,a)}{\sin(\pi s)(p-1)!s^q}.$$

It is obvious that the function $F_5(s)$ only has poles at $s=0,\pm n$ and $n+a-1$ for $n\in\mathbb{N}$. By Eqs. (2.3), (2.8)-(2.11), for positive integer n, the corresponding residues are

$$\mathrm{Res}[F_5(s),s=-n]=\frac{(-1)^{p+n+q}}{n^q}\left(\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^p}-\sum_{k=1}^{n}\frac{a_{k-n}}{(k+a)^p}+\frac{a_0}{(n+a)^p}\right),$$

$$\mathrm{Res}[F_5(s),s=n]=\frac{(-1)^{p+n}}{n^q}\left(\frac{a_n}{a^p}+\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^p}+(-1)^p\sum_{k=1}^{n}\frac{a_{n-k}}{(k+a)^p}\right),$$

$$\mathrm{Res}[F_5(s),s=n+a-1]=\frac{1}{(p-1)!}\lim_{s\to n+a-1}\frac{d^{p-1}}{ds^{p-1}}\frac{a_{n-1}\pi}{\sin(\pi s)s^q},$$

$$\mathrm{Res}[F_5(s),s=0]=(-1)^p\binom{p+q-1}{p-1}\left(\frac{a_0}{a^{p+q}}+\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{p+q}}\right)$$
$$+2(-1)^p\sum_{\substack{2k_1+k_2=q+1\\k_1,k_2\geq1}}\left[\binom{k_2+p-2}{p-1}\bar{\zeta}(2k_1)\left(\frac{a_0}{a^{k_2+p-1}}+\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^{k_2+p-1}}\right)\right].$$

Then applying Lemma 2.1 we have

$$\sum_{n=1}^{\infty}(\mathrm{Res}[F_5(s),s=n+a-1]+\mathrm{Res}[F_5(s),s=-n]+\mathrm{Res}[F_5(s),s=n])$$
$$+\mathrm{Res}[F_5(s),s=0]=0.$$

Summing these four contributions yields the statement of the theorem. □

Corollary 3.10. *For real* $0<|a|<1$, *we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(H_n^{(2)}(-a)-H_n^{(2)}(a)\right)\\&=-\frac{3}{a^4}-3\zeta(4;a+1)-2\frac{\ln(2)-\bar{\zeta}(1;a+1)}{a^3}\\&\quad+\bar{\zeta}(2;a+1)\left(\frac{1}{a^2}-\frac{\pi^2\cot(\pi a)}{\sin(\pi a)}\right)+\frac{2\pi\bar{\zeta}(3;a)}{\sin(\pi a)}+\frac{\pi^2\cot(\pi a)}{a^2\sin(\pi a)},\end{aligned} \tag{3.16}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\bar{H}_n^{(2)}(a)-\bar{H}_n^{(2)}(-a)\right)$$

$$
\begin{aligned}
&= \frac{1}{a^2}\left(\bar\zeta(2)+\bar\zeta(2;a+1)\right)-3\bar\zeta(4;a)-2\bar\zeta(2)\bar\zeta(2;a)\\
&\quad+2\pi\csc(a\pi)\zeta(3;a)+\pi^2\cot(a\pi)\csc(a\pi)\zeta(2;a)\\
&\quad-\frac{2}{a^3}\left(\bar\zeta(1)-\bar\zeta(1;a+1)\right)+\zeta(2)\left(2\bar\zeta(2;a+1)-\frac{1}{a^2}\right). \qquad (3.17)
\end{aligned}
$$

Proof. This corollary can be obtained immediately from (3.15) with $p=q=2$, $a_k=1$ and $p=q=2$, $a_k=(-1)^k$. □

Applying the partial fraction decomposition

$$
\frac{1}{n(n+a)^p}=\frac{1}{a^{p-1}}\frac{1}{n\,(n+a)}-\sum_{k=0}^{p-2}\frac{1}{a^{k+1}}\frac{1}{(n+a)^{p-k}},
$$

then (3.1) can be rewritten as

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{a_n}{(n+a)^p}\left(\sum_{k=1}^{n}\frac{b_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{b_k}{k+a}-\frac{b_{k+n}}{k}\right)\right)\\
&+\sum_{n=1}^{\infty}\frac{b_n}{(n+a)^p}\left(\sum_{k=1}^{n}\frac{a_{n-k}}{k}+\sum_{k=1}^{\infty}\left(\frac{a_k}{k+a}-\frac{a_{k+n}}{k}\right)\right)\\
&-p\sum_{n=1}^{\infty}\frac{a_nb_n}{(n+a)^{p+1}}-a_0\sum_{k=1}^{\infty}\frac{b_k}{k(k+a)^p}-b_0\sum_{k=1}^{\infty}\frac{a_k}{k(k+a)^p}\\
&+\sum_{\substack{j_1+j_2=p-1\\ j_1,j_2\ge1}}\left(\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^{j_1+1}}\right)\left(\sum_{k=1}^{\infty}\frac{b_k}{(k+a)^{j_2+1}}\right)=0. \qquad (3.18)
\end{aligned}
$$

Corollary 3.11 ([19])**.** *For integer $p>1$ and real $|a|<1$,*

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^p}&=\frac{p}{2}\zeta\,(p+1;a+1)-\frac12\sum_{j=1}^{p-2}\zeta\,(p-j;a+1)\zeta\,(j+1;a+1)\\
&\quad+a\zeta\,(p;a+1)\sum_{n=1}^{\infty}\frac{1}{n\,(n+a)}+\sum_{n=1}^{\infty}\frac{1}{n(n+a)^p},\\
\frac32\sum_{n=1}^{\infty}\frac{H_n^2-H_n^{(2)}}{(n+a)^p}&=p\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^{p+1}}+\sum_{n=1}^{\infty}\frac{H_n}{n\,(n+a)^p}\\
&\quad-\sum_{j=2}^{p-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^j}\right)\zeta\,(p+1-j;a+1)\\
&\quad+a\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^p}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n\,(n+a)}\right)
\end{aligned}
$$

$$+ a\zeta(p; a+1)\left(\sum_{n=1}^{\infty}\frac{H_n}{n(n+a)}\right),$$

$$\sum_{n=1}^{\infty}\frac{H_n^3 - 3H_nH_n^{(2)}}{(n+a)^p} = p\sum_{n=1}^{\infty}\frac{H_n^2}{(n+a)^{p+1}} - \sum_{j=2}^{p-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^j}\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^{p+1-j}}\right) + 2a\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+a)^p}\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n(n+a)}\right).$$

Proof. Taking $(a_k, b_k) = (1,1), (1, H_k)$ and (H_k, H_k) in (3.18), and noting that

$$\sum_{k=1}^{\infty}\left(\frac{H_k}{k+a} - \frac{H_{k+n}}{k}\right) = -a\sum_{n=1}^{\infty}\frac{H_n}{n(n+a)} - \frac{H_n^2 + H_n^{(2)}}{2}$$

by simple calculations, we can obtain the desired formulas. □

4. Further extensions

Similarly to the definition of classical (alternating) Hurwitz zeta function, let $r \in \mathbb{N}$, for any $p_j \in \mathbb{C}$ and $a_j \in \mathbb{C}\setminus\mathbb{N}^-$ $(1 \le j \le r)$ we define the Hurwitz zeta function with r-variables and the alternating Hurwitz zeta function with r-variables by

(4.1)
$$\zeta(p; a_1+1, \ldots, a_r+1) := \sum_{n=1}^{\infty}\frac{1}{(n+a_1)^{p_1}\cdots(n+a_r)^{p_r}} \quad (\Re(p_1+\cdots+p_r) > 1),$$

(4.2)
$$\bar{\zeta}(p; a_1+1, \ldots, a_r+1) := \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+a_1)^{p_1}\cdots(n+a_r)^{p_r}} \quad (\Re(p_1+\cdots+p_r) \ge 1).$$

In particular, Alzer and Choi gave the following evaluation that for $p, q \in \mathbb{N}$ and $a, b \in \mathbb{C}\setminus\mathbb{N}^-$ with $a \neq b$,

$$\zeta(p,q;a+1,b+1) = (-1)^q\binom{p+q-2}{p-1}\frac{\psi(b+1)-\psi(a+1)}{(a-b)^{p+q-1}} + (-1)^q\sum_{j=1}^{q-1}\frac{1}{j!}\binom{p+q-j-2}{p-1}\frac{\psi^{(j)}(b+1)}{(a-b)^{p+q-j-1}} + (-1)^p\sum_{j=1}^{p-1}\frac{1}{j!}\binom{p+q-j-2}{q-1}\frac{\psi^{(j)}(a+1)}{(b-a)^{p+q-j-1}}, \tag{4.3}$$

$$\bar{\zeta}(p,q;a+1,b+1) = \sum_{j=0}^{p-1}(-1)^j\binom{q+j-1}{q-1}\frac{\bar{\zeta}(p-j;a+1)}{(b-a)^{q-j}}$$
$$+\sum_{j=0}^{q-1}(-1)^j\binom{p+j-1}{p-1}\frac{\bar{\zeta}(q-j;b+1)}{(a-b)^{p-j}}, \tag{4.4}$$

where $\psi^{(j)}(a+1) = (-1)^{j+1}j!\zeta(j+1;a+1)$ for $j \in \mathbb{N}$.

By applying the same kernel functions to different base functions in Section 3, others general identities can be established.

Theorem 4.1. *For integer $p > 1$ and reals $0 < |a| < 1$, $b \notin \mathbb{Z}$ with $a \neq b$,*

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^p}{(n+b)^p}\sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a}\right)+\sum_{n=1}^{\infty}\left(\frac{(-1)^p}{(n+b)^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k-n}}{k+a}\right)$$
$$-\sum_{n=1}^{\infty}\frac{a_n}{a(n-b)^p}+\sum_{n=1}^{\infty}\left(\frac{1}{(n-b)^p}\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}\right)$$
$$+\sum_{n=1}^{\infty}\left(\frac{1}{(n-b)^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a}\right)+(-1)^{p+1}\frac{a_0}{ab^p}$$
$$+\frac{1}{(p-1)!}\frac{d^{p-1}}{ds^{p-1}}\left(\pi\cot(\pi s)\Psi(-s;A,a)\right)\Big|_{s=b}$$
$$+\sum_{n=0}^{\infty}\frac{\pi a_n}{\tan(a\pi)(n+a-b)^p}=0, \tag{4.5}$$

where if $n \to \pm\infty$, then $a_n = o\left(n^{\alpha}\right)$ with $\alpha < 1$.

Proof. The proof follows from the function

$$G_2(s) = \frac{\pi\cot(\pi s)\Psi(-s;A,a)}{(s-b)^p}$$

and the residue computation. By straightforward calculations, we can find the following residues that for positive integer n,

$$\mathrm{Res}[G_2(s), s=-n] = \frac{\sum_{k=1}^{n-1}\frac{a_{k-n}}{k+a}+\sum_{k=1}^{\infty}\frac{a_k-a_{k-n}}{k+a}}{(-1)^p(n+b)^p},$$
$$\mathrm{Res}[G_2(s), s=n] = -\frac{a_n}{a(n-b)^p}+\frac{\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}+\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a}}{(n-b)^p},$$
$$\mathrm{Res}[G_2(s), s=n+a-1] = \frac{\pi a_{n-1}}{\tan(a\pi)(n+a-b-1)^p},$$
$$\mathrm{Res}[G_2(s), s=b] = \frac{1}{(p-1)!}\frac{d^{p-1}}{ds^{p-1}}\left(\pi\cot(\pi s)\Psi(-s;A,a)\right)\Big|_{s=b},$$
$$\mathrm{Res}[G_2(s), s=0] = \frac{(-1)^{p+1}a_0}{ab^p}.$$

Then according to Lemma 2.1 we have

$$\sum_{n=1}^{\infty} (\mathrm{Res}[G_2(s), s = n + a - 1] + \mathrm{Res}[G_2(s), s = -n] + \mathrm{Res}[G_2(s), s = n]) + \mathrm{Res}[G_2(s), s = 0] + \mathrm{Res}[G_2(s), s = b] = 0.$$

Thus, by a simple calculation, we may deduce the desired result. □

Corollary 4.2. *For reals $0 < |a| < 1$ and $b \notin \mathbb{Z}$ with $a \neq b$, we have*

$$\begin{aligned}
S_{1,2}(a,b) &= S_{1,2}^{++}(-a,-b) + S_{1,2}^{++}(a,b) \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n(-a)}{(n-b)^2} + \frac{H_n(a)}{(n+b)^2} \right) \\
&= \frac{1}{ab^2} + \zeta(1,2;a+1,b+1) + \frac{\zeta(2;1-b)}{a} - \pi\cot(a\pi)\zeta(2;a-b) \\
&\quad + \pi\cot(b\pi)\zeta(2;a-b) + \pi^2\csc^2(b\pi)\left(\Psi(-b;a)+\gamma\right), \qquad (4.6)\\
S_{1,3}(a,b) &= S_{1,3}^{++}(-a,-b) - S_{1,3}^{++}(a,b) \\
&= \sum_{n=1}^{\infty} \left(\frac{H_n(-a)}{(n-b)^3} - \frac{H_n(a)}{(n+b)^3} \right) \\
&= \frac{\zeta(3;1-b)}{a} - \zeta(1,3;a+1,b+1) - \pi\cot(a\pi)\zeta(3;a-b) - \frac{1}{ab^3} \\
&\quad - \pi^3\cot(b\pi)\csc^2(b\pi)\left(\Psi(-b;a)+\gamma\right) \\
&\quad + \pi\cot(\pi b)\zeta(3;a-b) - \pi^2\csc^2(b\pi)\zeta(2;a-b), \qquad (4.7)\\
\bar{L}_{1,2}(a,b) &= S_{1,2}^{--}(-a,-b) + S_{1,2}^{--}(a,b) \\
&= \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(n+b)^2} \sum_{k=1}^{n} \frac{(-1)^k}{k+a} + \frac{(-1)^n}{(n-b)^2} \sum_{k=1}^{n} \frac{(-1)^k}{k-a} \right) \\
&= \zeta(1,2;a+1,b+1) + \frac{\bar{\zeta}(2;-b)}{a} + \pi\cot(b\pi)\bar{\zeta}(2;a-b) \\
&\quad - \pi\cot(a\pi)\bar{\zeta}(2;a-b) \\
&\quad + \bar{\zeta}(1;1+a)\left(\bar{\zeta}(2;1+b) + \zeta(2;1+b) + \bar{\zeta}(2;1-b) + \zeta(2;1-b)\right) \\
&\quad - \pi^2\csc^2(b\pi)\left(\bar{\zeta}(1;a-b) + \bar{\zeta}(1;a+1)\right). \qquad (4.8)
\end{aligned}$$

Proof. Letting $p = 2, 3, a_k = 1$ and $p = 2, a_k = (-1)^k$ in (4.5), by elementary calculations, we obtain the three results. □

Theorem 4.3. *For integer $p > 1$ and reals $0 < |a| < 1$, $b \notin \mathbb{Z}$ with $a \neq b$,*

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+p}}{(n+b)^p} \sum_{k=1}^{n-1} \frac{a_{k-n}}{k+a} \right) + \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+p}}{(n+b)^p} \sum_{k=1}^{\infty} \frac{a_k - a_{k-n}}{k+a} \right)$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n a_n}{a(n-b)^p}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{(n-b)^p}\sum_{k=1}^{\infty}\frac{a_k-a_{k+n}}{k+a}\right)+(-1)^{p+1}\frac{a_0}{ab^p}$$
$$+\frac{\pi}{(p-1)!}\frac{d^{p-1}}{ds^{p-1}}\left.\frac{\Psi(-s;A,a)}{\sin(\pi s)}\right|_{s=b}+\sum_{n=0}^{\infty}\frac{(-1)^n\pi a_n}{\sin(a\pi)(n+a-b)^p}$$
$$+\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{(n-b)^p}\sum_{k=1}^{n}\frac{a_{n-k}}{k-a}\right)=0, \tag{4.9}$$

where $n\to\pm\infty$, then $a_n=o\left(n^{\alpha}\right)$ with $\alpha<1$.

Proof. The theorem results from consider the function

$$G_3(s)=\frac{\pi\Psi(-s;A,a)}{\sin(\pi s)(s-b)^p},$$

and performing the residue computation. We leave the detail to the interested reader. □

Corollary 4.4. *For reals $0<|a|<1$ and $b\notin\mathbb{Z}$ with $a\neq b$, we have*

$$\begin{aligned}
L_{1,2}(a,b) &= -S_{1,2}^{+-}(-a,-b)-S_{1,2}^{+-}(a,b)\\
&= \sum_{n=1}^{\infty}\left(\frac{(-1)^n H_n(-a)}{(n-b)^2}+\frac{(-1)^n H_n(a)}{(n+b)^2}\right)\\
&= \frac{1}{ab^2}-\bar{\zeta}(1,2;a+1,b+1)-\frac{\bar{\zeta}(2;1-b)}{a}-\frac{\pi}{\sin(a\pi)}\bar{\zeta}(2;a-b)\\
&\quad +\frac{\pi}{\sin(b\pi)}\zeta(2;a-b)+\pi^2\csc(b\pi)\cot(b\pi)\left(\Psi(-b;a)+\gamma\right),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
L_{1,3}(a,b) &= S_{1,3}^{+-}(a,b)-S_{1,3}^{+-}(-a,-b)\\
&= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1} H_n(a)}{(n+b)^3}+\frac{(-1)^n H_n(-a)}{(n-b)^3}\right)\\
&= \bar{\zeta}(1,3;a+1,b+1)-\frac{\pi}{\sin(a\pi)}\bar{\zeta}(3;a-b)-\frac{1}{ab^3}-\frac{\bar{\zeta}(3;1-b)}{a}\\
&\quad -\pi^2\cot(b\pi)\csc(b\pi)\zeta(2;a-b)+\pi\csc(\pi b)\zeta(3;a-b)\\
&\quad -\left(\frac{\pi^3\cot^2(b\pi)\csc(b\pi)}{2}+\frac{\pi^3\csc^3(b\pi)}{2}\right)\left(\Psi(-b;a)+\gamma\right),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\bar{S}_{1,2}(a,b) &= -S_{1,2}^{-+}(a,b)-S_{1,2}^{-+}(-a,-b)\\
&= \sum_{n=1}^{\infty}\left(\frac{1}{(n+b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{k+a}+\frac{1}{(n-b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{k-a}\right)\\
&= -\bar{\zeta}(1,2;a+1,b+1)+\frac{1}{a}\zeta(2;-b)+\pi\csc(b\pi)\bar{\zeta}(2;a-b)\\
&\quad -\bar{\zeta}(1;1+a)\left(\bar{\zeta}(2;1+b)+\zeta(2;1+b)+\bar{\zeta}(2;1-b)+\zeta(2;1-b)\right)
\end{aligned}$$

$$-\pi\csc(a\pi)\zeta(2;a-b)-\pi^2\cot(b\pi)\csc(b\pi)\left(\bar\zeta(1;a+1)+\bar\zeta(1;a-b)\right). \tag{4.12}$$

Proof. It follows immediately from (4.9) with $p=2,3$, $a_k=1$ and $p=2$, $a_k=(-1)^k$. □

Theorem 4.5. *For any $0<|a|<1$ and $b\notin\mathbb{Z}$ with $a\neq b$ and positive integers $p\geq 1$, $q>1$,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{p+q}}{(n+b)^q}\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^p}-\sum_{n=1}^{\infty}\frac{(-1)^{p+q}}{(n+b)^q}\sum_{k=1}^{n-1}\frac{a_{k-n}}{(k+a)^p}+\sum_{n=1}^{\infty}\frac{(-1)^p a_n}{(n-b)^q a^p}\\&+\sum_{n=1}^{\infty}\frac{(-1)^p}{(n-b)^q}\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^p}+\sum_{n=1}^{\infty}\frac{1}{(n-b)^q}\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^p}\\&+\frac{(-1)^{p+q}a_0}{a^pb^q}+\frac{(-1)^{p+q}}{b^q}\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^p}+\sum_{n=0}^{\infty}\frac{a_n\pi}{(p-1)!}\frac{d^{p-1}}{ds^{p-1}}\frac{\cot(\pi s)}{(s-b)^q}\bigg|_{s=n+a}\\&+\frac{1}{(q-1)!}\frac{d^{q-1}}{ds^{q-1}}\frac{\pi\cot(\pi s)\Psi^{(p-1)}(-s;A,a)}{(p-1)!}\bigg|_{s=b}=0,\end{aligned}\tag{4.13}$$

where $n\to\pm\infty$, then $a_n=o\left(n^{\alpha}\right)$ with $\alpha<1$.

Proof. The theorem follows from the function

$$G_4(s):=\frac{\pi\cot(\pi s)\Psi^{(p-1)}(-s;A,a)}{(p-1)!(s-b)^q}$$

and the direct residue computation. □

Corollary 4.6. *For any $0<|a|<1$ and $b\notin\mathbb{Z}$ with $a\neq b$, we have*

$$\begin{aligned}S_{2,2}(a,b)&=S_{2,2}^{++}(-a,-b)-S_{2,2}^{++}(a,b)\\&=\sum_{n=1}^{\infty}\left(\frac{H_n^{(2)}(-a)}{(n-b)^2}-\frac{H_n^{(2)}(a)}{(n+b)^2}\right)\\&=-\zeta(2;a+1)\left(\zeta(2;1-b)+\zeta(2;1+b)\right)-\frac{\zeta(2;1-b)}{a^2}-\frac{\zeta(2;a+1)}{b^2}\\&\quad-\frac{1}{a^2b^2}+\frac{\pi^2}{\sin^2(\pi a)}\zeta(2;a-b)+2\pi\cot(\pi a)\zeta(3;a-b))\\&\quad+\frac{\pi^2}{\sin^2(\pi b)}\zeta(2;a-b-2\pi\cot(\pi b)\zeta(3;a-b)-\zeta(2,2;a+1,b+1),\end{aligned}\tag{4.14}$$

$$\begin{aligned}\bar L_{2,2}(a,b)&=S_{2,2}^{--}(-a,-b)-S_{2,2}^{--}(a,b)\\&=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{(n-b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{(k-a)^2}-\frac{(-1)^n}{(n+b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{(k+a)^2}\right)\\&=-\frac{1}{a^2}\bar\zeta(2;-b)+2\pi\cot(a\pi)\bar\zeta(3;a-b)+\pi^2\csc^2(a\pi)\bar\zeta(2;a-b)\\&\quad-2\pi\cot(b\pi)\bar\zeta(3;a-b)+\pi^2\csc^2(b\pi)\bar\zeta(2;a-b)\end{aligned}$$

$$-\zeta(2,2;a+1,b+1)-\bar{\zeta}(2;a+1)\left(\bar{\zeta}(2;b+1)-\bar{\zeta}(2;-b)\right). \tag{4.15}$$

Proof. The corollary follows immediately from (4.13) if we set $p=q=2$, $a_k=1$ and $p=q=2$, $a_k=(-1)^k$. □

Theorem 4.7. *For any $0<|a|<1$ and $b\notin\mathbb{Z}$ with $a\neq b$ and positive integers $p\geq 1$, $q>1$,*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^{n+p+q}}{(n+b)^q}\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^p}-\sum_{n=1}^{\infty}\frac{(-1)^{n+p+q}}{(n+b)^q}\sum_{k=1}^{n-1}\frac{a_{k-n}}{(k+a)^p}\\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n+p}a_n}{(n-b)^q a^p}+\sum_{n=1}^{\infty}\frac{(-1)^{n+p}}{(n-b)^q}\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^p}\\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n+2p}}{(n-b)^q}\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^p}+\sum_{n=0}^{\infty}\frac{a_n\pi}{(p-1)!}\frac{d^{p-1}}{ds^{p-1}}\left.\frac{1}{\sin(\pi s)(s-b)^q}\right|_{s=n+a}\\
&+\frac{(-1)^{p+q}a_0}{a^p b^q}+\frac{(-1)^{p+q}}{b^q}\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^p}\\
&+\frac{1}{(q-1)!}\frac{d^{q-1}}{ds^{q-1}}\left.\frac{\pi\Psi^{(p-1)}(-s;A,a)}{\sin(\pi s)(p-1)!}\right|_{s=b}=0,
\end{aligned}\tag{4.16}$$

where $n\to\pm\infty$, then $a_n=o\left(n^{\alpha}\right)$ with $\alpha<1$.

Proof. We consider the function

$$G_5(s):=\frac{\pi\Psi^{(p-1)}(-s;A,a)}{\sin(\pi s)(p-1)!(s-b)^q}.$$

By direct calculations, we compute that for positive integer n,

$$\begin{aligned}
&\mathrm{Res}[G_5(s),s=-n]=\frac{(-1)^{n+q+p}\left(\sum_{k=1}^{\infty}\frac{a_{k-n}}{(k+a)^p}-\sum_{k=1}^{n-1}\frac{a_{k-n}}{(k+a)^p}\right)}{(n+b)^q},\\
&\mathrm{Res}[G_5(s),s=n]=\frac{(-1)^{n+p}}{(n-b)^q}\left(\frac{a_n}{a^p}+\sum_{k=1}^{\infty}\frac{a_{k+n}}{(k+a)^p}+(-1)^p\sum_{k=1}^{n}\frac{a_{n-k}}{(k-a)^p}\right),\\
&\mathrm{Res}[G_5(s),s=n+a-1]=\frac{1}{(p-1)!}\lim_{s\to n+a-1}\frac{d^{p-1}}{ds^{p-1}}\frac{a_{n-1}\pi}{\sin(\pi s)(s-b)^q},\\
&\mathrm{Res}[G_5(s),s=b]=\frac{1}{(q-1)!}\lim_{s\to b}\frac{d^{q-1}}{ds^{q-1}}\frac{\pi\Psi^{(p-1)}(-s;A,a)}{\sin(\pi s)(p-1)!},\\
&\mathrm{Res}[G_5(s),s=0]=\frac{(-1)^{p+q}\left(\frac{a_0}{a^p}+\sum_{k=1}^{\infty}\frac{a_k}{(k+a)^p}\right)}{b^q}.
\end{aligned}$$

From Lemma 2.1 we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\mathrm{Res}[G_5(s),s=n+a]+\mathrm{Res}[G_5(s),s=-n]+\mathrm{Res}[G_5(s),s=n]\right)\\
&+\mathrm{Res}[G_5(s),s=0]+\mathrm{Res}[G_5(s),s=b]=0.
\end{aligned}$$

Summing these five contributions yields the statement of the theorem. □

Corollary 4.8. *For any $0 < |a| < 1$ and $b \notin \mathbb{Z}$ with $a \neq b$, we have*

$$\begin{aligned} L_{2,2}(a,b) =& -S^{+-}_{2,2}(-a,-b) + S^{+-}_{2,2}(a,b) \\ =& \sum_{n=1}^{\infty}\left(\frac{(-1)^n H_n^{(2)}(-a)}{(n-b)^2} - \frac{(-1)^n H_n^{(2)}(a)}{(n+b)^2}\right) \\ =& \zeta(2;a+1)(\bar\zeta(2;1-b)+\bar\zeta(2;1+b)) + \bar\zeta(2,2;a+1,b+1) \\ &+ \frac{\bar\zeta(2;1-b)}{a^2} + \frac{2\pi}{\sin(a\pi)}\bar\zeta(3;a-b) - \frac{\zeta(2;a+1)}{b^2} \\ &+ \cot(a\pi)\csc(\pi a)\pi^2\bar\zeta(2;a-b) - \frac{1}{a^2b^2} - 2\pi\csc(\pi b)\zeta(3;a-b) \\ &+ \cot(b\pi)\csc(\pi b)\pi^2\zeta(2;a-b), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \bar S_{2,2}(a,b) =& -S^{-+}_{2,2}(-a,-b) + S^{-+}_{2,2}(a,b) \\ =& \sum_{n=1}^{\infty}\left(\frac{1}{(n-b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{(k-a)^2} - \frac{1}{(n+b)^2}\sum_{k=1}^{n}\frac{(-1)^k}{(k+a)^2}\right) \\ =& \bar\zeta(2;1+a)\left(\zeta(2;1+b)+\zeta(2;-b)\right) - \frac{1}{a^2}\zeta(2;-b) \\ &+ 2\pi\csc(a\pi)\zeta(3;a-b) + \pi^2\cot(a\pi)\csc(a\pi)\zeta(2;a-b) \\ &- 2\pi\csc(b\pi)\bar\zeta(3;a-b) + \bar\zeta(2,2;a+1,b+1) \\ &+ \pi^2\cot(b\pi)\csc(b\pi)\bar\zeta(2;a-b). \end{aligned} \tag{4.18}$$

Proof. Setting $p = q = 2$, $a_k = 1$ and $p = q = 2$, $a_k = (-1)^k$ in (4.16) yield the two desired evaluations with elementary calculations. □

References

[1] H. Alzer and J. Choi, *Four parametric linear Euler sums*, J. Math. Anal. Appl. **484** (2020), no. 1, 123661, 22 pp. https://doi.org/10.1016/j.jmaa.2019.123661

[2] D. H. Bailey, J. M. Borwein, and R. Girgensohn, *Experimental evaluation of Euler sums*, Experiment. Math. **3** (1994), no. 1, 17–30. http://projecteuclid.org/euclid.em/1062621000

[3] B. C. Berndt, *Ramanujan's Notebooks. Part II*, Springer, New York, 1989. https://doi.org/10.1007/978-1-4612-4530-8

[4] D. Borwein, J. M. Borwein, and R. Girgensohn, *Explicit evaluation of Euler sums*, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 2, 277–294. https://doi.org/10.1017/S0013091500019088

[5] J. Choi, *Certain summation formulas involving harmonic numbers and generalized harmonic numbers*, Appl. Math. Comput. **218** (2011), no. 3, 734–740. https://doi.org/10.1016/j.amc.2011.01.062

[6] J. Choi and H. M. Srivastava, *Some summation formulas involving harmonic numbers and generalized harmonic numbers*, Math. Comput. Modelling **54** (2011), no. 9-10, 2220–2234. https://doi.org/10.1016/j.mcm.2011.05.032

[7] A. Dil and K. N. Boyadzhiev, *Euler sums of hyperharmonic numbers*, J. Number Theory **147** (2015), 490–498. https://doi.org/10.1016/j.jnt.2014.07.018

[8] A. Dil, I. Mező, and M. Cenkci, *Evaluation of Euler-like sums via Hurwitz zeta values*, Turkish J. Math. **41** (2017), no. 6, 1640–1655. https://doi.org/10.3906/mat-1603-4
[9] P. Flajolet and B. Salvy, *Euler sums and contour integral representations*, Experiment. Math. **7** (1998), no. 1, 15–35. http://projecteuclid.org/euclid.em/1047674270
[10] I. Mező, *Nonlinear Euler sums*, Pacific J. Math. **272** (2014), no. 1, 201–226. https://doi.org/10.2140/pjm.2014.272.201
[11] I. Mező and A. Dil, *Hyperharmonic series involving Hurwitz zeta function*, J. Number Theory **130** (2010), no. 2, 360–369. https://doi.org/10.1016/j.jnt.2009.08.005
[12] X. Si, *Euler-type sums involving multiple harmonic sums and binomial coefficients*, Open Math. **19** (2021), no. 1, 1612–1619. https://doi.org/10.1515/math-2021-0124
[13] X. Si, C. Xu, and M. Zhang, *Quadratic and cubic harmonic number sums*, J. Math. Anal. Appl. **447** (2017), no. 1, 419–434. https://doi.org/10.1016/j.jmaa.2016.10.026
[14] A. Sofo, *Quadratic alternating harmonic number sums*, J. Number Theory **154** (2015), 144–159. https://doi.org/10.1016/j.jnt.2015.02.013
[15] A. Sofo and J. Choi, *Extension of the four Euler sums being linear with parameters and series involving the zeta functions*, J. Math. Anal. Appl. **515** (2022), no. 1, Paper No. 126370, 23 pp. https://doi.org/10.1016/j.jmaa.2022.126370
[16] A. Sofo and H. M. Srivastava, *Identities for the harmonic numbers and binomial coefficients*, Ramanujan J. **25** (2011), no. 1, 93–113. https://doi.org/10.1007/s11139-010-9228-3
[17] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012. https://doi.org/10.1016/B978-0-12-385218-2.00001-3
[18] W. Wang and Y. Lyu, *Euler sums and Stirling sums*, J. Number Theory **185** (2018), 160–193. https://doi.org/10.1016/j.jnt.2017.08.037
[19] C. Xu, *Some evaluation of parametric Euler sums*, J. Math. Anal. Appl. **451** (2017), no. 2, 954–975. https://doi.org/10.1016/j.jmaa.2017.02.047
[20] C. Xu, *Multiple zeta values and Euler sums*, J. Number Theory **177** (2017), 443–478. https://doi.org/10.1016/j.jnt.2017.01.018
[21] C. Xu, *Some evaluations of infinite series involving parametric harmonic numbers*, Int. J. Number Theory **15** (2019), no. 7, 1531–1546. https://doi.org/10.1142/S179304211950088X
[22] C. Xu, *Extensions of Euler-type sums and Ramanujan-type sums*, Kyushu J. Math. **75** (2021), no. 2, 295–322. https://doi.org/10.2206/kyushujm.75.295
[23] C. Xu and W. Wang, *Explicit formulas of Euler sums via multiple zeta values*, J. Symbolic Comput. **101** (2020), 109–127. https://doi.org/10.1016/j.jsc.2019.06.009
[24] J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on Number Theory and its Applications, 12, World Sci. Publ., Hackensack, NJ, 2016. https://doi.org/10.1142/9634

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