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THE BRUHAT ORDER OF GENERALIZED ALTERNATING SIGN MATRICES AND ITS RANK

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Abstract. We continue the investigations in [7] extending the Bruhat order on $n \times n$ alternating sign matrices to a more general setting. We show that the resulting partially ordered set is a lattice and also investigate its rank.

1. Introduction

An $n \times n$ *alternating sign matrix* (abbreviated to ASM) is a $(0, \pm 1)$ matrix such that, ignoring 0's, the $+1$'s and in each row and each column alternate beginning and ending with $+1$. The origins and properties of ASMs can be found in [1].

In [7] a generalization of alternating sign matrices was defined as follows: Let $u = (u_1, u_2, \ldots, u_n), u' = (u'_1, u'_2, \ldots, u'_n), v = (v_1, v_2, \ldots, v_m)$, and $v' = (v'_1, v'_2, \ldots, v'_m)$ be vectors of ± 1's. If A is an $m \times n$ matrix, we define $A(u, u'|v, v')$ to be the $(m+2) \times (n+2)$ matrix (1.1) with rows indexed by $0, 1, \ldots, m+1$ and columns indexed by $0, 1, \ldots n+1$.

$$A(u,u'|v,v') = \begin{array}{c||ccccc||c} 0 & u_1 & u_2 & \cdots & u_{n-1} & u_n & 0 \\ \hline\hline v_1 & & & & & & v'_1 \\ v_2 & & & & & & v'_2 \\ \vdots & & & A & & & \vdots \\ v_{m-1} & & & & & & v'_{m-1} \\ v_m & & & & & & v'_m \\ \hline\hline 0 & u'_1 & u'_2 & \cdots & u'_{n-1} & u'_n & 0 \end{array}. \tag{1.1}$$

We then write $A = A(u, u'|v, v')[1, 2, \ldots, m|1, 2, \ldots, n]$ to denote that A is the middle $m \times n$ submatrix of $A(u, u'|v, v')$, and also denote this by $A(u, u'|v, v') \to A$.

A $(u, u'|v, v')$-*ASM* is an $m \times n$ $(0, \pm 1)$-matrix A as in (1.1) such that, ignoring 0's, the $+1$'s and -1's in rows $0, 1, 2, \ldots, m+1$ and columns $0, 1, 2, \ldots, n+1$ of

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the $(0,\pm1)$-matrix $A(u,u'|v,v')$ alternate. Thus the condition that the first and last nonzero entry in rows and columns of an ASM is $+1$ is relaxed, where now the first and last nonzero of each row is determined by v and v', respectively, and the first and last nonzero of each column is determined by u and u', respectively. Note also that, unlike ASMs, the first and last rows and columns of a $(u,u'|v,v')$-ASM depend on u,u',v,v' and so may contain more than one nonzero entry. We denote the set of $(u,u'|v,v')$-ASMs by $\mathcal{A}_{m,n}(u,u'|v,v')$. If $u=u'$ and $v=v'$ and $m=n$, we often abbreviate these notations to: (u,v)-ASM and $\mathcal{A}_n(u,v)$, respectively. If $u=u'=v=v'$, we also use the abbreviations (u)-ASM and $\mathcal{A}_n(u)$. Observe that if $u=u'=v=v'=(-1,-1,\ldots,-1)$, then a (u)-ASM is an ordinary ASM, and $\mathcal{A}_n(u)$ is the usual set $\mathcal{A}_n$ of $n\times n$ ASMs.

Example 1.1. Let $m=2$ and $n=3$, and let $u=(1,-1,1)$, $u'=(-1,1,-1)$, $v=(1,-1)$ and $v=(1,-1)$. Then a $(u,u'|v,v')$-ASM is given below:

$$\left[\begin{array}{r||rrr||r} 0 & 1 & -1 & 1 & 0 \\ \hline\hline 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ \hline\hline 0 & -1 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{rrr} -1 & 1 & -1 \\ 1 & -1 & 1 \end{array}\right].$$

In [7], necessary and sufficient conditions are given for the nonemptiness of $\mathcal{A}_{m,n}(u,u'|v,\,v')$. Let $u=(u_1,u_2,\ldots,u_n), u'=(u'_1,u'_2,\ldots,u'_n), v=(v_1,v_2,\ldots,v_m)$, and $v'=(v'_1,v'_2,\ldots,\ v'_m)$ be vectors of ±1s. We define m-vectors $r^+(v,v')=(r^+_1,r^+_2,\ldots,r^+_m)$ and $r^-(v,v')=(r^-_1,r^-_2,\ldots,r^-_m)$, and n-vectors $c^+(u,u')=(c^+_1,c^+_2,\ldots,c^+_n)$ and $c^-(u,u')=(c^-_1,c^-_2,\ldots,c^-_n)$ as follows:

$r^+_k=r^+_k(v,v')$ is the number of $i\le k$ such that $v_i=v'_i=+1$;
$r^-_k=r^-_k(v,v')$ is the number of $i\le k$ such that $v_i=v'_i=-1$;
$c^+_l=c^+_l(u,u')$ is the number of $j\le l$ such that $u_j=u'_j=+1$; and
$c^-_l=c^-_l(u,u')$ is the number of $j\le l$ such that $u_j=u'_j=-1$.

In particular, we have:

$r^+_m=r^+_m(v,v')$ is the total number of i such that $v_i=v'_i=+1$;
$r^-_m=r^-_m(v,v')$ is the total number of i such that $v_i=v'_i=-1$;
$c^+_n=c^+_n(u,u')$ is the total number of j such that $u_j=u'_j=+1$; and
$c^-_n=c^-_n(u,u')$ is the total number of j such that $u_j=u'_j=-1$.

$$r^-_m(v,v')-r^+_m(v,v')=c^-_n(u,u')-c^+_n(u,u'). \tag{1.2}$$

We set

$$u^+ = |\{j : u_j = +1\}| \text{ and } u^- = |\{j : u_j = -1\}|,$$
$$v^+ = |\{i : v_i = +1\}| \text{ and } v^- = |\{i : v_i = -1\}|.$$

If we consider the sum of the first k rows of $(u, u'|v, v')$-ASM, then we obtain

$$-u^+ \leq r_k^-(v, v') - r_k^+(v, v') \leq u^- \quad (k = 1, 2, \ldots, m). \tag{1.3}$$

In a similar way, we obtain

$$-v^+ \leq c_l^-(u, u') - c_l^+(u, u') \leq v^- \quad (l = 1, 2, \ldots, n). \tag{1.4}$$

In [7], a $(u, u'|v, v')$-ASM exists if and only if (1.2), (1.3), and (1.4) hold. In this paper we primarily consider $\mathcal{A}(u, u'|v, v')$ where (1.2), (1.3), and (1.4) hold.

For any $m \times n$ matrix $A = [a_{ij}]$, the *sum-matrix* $\Sigma(A) = [\sigma_{ij}]$ of A is the $m \times n$ matrix where

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{k \leq i} \sum_{l \leq j} a_{kl} \quad (1 \leq i \leq m, 1 \leq j \leq n),$$

the sum of the entries in the leading $i \times j$ submatrix of A. Define $\sigma_{ij} = 0$ if $i = 0$ or $j = 0$. Then the matrix A is uniquely determined by its sum-matrix $\Sigma(A)$, namely,

$$a_{ij} = \sigma_{ij} - \sigma_{i,j-1} - \sigma_{i-1,j} + \sigma_{i-1,j-1} \quad (1 \leq i \leq m, 1 \leq j \leq n).$$

The sum-matrix $\Sigma(A) = [\sigma_{ij}]$ of a matrix $A \in \mathcal{A}(u, u'|v, v')$ has the following properties which are easily verified:

- The integers in row i and column j of $\Sigma(A)$ are taken from the set $\{0, \pm 1, \pm 2, \ldots\}$; moreover, $\sigma_{i1} \in \{0, -v_i\}$ and $\sigma_{1j} \in \{0, -u_j\}$ so that $\sigma_{i1} \in \{0, \pm 1\}$ for $1 \leq i \leq m$ and $\sigma_{1j} \in \{0, \pm 1\}$ for $1 \leq j \leq n$.
- Consecutive entries in a row or column differ in absolute value by at most 1.
- $\sigma_{in} = -(v_1 + v_2 + \cdots + v_i)$ for $1 \leq i \leq m$.
- $\sigma_{nj} = -(u_1 + u_2 + \cdots + u_j)$ for $1 \leq j \leq n$.

Example 1.2. Let $u = u' = (1, -1, 1, -1, 1)$ and $v = v' = (1, -1, 1)$. For

$$A = \left[\begin{array}{c|c|c|c|c} & & & & -1 \\ \hline & 1 & -1 & 1 & \\ \hline -1 & & & & \end{array}\right] \in \mathcal{A}_{3,5}(u, v), \text{ we have } \Sigma(A) = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ \hline -1 & 0 & -1 & 0 & -1 \end{array}\right].$$

In the next section we generalize the Bruhat order on the set $\mathcal{A}_n$ of $n \times n$ ASMs to the set $\mathcal{A}_{m,n}(u, u'|v, v')$. As with $\mathcal{A}_n$ we obtain a ranked lattice. In [8], we have $\mathcal{A}_n(u, v)$ is a ranked lattice where $\mathcal{A}_n(u, v)$ denote $\mathcal{A}_{m,n}(u, u'|v, v')$ for $u = u' =$

$(u_1, u_2, \ldots, u_n)$ and $v = v' = (v_1, v_2, \ldots, v_n)$ which are vectors of ± 1's where u and v have the same number of $+1$'s and the same number of -1's.

2. Bruhat Order and Lattice

There is a partial order, called the *Bruhat order* and denoted by $\preceq_B$, defined on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$, equivalently, the set $\mathcal{P}_n$ of $n \times n$ permutation matrices, which has also been extended to the set $\mathcal{A}_n$ of $n \times n$ ASMs [10]. We briefly describe this partial order in its various equivalent forms [8].

(i) For $\pi, \tau \in \mathcal{S}_n$, $\pi \preceq_B \tau$ provided π can be obtained from τ by a sequence of transpositions each of which reduces the number of inversions, not necessarily the set of inversions. There is such a sequence of transpositions each of which reduces the number of inversions by exactly one (but not, in general, by removing one inversion). The identity $\iota_n = (1, 2, \ldots, n)$ (the identity matrix I_n using $\mathcal{P}_n$) is the unique minimal permutation in the Bruhat order on $\mathcal{S}_n$; the anti-identity $\zeta_n = (n, \ldots, 2, 1)$ (the anti-identity matrix L_n using $\mathcal{P}_n$) is the unique maximal permutation. In terms of permutation matrices, for $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if P can be obtained from Q by a sequence of *interchanges* involving 2×2 submatrices (not necessarily with consecutive rows and consecutive columns):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) Another characterization of the Bruhat order is: For $P, Q \in \mathcal{P}_n$, $P \preceq_B Q$ if and only if $\Sigma(P) \geq \Sigma(Q)$ (entrywise).

(iii) The Bruhat order extends to $\mathcal{A}_n$ by defining for $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$. Then $(\mathcal{A}_n, \preceq_B)$ is a graded lattice extending the partially ordered set $(\mathcal{P}_n, \preceq_B)$, and indeed is the (unique up to isomorphism) smallest lattice extending $(\mathcal{P}_n, \preceq_B)$ (the *Dedekind-MacNeille completion* of $(\mathcal{P}_n, \preceq_B)$) [10]. The minimal element of $(\mathcal{A}_n, \preceq_B)$ is I_n, and the maximal element is L_n.

(iv) For $A_1, A_2 \in \mathcal{A}_n$, $A_1 \preceq_B A_2$ if and only if there is a sequence of ASMs, $X_1 = A_1, X_2, \ldots, X_p = A_2$, such that for $t = 1, 2, \ldots, p-1$, X_t can be obtained from X_{t+1} by an *interchange* which adds T where the 2×2 submatrix

$T[i,j|k,l]$ $(i < j, k < l)$ of T is the 2×2 matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and all other entries are zero. In order that the resulting matrices be ASMs, the 2×2 submatrix of X_{t+1} must equal

$$\left[\begin{array}{c|c} 0 \text{ or } -1 & 0 \text{ or } 1 \\ \hline 0 \text{ or } 1 & 0 \text{ or } -1 \end{array}\right].$$

An interchange is the analogue for ASMs of a interchange of a permutation matrix.

(v) Let $P, Q \in \mathcal{P}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{P}_n$ is defined by $P \preceq_b Q$ provided P can be obtained from Q by a sequence of *adjacent interchanges*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

applied to 2×2 submatrices with consecutive rows and consecutive columns. Since interchanges reduce the number of inversions of the corresponding permutations by exactly 1, $(\mathcal{P}_n, \preceq_b)$ is a subpartially ordered set of $(\mathcal{P}_n, \preceq_B)$ and, in fact, $(\mathcal{P}_n, \preceq_b)$ is a lattice. This lattice can be equivalently described in terms of the corresponding permutations in $\mathcal{S}_n$ as follows: For $\pi_1, \pi_2 \in \mathcal{S}_n$, $\pi_1 \preceq_b \pi_2$ if and only if the set $\text{inv}(\pi_1)$ of inversions of π_1 is a subset of the set $\text{inv}(\pi_2)$ of inversions of π_2. In general, $\text{inv}(\pi_1) \cap \text{inv}(\pi_2)$ is the set of inversions of a unique permutation $\pi_1 \wedge \pi_2$, the *meet* of π_1 and π_2, and $\text{inv}(\pi_1) \cup \text{inv}(\pi_2)$ is the set of inversions of a unique permutation $\pi_1 \vee \pi_2$, the *join* of π_1 and π_2 [2].

(vi) Let $A_1, A_2 \in \mathcal{A}_n$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{A}_n$ can be defined by writing $A_1 \preceq_b A_2$ provided A_1 can be obtained from A_2 by a sequence of *adjacent interchanges* each of which adds to a 2×2 submatrix $A_2[i, i+1|j, j+1]$ of A_2 with consecutive rows and columns, the $n \times n$ matrix $T_{i,j}$ which is all 0's except for its 2×2 submatrix determined by rows i and $i+1$ and columns j and $j+1$ which equals

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and where the result is also an ASM. Unlike for $\mathcal{P}_n$, the Bruhat order and weak Bruhat order coincide on $\mathcal{A}_n$.

(vii) Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be vectors of ± 1's where u and v have the same number of $+1$'s and the same number of -1's. Let

$A_1, A_2 \in \mathcal{A}_n(u,v)$. The *weak Bruhat order* $\preceq_b$ on $\mathcal{A}_n(u,v)$ can be defined by writing $A_1 \preceq_b A_2$ provided A_1 can be obtained from A_2 by a sequence of *adjacent interchanges* each of which adds to a 2×2 submatrix $A_2[i,i+1|j,j+1]$ of A_2 with consecutive rows and columns, the $n\times n$ matrix $T_{i,j}$ which is all 0's except for its 2×2 submatrix determined by rows i and $i+1$ and columns j and $j+1$ which equals

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and where the result is also an element in $\mathcal{A}_n(u,v)$.

These and other properties of ASMs can be found in several places including [2, 3, 4, 5, 6, 7, 8, 9, 10].

Let $u = (u_1, u_2, \ldots, u_n), u' = (u'_1, u'_2, \ldots, u'_n), v = (v_1, v_2, \ldots, v_m)$, and $v' = (v'_1, v'_2, \ldots, v'_m)$ be vectors of ± 1's. There is a partial order, called the *Bruhat order* and denoted by $\preceq_B$, on $\mathcal{A}_{m,n}(u,v|u',v')$ of $(u,u'|v,v')$-ASMs. There are three possibilities for defining $A_1 \preceq_B A_2$ for $A_1, A_2 \in \mathcal{A}_{m,n}(u,v|u',v')$, namely:

(a) $\Sigma(A_1) \geq \Sigma(A_2)$ (entrywise).
(b) A_1 can be obtained from A_2 by a sequence of interchanges.
(c) A_1 can be obtained from A_2 by a sequence of adjacent interchanges.

We will prove that these three possibilities are also equivalent for $\mathcal{A}_{m,n}(u,v|u',v')$. If A_1 can be obtained from A_2 by a sequence of adjacent interchanges, then we denote $A_2 \to A_1$.

Example 2.1. Let $u = (-1,1,1,1,-1)$, $u' = (-1,-1,-1,-1,-1)$ and $v = v' = (-1,-1)$. Let

$$A_1 : \begin{array}{c||c|c|c|c|c||c} 0 & - & + & + & + & - & 0 \\ \hline\hline - & + & & & & & - \\ \hline - & & & & & + & - \\ \hline\hline 0 & - & - & - & - & - & 0 \end{array} \text{ and } A_2 : \begin{array}{c||c|c|c|c|c||c} 0 & - & + & + & + & - & 0 \\ \hline\hline - & & & & & + & - \\ \hline - & + & & & & & - \\ \hline\hline 0 & - & - & - & - & - & 0 \end{array}.$$

Since $\Sigma(A_1) \geq \Sigma(A_2)$, $A_1 \preceq_B A_2$ by definition (a). Also, A_1 can be obtained from A_2 by a sequence of adjacent interchanges such as;

$$A_2: \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & & & & & + & - \\ \hline - & + & & & & & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array} \to \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & + & - & & & + & - \\ \hline - & & + & & & & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}$$

$$\to \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & + & & - & & + & - \\ \hline - & & & + & & & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}$$

$$\to \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & + & & & - & + & - \\ \hline - & & & & + & & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}$$

$$\to \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & + & & & & & - \\ \hline - & & & & & + & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array} : A_1$$

Theorem 2.2. *Let $u = (u_1, u_2, \ldots, u_n), u' = (u'_1, u'_2, \ldots, u'_n), v = (v_1, v_2, \ldots, v_m)$, and $v' = (v'_1, v'_2, \ldots, v'_m)$ be vectors of ± 1's. Let A_1 and A_2 be in $\mathcal{A}_{m,n}(u, v|u', v')$. Then there exists unique $M \in \mathcal{A}_{m,n}(u, v|u', v')$ such that*

$$\Sigma(M) = \max\{\Sigma(A_1), \Sigma(A_2)\} \ (\textit{entrywise maximum}).$$

Moreover, M can be obtained from each of A_1 and A_2 by a sequence of adjacent interchanges, and hence $M \preceq_B A_1, A_2$.

Proof. Let $A_1 = [a^1_{ij}], A_2 = [a^2_{ij}] \in \mathcal{A}_{m,n}(u, u'|v, v')$, and let $D = [d_{ij}] = \Sigma(A_1) - \Sigma(A_2)$, an integral matrix with last row and last column all 0's. If $D = 0$, then $A_1 = A_2$ and there is nothing to prove. So assume that $D \neq 0$.

If $u_j = +1$ then, since the ± 1's alternate, the partial sum of column j of a matrix in $\mathcal{A}_{m,n}(u, u'|v, v')$ down to a row i is either 0 or -1; if $u_j = -1$, this partial sum is either 0 or 1. Let $k \geq 1$ be the smallest integer such that row k of D is nonzero. Then the following *k-partial column sum property* holds: *The partial column sums of A_1 and A_2 down to row $(k-1)$ are equal.* It thus follows that row k of D is a $(0, \pm 1)$-vector. Let $l_1 \geq 1$ be the smallest integer such that $d_{kl_1} \neq 0$. Then $d_{kl_1} = \pm 1$, and since we may interchange A_1 and A_2, we may assume that $d_{kl_1} = +1$. Let $l_2 \geq l_1$ be the smallest integer such that $d_{kj} = +1$ for $l_1 \leq j \leq l_2$ and $d_{k,l_2+1} \neq +1$.

Since $d_{kn} = 0$, such an l_2 exists. The k-partial column sum property implies that $d_{k,l_2+1} = 0$. Thus we have either

Case (i) $a^1_{kl_1} = a^2_{k,l_2+1} = +1$ and $a^2_{kl_1} = a^1_{k,l_2+1} = 0$, or
Case (ii) $a^1_{kl_1} = a^2_{k,l_2+1} = 0$ and $a^2_{kl_1} = a^1_{k,l_2+1} = -1$.

We only argue the first case (i) with the argument for the second case (ii) being very similar. So assume (i) holds.

Consider the sequences determined by columns l_1 and l_2 of A_2:

$$u_{l_1}, a^2_{1l_1}, a^2_{2l_1}, \ldots, a^2_{k-1,l_1}, (a^2_{k,l_1} + 1)$$

and

$$u_{l_2+1}, a^2_{1,l_2+1}, a^2_{2,l_2+1}, \ldots, a^2_{k-1,l_2+1}, (a^2_{k,l_2+1} - 1).$$

Since $a^1_{kl_1} = 1$ and $a^2_{k,l_1} = 0$, the $+1$'s and -1's in the first of these sequences alternate. Since $a^1_{k,l_2+1} = 0$ and $a^2_{k,l_2+1} = +1$, the $+1$'s and -1's in the second of these sequences alternate. Thus there exists a q with $l_1 \leq q \leq l_2$ such that in the sequences

$$u_q, a^2_{1q}, a^2_{2q}, \ldots, a^2_{k-1,q}, (a^2_{k,q} + 1) \text{ and } u_{q+1}, a^2_{1,q+1}, a^2_{2,q+1}, \ldots, a^2_{k-1,q+1}, (a^2_{k,q+1} - 1)$$

the $+1$'s and -1's alternate. We choose the smallest such q.

Now consider the alternating sequences

$$v_i, a^2_{i1}, a^2_{i2}, \ldots, a^2_{iq} \text{ for } i > k. \tag{2.1}$$

Suppose that the last nonzero entry each of these sequences is -1 for all $i > k$. Then for each such i we compute

$$\begin{aligned} d_{iq} &= \Sigma(A_1)_{iq} - \Sigma(A_2)_{iq} \\ &= \left(\Sigma(A_1)_{i-1,q} + \sum_{j=1}^{q} a^1_{ij}\right) - \left(\Sigma(A_2)_{i-1,q} + \sum_{j=1}^{q} a^2_{ij}\right) \\ &= (\Sigma(A_1)_{i-1,q} - \Sigma(A_2)_{i-1,q}) + \left(\sum_{j=1}^{q} a^1_{ij} - \sum_{j=1}^{q} a^2_{ij}\right) \\ &= d_{i-1,q} + \sum_{j=1}^{q} a^1_{ij} - \sum_{j=1}^{q} a^2_{ij}. \end{aligned}$$

Since the last nonzero of each of the sequences (2.1) is -1, we have that

$$\sum_{j=1}^{q} a^1_{ij} \geq \sum_{j=1}^{q} a^2_{ij}.$$

Therefore $d_{iq} \geq d_{i-1,q}$ and so the sequence $d_{kq}, d_{k+1,q}, \ldots, d_{nq}$ is nondecreasing. This contradicts the fact that $d_{kq} = 1$ and $d_{nq} = 0$. Let $p(\geq k)$ be the smallest integer such that the last nonzero of the sequence $v_{p+1}, a^2_{p+1,1}, a^2_{p+1,2}, \ldots, a^2_{p+1,q}$ is $+1$ and let $A_2^1 = A_2 + T_{p,q}$, where $T_{p,q}$ is the matrix which is all zeros except for its 2×2 submatrix, determined by rows p and $p+1$ and columns q and $q+1$, equal to

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then $A_2^1 \in \mathcal{A}_{m,n}(u, v|u', v')$ such that $D - \big(\Sigma(A_1) - \Sigma(A_2^1)\big)$ has just one nonzero entry $+1$.

As already indicated, Case (ii) proceeds in a very similar way. Thus continuing, we obtain for some k_1 and k_2, that there are adjacent interchanges $A_1 \to A_1^1 \to A_1^2 \to \cdots A_1^{k_1}$ and $A_2 \to A_2^1 \to A_2^2 \to \cdots A_2^{k_2}$, with $A_1^{k_1} = A_2^{k_2}$ and $\Sigma(A_1^{k_1}) = \Sigma(A_2^{k_2}) = \max\{\Sigma(A_1), \Sigma(A_2)\}$(entrywise). □

Example 2.3. Let A and B be determined, respectively, by:

0	−	−	−	−	−	0
−				+		−
+						−
−			+	−		+
−	+		−	+		−
+				−	+	−
0	−	+	+	+	−	0

and

0	−	−	−	−	−	0
−	+					−
+	−				+	−
−	+				−	+
−					+	−
+						−
0	−	+	+	+	−	0

.

Then we have

$$\Sigma(A) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}, \quad \Sigma(B) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix},$$

and

$$D = \Sigma(A) - \Sigma(B) = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

First, we find the first nonzero row in D:

$$D = \left[\begin{array}{c|c|c|c|c} -1 & -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Then for $k = 1$ we can find $q = 3$ and $p = 1$ in A.

$$\begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & & & & + & & - \\ \hline + & & & & & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}.$$

Then we have $A + T_{1,3} = A_1 \in \mathcal{A}_{5,5}(u, v|u', v')$ such as

$$\begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & & & & + & & - \\ \hline + & & & & & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array} + \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & & & + & - & & - \\ \hline + & & & - & + & & - \\ \hline - & & & & & & + \\ \hline - & & & & & & - \\ \hline + & & & & & & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}$$

$$= \begin{array}{c||c|c|c|c|c||c} 0 & - & - & - & - & - & 0 \\ \hline\hline - & & & + & & & - \\ \hline + & & & - & + & & - \\ \hline - & & & + & - & & + \\ \hline - & + & & - & + & & - \\ \hline + & & & & - & + & - \\ \hline\hline 0 & - & + & + & + & - & 0 \end{array}.$$

If we denote $A + T_{p,q} = A_1$ by $A \xrightarrow{(p,q)} A_1$, then we have

$$A \xrightarrow{(1,3)} A_1 \xrightarrow{(1,2)} A_2 \xrightarrow{(1,1)} A_3,$$

$$B \xrightarrow{(2,4)} B_1,$$

$$A_3 \xrightarrow{(3,2)} A_4 \xrightarrow{(3,1)} A_5,$$

and

$$B_1 \xrightarrow{(4,4)} B_2,$$

where $A_5 = B_2$ such as

0	−	−	−	−	−	0
−	+					−
+	−			+		−
−	+			−		+
−				+		−
+				−	+	−
0	−	+	+	+	−	0

.

Therefore we have $A \vee B = A_5 = B_2$.

Corollary 2.4. *The three possibilities* (a), (b), *and* (c) *are equivalent for* $\mathcal{A}_{m,n}(u, v|u', v')$ *and each defines the Bruhat order* $\preceq_B$ *on* $\mathcal{A}_{m,n}(u, v|u', v')$.

Proof. Let $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$ with $\Sigma(A_1) \geq \Sigma(A_2)$. Then $\max\{\Sigma(A_1), \Sigma(A_2)\} = \Sigma(A_1)$, and by Theorem 2.2 A_1 can be obtained from A_2 by a sequence of adjacent interchanges. The corollary now follows from the discussion preceding Theorem 2.2. □

Note that this corollary implies that, as with ordinary ASMs, there is no difference between the Bruhat order and the weak Bruhat order on $\mathcal{A}_{m,n}(u, v|u', v')$.

Corollary 2.5. *The partially ordered set* $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ *is a distributive lattice where for* $A_1, A_2 \in \mathcal{A}_{m,n}(u, v|u', v')$, *the meet and join are given by*

(i) $A_1 \wedge A_2 = B$ *where* $B \in \mathcal{A}_{m,n}(u, v|u', v')$ *such that* $\Sigma(B) = \max\{\Sigma(A_1), \Sigma(A_2)\}$.

(ii) $A_1 \vee A_2 = C$ *where* $C \in \mathcal{A}_{m,n}(u, v|u', v')$ *such that* $\Sigma(C) = \min\{\Sigma(A_1), \Sigma(A_2)\}$.

Proof. By Theorem 2.2 $\mathcal{A}_{m,n}(u, v|u', v')$ has a well-defined meet and assertion (i) follows. Assertion (ii) can be obtained from the analogue of Theorem 2.2 with minimum replacing maximum; it also follows from the fact that $\mathcal{A}_{m,n}(u, v|u', v')$ is finite and so the existence of joins follows from the existence of meets.

In order that $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ be distributive, we must have

$$A_1 \wedge (A_2 \vee A_3) = (A_1 \vee A_2) \wedge (A_1 \vee A_3)$$

for all $A_1, A_2, A_3 \in \mathcal{A}_{m,n}(u, v|u', v')$. The distributive property for the real numbers with the usual $\leq$ order relation holds trivially. It then follows from (i) and (ii) that $(\mathcal{A}_{m,n}(u, v|u', v'), \preceq_B)$ is also distributive. □

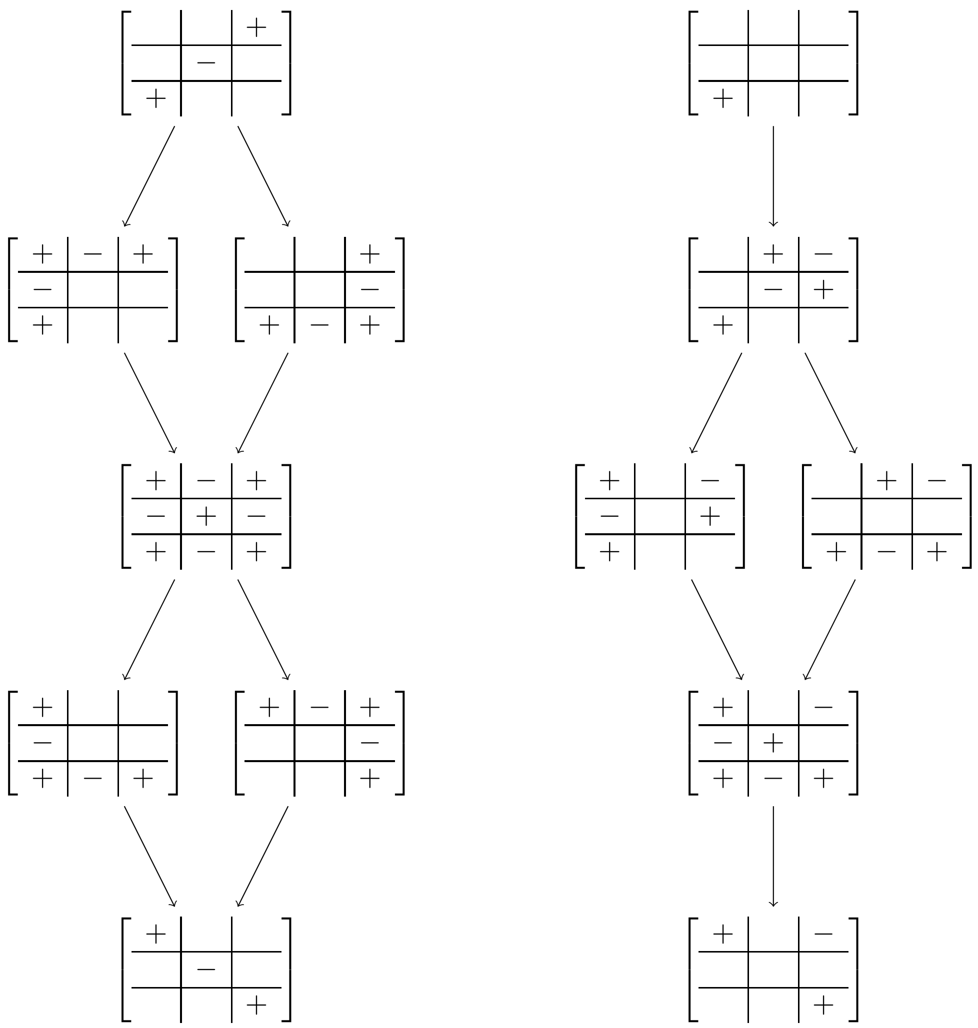


Figure 1: Hasse diagrams of $\mathcal{K}_1$ and $\mathcal{K}_2$.

In [8], we have the (u, v)-identity matrix $I_n(u, v)$ is the unique minimal element in $\mathcal{A}_n(u, v)$, and the (u, v)-anti-identity matrix $L_n(u, v)$ is the unique maximal element in $\mathcal{A}_n(u, v)$. We know that there is the unique minimal element $I_{m,n}(u, v|u', v')$ and the unique maximal element $L_{m,n}(u, v|u', v')$ in $\mathcal{A}_{m,n}(u, v|u', v')$, but do not characterize $I_{m,n}(u, v|u', v')$ and $L_{m,n}(u, v|u', v')$ yet. For a matrix $A \in \mathcal{A}_{m,n}(u, v|u', v')$, let $\rho'(A)$ equal the sum of the entries of $\Sigma(A)$, and let $\rho(A) = \rho'(I_{m,n}(u, v|u', v')) - \rho'(A)$.

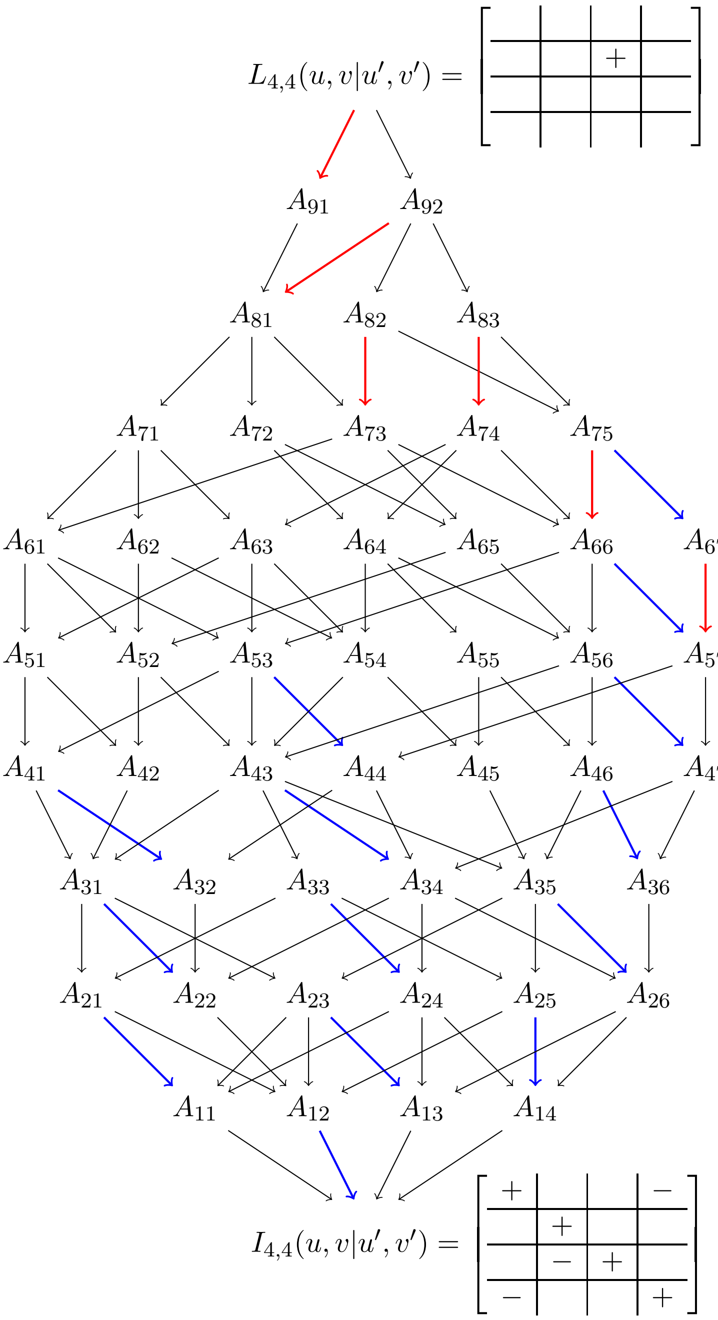
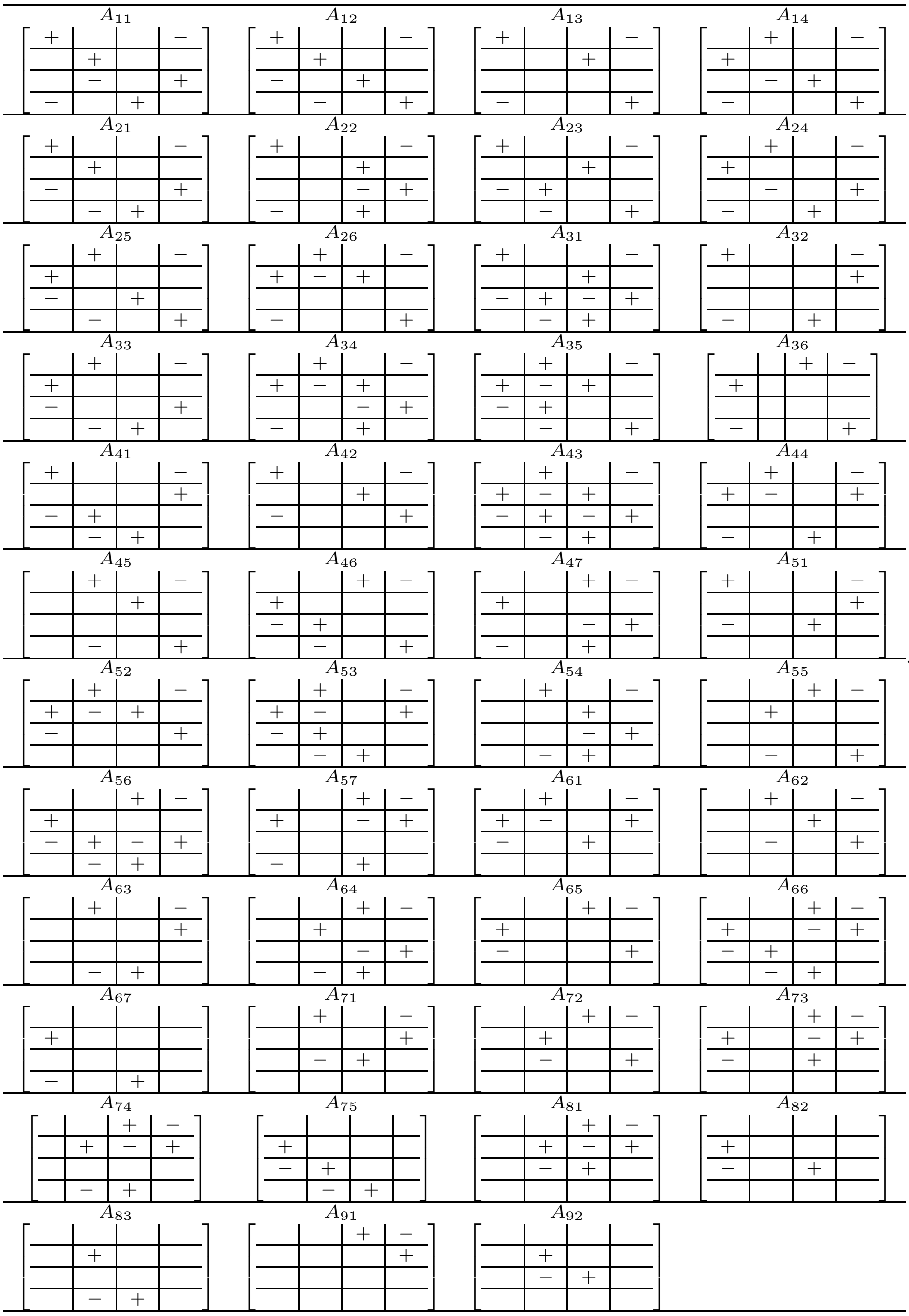


Figure 2: Hasse diagram of $(\mathcal{A}_{4,4}(u,v|u',v'), \preceq_B)$



Corollary 2.6. *The partially ordered set* $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$ *is a graded lattice where the rank of* $A\in(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$ *equals* $\rho(A)$.

Proof. By Corollary 2.5 $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$ is a lattice. Let $A_1, A_2 \in \mathcal{A}_{m,n}(u,v|u',v')$ where A_2 covers A_1. Then $\Sigma(A_1) - \Sigma(A_2) \geq 0$ and $A_1 \wedge A_2 = A_1$. By Theorem 2.2, A_1 can be obtained from A_2 by a sequence of adjacent interchanges. Each interchange increases the sum of the entries of the corresponding Σ-matrix by 1 and gives a matrix $A' \in \mathcal{A}_{m,n}(u,v|u',v')$ with $A_1 \preceq_B A' \preceq_B A_2$. Since A_2 covers A_1, we have that $A_1 = A'$ and $\rho'(A_1) = \rho'(A_2) + 1$. Hence A_1 can be obtained from A_2 by one adjacent interchange and $\rho(A_1) = \rho(A_2) - 1$. Hence $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$ is graded with rank function $\rho(\cdot)$ where $\rho(I_{m,n}(u,v|u',v')) = 0$ and $\rho(L_{m,n}(u,v|u',v')) = \rho'(I_{m,n}(u,v|u',v')) - \rho'(L_{m,n}(u,v|u',v'))$. □

Example 2.7. Let

$$\mathcal{K}_1 = (\mathcal{A}_{3,3}(-1,+1,-1),\preceq_B)$$

and

$$\mathcal{K}_2 = (\mathcal{A}_{3,3}(u,v|u',v'),\preceq_B)$$

where

$$u = (-1,-1,+1), v = (-1,+1,-1), u' = (-1,+1,-1), v' = (+1,-1,-1).$$

Then we have

$$\rho(\mathcal{K}_1) = \rho(\mathcal{K}_2) = 4$$

and Hasse diagrams of $\mathcal{K}_1$ and $\mathcal{K}_2$ are Figure 1.

Hasse diagram of $\mathcal{K}_1$ is isomorphic to Hasse diagram of $(\mathcal{A}_3,\preceq_B)$ in [8], but Hasse diagram of $\mathcal{K}_2$ is not isomorphic.

Let

$$u = (-1,-1,-1,+1), v = (-1,-1,+1,+1), u' = (+1,+1,-1,-1)$$

and

$$v' = (+1,-1,-1,-1).$$

Then we have

$$(\mathcal{A}_{4,4}(u,v|u',v'),\preceq_B)$$

as Figure 2. Also we have

$$\rho(\mathcal{A}_{4,4}(u,v|u',v')) = 10 = \rho(\mathcal{A}_4)$$

and

$$\left|\mathcal{A}_{4,4}(u,v|u',v')\right| = 49 \neq 42 = |\mathcal{A}_4|.$$

3. RANK OF THE LATTICE $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$

Let $\rho(\cdot)$ denote the rank function of the graded lattice $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B)$. Let $L_{m,n}(u,v|u',v')$ and $I_{m,n}(u,v|u',v')$ the unique maximum element and the unique minimum element of this lattice, respectively. The rank of the lattice $(\mathcal{A}_{m,n}(u,v|u',v'),\preceq_B$) equals $\rho(L_{m,n}(u,v|u',v'))$. Moreover, for $A\in\mathcal{A}_{m,n}(u,v|u',v')$, we have

$$\rho(A)=\rho'(I_{m,n}(u,v))-\rho'(A)=\rho'(I_{m,n}(u,v)-A)$$

where recall that $\rho'(\cdot)$ denotes the sum of the entries of $\Sigma(\cdot)$. We have $\rho(\mathcal{A}_n)=\binom{n+1}{3}$, see e.g. [4].

Theorem 3.1. *Let $u=(u_1,u_2,\ldots,u_n),u'=(u'_1,u'_2,\ldots,u'_n),v=(v_1,v_2,\ldots,v_m)$, and $v'=(v'_1,v'_2,\ldots,v'_m)$ be vectors of ±1's. Suppose $m\le n$. Then we have*

$$\rho(\mathcal{A}_{m,n}(u,v|u',v'))\le\begin{cases}\frac{1}{12}(m-1)(m+1)(3n-m), & \text{if } m \text{ odd,}\\ \\ \frac{1}{12}m(3mn-m^2-2), & \text{if } m \text{ even}\end{cases}.$$

Proof. Let $\rho'(A)$ equal the sum of the entries of $\Sigma(A)$ and $\rho(A)=\rho'(I_{m,n}(u,v|u',v'))-\rho'(A)$ for $A\in\mathcal{A}_{m,n}(u,v|u',v')$. Let $D(A)=[d_{ij}]=\Sigma(A)-\Sigma(I_{m,n}(u,v|u',v'))$. Then we have for each integer i,j such that $1\le i\le m$ and $1\le j\le n$,

(1) $d_{ij}\ge0$,
(2) $d_{in}=d_{mj}=0$,
(3) $d_{i1}=d_{1j}\in\{0,1\}$,
(4) $|d_{ij}-d_{i,j+1}|\le1$,
(5) $|d_{ij}-d_{i+1,j}|\le1$.

Therefore we have

$$D(A)\le D^*=[d^*_{ij}]=\left[\begin{array}{ccccccc|c}1&1&1&\cdots&1&1&1&0\\1&2&2&\cdots&2&2&1&0\\1&2&3&\cdots&3&2&1&0\\\vdots&\vdots&\vdots&\cdots&\vdots&\vdots&\vdots&\vdots\\1&2&3&\cdots&3&2&1&0\\1&2&2&\cdots&2&2&1&0\\1&1&1&\cdots&1&1&1&0\\\hline 0&0&0&\cdots&0&0&0&0\end{array}\right].$$

Therefore $\rho(A)=\sum d_{ij}\le\sum d^*_{ij}$.

Case (i) m is odd.

$$\rho(A)\le\sum d^*_{ij}\ =\ (m-1)(n-1)+(m-3)(n-3)+\ldots+2\cdot(n-m+2)$$

$$\begin{aligned}
&= \sum_{k=1}^{\frac{m-1}{2}} 2k \cdot (n-m+2k) \\
&= \sum_{k=1}^{\frac{m-1}{2}} \left\{4k^2 + 2k(n-m)\right\} \\
&= \frac{(m^2-1)m}{6} + \frac{(m^2-1)(n-m)}{4} \\
&= \frac{m^2-1}{12}(3m-m) = \frac{1}{12}(m-1)(m+1)(3n-m)
\end{aligned}$$

Case (ii) m is even.

$$\begin{aligned}
\rho(A) \leq \sum d_{ij}^* &= (m-1)(n-1) + (m-3)(n-3) + \ldots + 1 \cdot (n-m+1) \\
&= \sum_{k=1}^{\frac{m}{2}} (2k-1)(n-m+2k-1) \\
&= \sum_{k=1}^{\frac{m}{2}} \left\{4k^2 + 2k(n-m-2) + (-n+m+1)\right\} \\
&= \frac{m(m+2)(m+1)}{6} + \frac{m(m+2)(n-m-3)}{4} + \frac{m(-n+m+1)}{2} \\
&= \frac{m}{12}(3mn - m^2 - 2)
\end{aligned}$$

Therefore we have the result. □

Corollary 3.2. *Let* $u = (u_1, u_2, \ldots, u_n), u' = (u'_1, u'_2, \ldots, u'_n), v = (v_1, v_2, \ldots, v_n)$, *and* $v' = (v'_1, v'_2, \ldots, v'_n)$ *be vectors of* ± 1*'s. Then we have*

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \binom{n+1}{3}.$$

Proof. If n is odd, then we have by Theorem 3.1,

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \frac{1}{12}(n-1)(n+1)(3n-n) = \frac{(n-1)n(n-2)}{6} = \binom{n+1}{3}.$$

If n is even, then we have by Theorem 3.1,

$$\rho(\mathcal{A}_{n,n}(u, v|u', v')) \leq \frac{n}{12}(3n^2 - n^2 - 2) = \frac{(n-1)n(n-2)}{6} = \binom{n+1}{3}.$$

□

4. Closing Remarks

In this paper, we extend the result in [8] to a more general situation. We show another example of the lattice $(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$ for $u = v = (+1, -1, +1, -1)$ and $u' = v' = (-1, +1, -1, +1)$ in Figure 3.

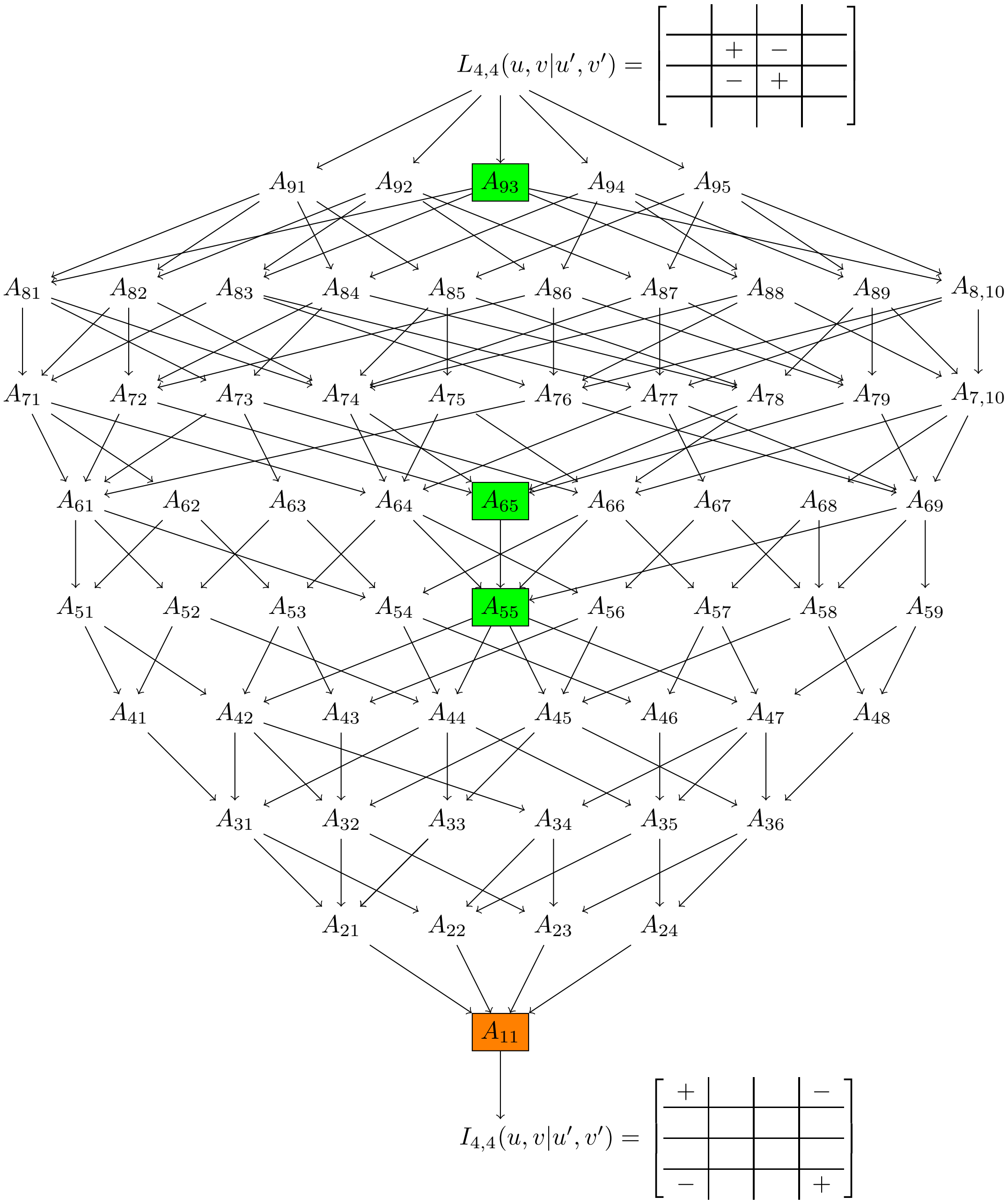
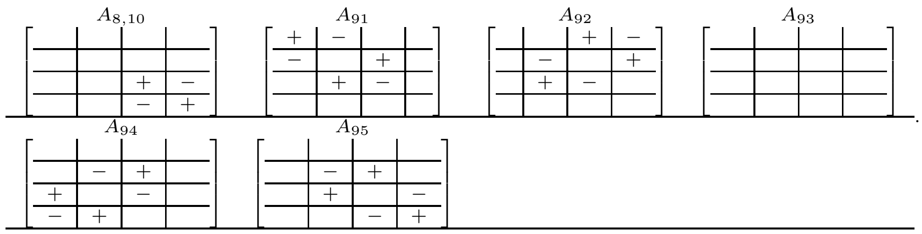


Figure 3: Hasse diagram of $(\mathcal{A}_{4,4}(u, v|u', v'), \preceq_B)$

A_{11} $\begin{bmatrix} + & & & - \\ & - & + & \\ & + & - & \\ - & & & + \end{bmatrix}$	A_{21} $\begin{bmatrix} + & & & - \\ & - & + & \\ & & & \\ - & + & - & + \end{bmatrix}$	A_{22} $\begin{bmatrix} + & & & - \\ & - & & + \\ & + & & - \\ - & & & + \end{bmatrix}$	A_{23} $\begin{bmatrix} + & & & - \\ - & & + & \\ + & & - & \\ - & & & + \end{bmatrix}$
A_{24} $\begin{bmatrix} + & - & + & - \\ & & & \\ & + & - & \\ - & & & + \end{bmatrix}$	A_{31} $\begin{bmatrix} + & & & - \\ & - & & + \\ & & + & - \\ - & + & - & + \end{bmatrix}$	A_{32} $\begin{bmatrix} + & & & - \\ - & & + & \\ + & - & & \\ - & + & - & + \end{bmatrix}$	A_{33} $\begin{bmatrix} + & - & + & - \\ & & & \\ & & & \\ - & + & - & + \end{bmatrix}$
A_{34} $\begin{bmatrix} + & & & - \\ - & & & + \\ + & & & - \\ - & & & + \end{bmatrix}$	A_{35} $\begin{bmatrix} + & - & + & - \\ & & - & + \\ & + & & - \\ - & & & + \end{bmatrix}$	A_{36} $\begin{bmatrix} + & - & + & - \\ - & + & & \\ + & & - & \\ - & & & + \end{bmatrix}$	A_{41} $\begin{bmatrix} + & & & - \\ & - & & + \\ & & & \\ - & + & & \end{bmatrix}$
A_{42} $\begin{bmatrix} + & & & - \\ - & & & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$	A_{43} $\begin{bmatrix} + & & & - \\ - & & + & \\ & & & \\ & & - & + \end{bmatrix}$	A_{44} $\begin{bmatrix} + & - & + & - \\ & & - & + \\ & & + & - \\ - & + & - & + \end{bmatrix}$	A_{45} $\begin{bmatrix} + & - & + & - \\ - & + & & \\ + & - & & \\ - & + & - & + \end{bmatrix}$
A_{46} $\begin{bmatrix} + & - & & \\ & & & \\ & + & & - \\ - & & & + \end{bmatrix}$	A_{47} $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & & & - \\ - & & & + \end{bmatrix}$	A_{48} $\begin{bmatrix} & & + & - \\ & & & \\ + & & - & \\ - & & & + \end{bmatrix}$	A_{51} $\begin{bmatrix} + & & & - \\ - & & & + \\ + & - & & \\ - & + & & \end{bmatrix}$
A_{52} $\begin{bmatrix} + & - & + & - \\ & & - & + \\ & & & \\ - & + & & \end{bmatrix}$	A_{53} $\begin{bmatrix} + & & & - \\ - & & & + \\ & & + & - \\ & & - & + \end{bmatrix}$	A_{54} $\begin{bmatrix} + & - & & \\ & & & \\ & & + & - \\ - & + & - & + \end{bmatrix}$	A_{55} $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$
A_{56} $\begin{bmatrix} + & - & + & - \\ - & + & & \\ & & & \\ & & - & + \end{bmatrix}$	A_{57} $\begin{bmatrix} + & - & & \\ - & + & & \\ + & & & - \\ - & & & + \end{bmatrix}$	A_{58} $\begin{bmatrix} & & + & - \\ & & & \\ + & - & & \\ - & + & - & + \end{bmatrix}$	A_{59} $\begin{bmatrix} & & + & - \\ & & - & + \\ + & & & - \\ - & & & + \end{bmatrix}$.
A_{61} $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & & \\ - & + & & \end{bmatrix}$	A_{62} $\begin{bmatrix} + & & & - \\ - & & & + \\ & & & \\ & & & \end{bmatrix}$	A_{63} $\begin{bmatrix} + & - & & \\ & & & \\ & & & \\ + & - & & \end{bmatrix}$	A_{64} $\begin{bmatrix} + & - & + & - \\ + & + & - & + \\ & & + & - \\ & & - & + \end{bmatrix}$
A_{65} $\begin{bmatrix} + & - & + & - \\ - & & & + \\ + & & & - \\ - & + & - & + \end{bmatrix}$	A_{66} $\begin{bmatrix} + & - & & \\ - & + & & \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$	A_{67} $\begin{bmatrix} & & + & - \\ & & & \\ & & & \\ & & - & + \end{bmatrix}$	A_{68} $\begin{bmatrix} & & & \\ & & & \\ + & & & - \\ - & & & + \end{bmatrix}$
A_{69} $\begin{bmatrix} & & + & - \\ & & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$	A_{71} $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ & & & \\ & & & \end{bmatrix}$	A_{72} $\begin{bmatrix} + & - & + & - \\ - & & & + \\ + & & - & \\ - & + & & \end{bmatrix}$	A_{73} $\begin{bmatrix} + & - & & \\ - & + & & \\ + & - & & \\ - & + & & \end{bmatrix}$
A_{74} $\begin{bmatrix} + & - & + & - \\ - & & & + \\ & + & & - \\ & & - & + \end{bmatrix}$	A_{75} $\begin{bmatrix} + & - & & \\ - & + & & \\ & & + & - \\ & & - & + \end{bmatrix}$	A_{76} $\begin{bmatrix} & & + & - \\ & & - & + \\ + & - & & \\ - & + & & \end{bmatrix}$	A_{77} $\begin{bmatrix} & & + & - \\ & & - & + \\ & & + & - \\ & & - & + \end{bmatrix}$
A_{78} $\begin{bmatrix} + & - & & \\ - & & + & \\ + & & & - \\ - & + & - & + \end{bmatrix}$	A_{79} $\begin{bmatrix} & & + & - \\ & - & & + \\ + & & & - \\ - & + & - & + \end{bmatrix}$	$A_{7,10}$ $\begin{bmatrix} & & & \\ & & & \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$	A_{81} $\begin{bmatrix} + & - & & \\ - & + & & \\ & & & \\ & & & \end{bmatrix}$
A_{82} $\begin{bmatrix} + & - & + & - \\ - & & & + \\ & + & - & \\ & & & \end{bmatrix}$	A_{83} $\begin{bmatrix} & & + & - \\ & & - & + \\ & & & \\ & & & \end{bmatrix}$	A_{84} $\begin{bmatrix} + & - & & \\ - & & + & \\ + & & - & \\ - & + & & \end{bmatrix}$	A_{85} $\begin{bmatrix} + & - & & \\ - & & + & \\ & + & & - \\ & & - & + \end{bmatrix}$
A_{86} $\begin{bmatrix} & & + & - \\ & - & & + \\ + & & - & \\ - & + & & \end{bmatrix}$	A_{87} $\begin{bmatrix} & & + & - \\ & - & & + \\ & + & & - \\ & & - & + \end{bmatrix}$	A_{88} $\begin{bmatrix} & & & \\ & & & \\ + & - & & \\ - & + & & \end{bmatrix}$	A_{89} $\begin{bmatrix} & & & \\ & - & + & \\ + & & & - \\ - & + & - & + \end{bmatrix}$



REFERENCES

1. D.M. Bressoud: Proofs and Confirmations; The Story of the Alternating Sign Matrix Conjecture. Math. Assoc. America, Washington, D.C., 1999. `https://doi.org/10.1017/ CBO9780511613449.002`
2. A. Björner & F. Brenti: Combinatorics of Coxeter Groups. Graduate Texts in Math. #231, Springer (2003). `https://doi.org/10.1007/3-540-27596-7`
3. R.A. Brualdi, K.P. Kiernan, S.A. Meyer & M.W. Schroeder: Patterns of alternating sign matrices. Linear Algebra Appl. **438** (2013), 3967-3990. `https://doi.org/10.1016/j.laa. 2012.03.009`
4. R.A. Brualdi & M.W. Schroeder: Alternating sign matrices and their Bruhat order. Discrete Math. **340** (2017), 1996-2019. `https://doi.org/10.1016/j.disc.2016.10.010`
5. R.A. Brualdi & G. Dahl: Alternating sign matrices, extensions, and related cones. Adv. in Appl. Math. **86** (2017), 19-49. `https://doi.org/10.1016/j.aam.2016.12.001`
6. R.A. Brualdi & G. Dahl: Alternating sign matrices: Extensions, König-properties, and primary sum-sequences. Graph and Combinatorics **36** (2020), 63-92. `https://doi.org/10. 1007/s00373-019-02119-x`
7. R.A. Brualdi & H.K. Kim: A generalization of alternating sign matrices. J. Combin. Des. **23** (2015), 204-215. `https://doi.org/10.1002/jcd.21397`
8. R.A. Brualdi & H.K. Kim: A generalized alternating sign matrices and signed permuation matrices. J. Korean Math. Soc. **58** (2021), no. 4, 921-948. `https://doi.org/10. 4134/JKMS.j200318`
9. R.A. Brualdi & H.K. Kim: Completions of alternating sign matrices. Graphs and Combinatorics **31** (2015), no. 3, 507-522. `https://doi.org/10.1007/s00373-014-1409-1`
10. A. Lascoux & M. Schützenberger: Treillis et bases des groupes de Coxeter. (French) [Lattices and bases of Coxeter groups] Electron. J. Combin. **3** (1996), no. 2, Research paper 27. `https://doi.org/10.37236/1285`

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