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CHOOSER OPTIONS ON VARIOUS UNDERLYING OPTIONS

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Abstract. We consider chooser options written on various underlying assets other than vanilla call and put options. Specifically, we deal with (i) the chooser option written on the power call and put options, and (ii) the chooser option written on the exchange options. We provide explicit formulas for the prices of these chooser options whose underlying assets are either power options or exchange options, rather than the vanilla call and put options.

1. Introduction

Financial derivatives have become essential tools for investors and traders in modern financial markets, offering effective risk management, volatility hedging, and exposure to diverse asset classes.

Among the various types of derivatives, chooser options hold significant importance. Chooser options, also referred to as "you-choose" or "as-you-like" options, allow holders to exercise their right to determine, at a predetermined future date before the option's maturity, whether the option should be a call or a put. This date is commonly known as the choice date. Evaluating the expected payoff of a call option against that of a put option, the holder decides the option type based on the price movement from the present to the choice date. If the call and put options possess identical strike prices and maturities, they are classified as standard, simple, or regular chooser options. Conversely, if they differ, they are categorized as complex chooser options (Zhang [12]; Hull [4]; Whaley [11]; Rubinstein and Reiner [9]).

Chooser options emerged in the late 1980s, attracting the attention of researchers. Rubinstein [8, 9] provided good explanations about chooser options. In Rubinstein's formula, a nonlinear equation has to be solved numerically using the iteration method. Buchen [2] introduced a new pricing technique utilizing the partial differential equation for a class of exotic options, such as chooser options, which involve two expiry dates.

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The solutions for chooser options under deterministic interest rates on stocks were extended by Sandmann and Wittke [10] into a unified framework capable of pricing different lognormal assets under deterministic or stochastic interest rates. Martinkut-Kaulien [7] discussed the characteristics and applications of chooser options, highlighting their flexibility in adapting to price movements of underlying assets.

In this paper, we study chooser options written on various underlying assets other than vanilla call and put options. Specifically, we deal with (i) the chooser option written on the power call and put options, and (ii) the chooser option written on exchange options, and derive the prices of these chooser options.

For the derivation of the prices of the chooser options whose underlying assets are either power options or exchange options, rather than the vanilla call and put options, we need to apply the method used for the derivation of the chooser option written on the vanilla call and put options. Hence, we first explain the method for the derivation of the price of the chooser option written on the vanilla call and put options in detail, and then we apply the method to the derivation of the prices of the chooser options written on either power options or exchange options.

The rest of the paper is organized as follows. In Section 2, we review the method for the derivation of the price of the chooser option on the vanilla call and put options. In Section 3, we derive an explicit formula for the price of the chooser option written on the power call and put options. In Section 4, we derive an explicit formula for the price of the chooser option written on the exchange options.

2. Preliminary: Chooser options on vanilla call and put options

In this section, we present known results on the chooser option written on the vanilla call and put. The holder of this chooser option can choose between the vanilla call and put prior to the expiration. We explain how to derive the price of the chooser option and present an explicit formula for the price. The method explained in this section will be used in Sections 3 and 4 for the derivation of the price of chooser options written on various underlying options other than vanilla call and put.

Suppose that under a risk-neutral probability $\mathbb{P}$, the stock price process $S(t)$, $t \geq 0$, satisfies the standard Black-Scholes model:

$$dS(t) = S(t)(rdt + \sigma dW(t)),$$

where r is the risk-free interest rate, σ is the volatility of the stock price and $W(t)$, $t \geq 0$, is a Brownian motion under $\mathbb{P}$.

We consider the call and put options with strike price K and maturity $T > 0$ written on the stock. The payoffs of the call and put options at maturity T are given by

$$(S(T) - K)^+ \quad \text{and} \quad (K - S(T))^+,$$

FIGURE 1. Expiration time of a chooser option.

respectively. For $0 \le t < T$, $s > 0$ and $K > 0$, let $c(t, s, K, T)$ and $p(t, s, K, T)$ be the prices at time t of the call and put options with strike price K and maturity T, respectively, given $S(t) = s$. By the risk-neutral pricing, the prices $c(t, s, K, T)$ and $p(t, s, K, T)$ are given by

$$\begin{aligned} c(t, s, K, T) &= e^{-r(T-t)}\mathbb{E}[(S(T) - K)^+ | S(t) = s], \\ p(t, s, K, T) &= e^{-r(T-t)}\mathbb{E}[(K - S(T))^+ | S(t) = s], \end{aligned}$$

respectively. The Black-Scholes formula provides the explicit expressions for $c(t, s, K, T)$ and $p(t, s, K, T)$, as shown below.

Theorem 2.1 (Black and Scholes [1]). *Let $0 \le t < T$. Given $S(t) = s$, the call price at t, $c(t, s, K, T)$, and the put price at t, $p(t, s, K, T)$ with strike price K and maturity T are given by*

$$\begin{aligned} c(t, s, K, T) &= sN(d_1) - Ke^{-r(T-t)}N(d_2), \\ p(t, s, K, T) &= Ke^{-r(T-t)}N(-d_2) - sN(-d_1), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{s}{K} + (T - t)(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T - t}}, \\ d_2 &= d_1 - \sigma\sqrt{T - t}. \end{aligned}$$

Now, we describe a chooser option on vanilla call and put options. Let $0 \le t < T_0 < T$. We consider a chooser option with vanilla call and put options as underlying assets. The vanilla call and put options have strike price K and maturity T. If the maturity of the chooser option is T_0, then the holder of the chooser option can choose at T_0 between the call and put options; see Figure 1. Since the prices at T_0 of the call and put options with strike price K and maturity T are $c(T_0, S(T_0), K, T)$ and $p(T_0, S(T_0), K, T)$, respectively, the payoff of the chooser option at maturity T_0 is given by

$$\max\{c(T_0, S(T_0), K, T),\ p(T_0, S(T_0), K, T)\}.$$

Therefore, given $S(t) = s$, the price at t, $\mathrm{CH}(t, s, T_0, K, T)$ of the chooser option is given by

$$\begin{aligned} &\mathrm{CH}(t, s, T_0, K, T) \\ &= e^{-r(T_0-t)}\mathbb{E}[\max\{c(T_0, S(T_0), K, T),\ p(T_0, S(T_0), K, T)\} \,|\, S(t) = s]. \end{aligned}$$

Note that

$$\begin{aligned}&\max\{c(T_0,S(T_0),K,T),\ p(T_0,S(T_0),K,T)\}\\&=(c(T_0,S(T_0),K,T)-p(T_0,S(T_0),K,T))^+ + p(T_0,S(T_0),K,T).\end{aligned}$$

Since

$$c(T_0,S(T_0),K,T)-p(T_0,S(T_0),K,T)=S(T_0)-e^{-r(T-T_0)}K$$

by the put-call parity, we have

$$\begin{aligned}&\max\{c(T_0,S(T_0),K,T),\ p(T_0,S(T_0),K,T)\}\\&=(S(T_0)-e^{-r(T-T_0)}K)^+ + p(T_0,S(T_0),K,T).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{CH}(t,s,T_0,K,T)\\&=e^{-r(T_0-t)}\mathbb{E}[(S(T_0)-e^{-r(T-T_0)}K)^+ + p(T_0,S(T_0),K,T)\,|\,S(t)=s]\\&=e^{-r(T_0-t)}\mathbb{E}[(S(T_0)-e^{-r(T-T_0)}K)^+\,|\,S(t)=s]\\&\quad+e^{-r(T_0-t)}\mathbb{E}[p(T_0,S(T_0),K,T)\,|\,S(t)=s].\end{aligned}$$

Since

$$e^{-r(T_0-t)}\mathbb{E}[(S(T_0)-e^{-r(T-T_0)}K)^+\,|\,S(t)=s]=c(t,s,e^{-r(T-T_0)}K,T_0)$$

and

$$\begin{aligned}&e^{-r(T_0-t)}\mathbb{E}[p(T_0,S(T_0),K,T)\,|\,S(t)=s]\\&=e^{-r(T_0-t)}\mathbb{E}[e^{-r(T-T_0)}\mathbb{E}[(K-S(T))^+\,|\,S(T_0)]\,|\,S(t)=s]\\&=e^{-r(T-t)}\mathbb{E}[(K-S(T))^+\,|\,S(t)=s]\\&=p(t,s,K,T),\end{aligned}$$

we have the following theorem.

Theorem 2.2 (Rubinstein [8])**.** *Let $0\le t<T_0<T$. The price at t of the chooser option that chooses at T_0 between the call and put options with strike price K and maturity T is given by*

$$\mathrm{CH}(t,s,T_0,K,T)=c(t,s,e^{-r(T-T_0)}K,T_0)+p(t,s,K,T). \tag{1}$$

Remark 1. By using the relation

$$\begin{aligned}&\max\{c(T_0,S(T_0),K,T),p(T_0,S(T_0),K,T)\}\\&=(p(T_0,S(T_0),K,T)-c(T_0,S(T_0),K,T))^+ + c(T_0,S(T_0),K,T),\end{aligned}$$

we can obtain another expression for $\mathrm{CH}(t,s,T_0,K,T)$:

$$\mathrm{CH}(t,s,T_0,K,T)=p(t,s,e^{-r(T-T_0)}K,T_0)+c(t,s,K,T).$$

This can also be obtained from (1) by applying the put-call parity:

$$c(t,s,e^{-r(T-T_0)}K,T_0)-p(t,s,e^{-r(T-T_0)}K,T_0)=s-e^{-r(T-t)}K,$$

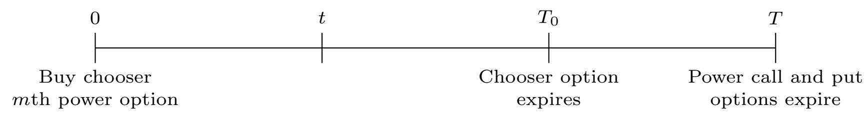


FIGURE 2. Expiration time of a chooser power option.

$$c(t, s, K, T) - p(t, s, K, T) = s - e^{-r(T-t)}K.$$

3. Chooser options on power call and put options

We consider the mth power call and put options with strike price K and maturity $T > 0$ written on the stock. The payoffs of the mth power call and put options at maturity T are given by

$$((S(T))^m - K^m)^+ \quad \text{and} \quad (K^m - (S(T))^m)^+,$$

respectively. For $0 \le t < T$, $s > 0$, $m > 0$ and $K > 0$, let $c^{(m)}(t, s, K, T)$ and $p^{(m)}(t, s, K, T)$ be the prices at t of the mth power call and put options with strike price K and maturity T, respectively, given $S(t) = s$. By the risk-neutral pricing, the prices $c^{(m)}(t, s, K, T)$ and $p^{(m)}(t, s, K, T)$ are given by

$$\begin{aligned} c^{(m)}(t, s, K, T) &= e^{-r(T-t)}\mathbb{E}[(S(T))^m - K^m)^+ \mid S(t) = s], \\ p^{(m)}(t, s, K, T) &= e^{-r(T-t)}\mathbb{E}[(K^m - (S(T))^m)^+ \mid S(t) = s]. \end{aligned}$$

The following theorem provides the explicit expressions for $c^{(m)}(t, s, K, T)$ and $p^{(m)}(t, s, K, T)$:

Theorem 3.1 (Heynen and Kat [3])**.** *Let $0 \le t < T$. Given $S(t) = s$, the mth power call price at t, $c^{(m)}(t, s, K, T)$, and the mth power put price at t, $p^{(m)}(t, s, K, T)$ with strike price K and maturity T are given by*

$$\begin{aligned} c^{(m)}(t, s, K, T) &= s^m e^{(m-1)(r+\frac{m\sigma^2}{2})(T-t)}N(d_1) - K^m e^{-r(T-t)}N(d_2), \\ p^{(m)}(t, s, K, T) &= K^m e^{-r(T-t)}N(-d_2) - s^m e^{(m-1)(r+\frac{m\sigma^2}{2})(T-t)}N(-d_1), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{s}{K} + (T-t)(r - \frac{\sigma^2}{2} + m\sigma^2)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - m\sigma\sqrt{T-t}. \end{aligned}$$

Now, we describe a chooser option on power call and put options. Let $0 \le t < T_0 < T$. If the maturity of the chooser option is T_0, then the holder of the chooser option can choose at T_0 between the mth power call and put options; see Figure 2. Note that the prices at T_0 of the mth power call and put options with strike price K and maturity T are $c^{(m)}(T_0, S(T_0), K, T)$ and

$p^{(m)}(T_0, S(T_0), K, T)$, respectively. The chooser option that chooses at T_0 between the mth power call and put options with strike price K and maturity T has payoff

$$\max\{c^{(m)}(T_0, S(T_0), K, T),\ p^{(m)}(T_0, S(T_0), K, T)\}$$

at T_0. Therefore, given $S(t) = s$, the price at t, $\mathrm{CH}^{(m)}(t, s, T_0, K, T)$ of the chooser power option is given by

$$\begin{aligned}&\mathrm{CH}^{(m)}(t, s, T_0, K, T)\\&= e^{-r(T_0-t)}\mathbb{E}[\max\{c^{(m)}(T_0, S(T_0), K, T),\ p^{(m)}(T_0, S(T_0), K, T)\} \mid S(t) = s].\end{aligned}$$

Note that

$$\begin{aligned}&\max\{c^{(m)}(T_0, S(T_0), K, T),\ p^{(m)}(T_0, S(T_0), K, T)\}\\&= (c^{(m)}(T_0, S(T_0), K, T) - p^{(m)}(T_0, S(T_0), K, T))^+ + p^{(m)}(T_0, S(T_0), K, T).\end{aligned}$$

Since

$$\begin{aligned}&c^{(m)}(T_0, S(T_0), K, T) - p^{(m)}(T_0, S(T_0), K, T)\\&= e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)}(S(T_0))^m - e^{-r(T-T_0)}K^m\end{aligned}$$

by the power put-call parity (see, for example, Lemma II.1 of [5]), we have

$$\begin{aligned}&\max\{c^{(m)}(T_0, S(T_0), K, T),\ p^{(m)}(T_0, S(T_0), K, T)\}\\&= (e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)}(S(T_0))^m - e^{-r(T-T_0)}K^m)^+ + p^{(m)}(T_0, S(T_0), K, T).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{CH}^{(m)}(t, s, T_0, K, T)\\&= e^{-r(T_0-t)}\mathbb{E}[(e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)}(S(T_0))^m - e^{-r(T-T_0)}K^m)^+\\&\quad + p^{(m)}(T_0, S(T_0), K, T) \mid S(t) = s]\\&= e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)}e^{-r(T_0-t)}\mathbb{E}[((S(T_0))^m - \big(e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K\big)^m)^+ \mid S(t)=s]\\&\quad + e^{-r(T_0-t)}\mathbb{E}[p^{(m)}(T_0, S(T_0), K, T) \mid S(t) = s].\end{aligned}$$

Since

$$\begin{aligned}&e^{-r(T_0-t)}\mathbb{E}[((S(T_0))^m - \big(e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K\big)^m)^+ \mid S(t) = s]\\&= c^{(m)}(t, s, e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K, T_0),\end{aligned}$$

and

$$\begin{aligned}&e^{-r(T_0-t)}\mathbb{E}[p^{(m)}(T_0, S(T_0), K, T) \mid S(t) = s]\\&= e^{-r(T_0-t)}\mathbb{E}[e^{-r(T-T_0)}\mathbb{E}[(K^m - (S(T))^m)^+ \mid S(T_0)] \mid S(t) = s]\\&= e^{-r(T-t)}\mathbb{E}[(K^m - (S(T))^m)^+ \mid S(t) = s]\\&= p^{(m)}(t, s, K, T),\end{aligned}$$

we have the following theorem.

Theorem 3.2. *Let $0 \le t < T_0 < T$. The price at t of the chooser power option that chooses at T_0 between the mth power call and put options with strike price K and maturity T is given by*

$$
\begin{aligned}
(2)\quad & \mathrm{CH}^{(m)}(t,s,T_0,K,T) \\
&= e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)} c^{(m)}(t,s,e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K,T_0) + p^{(m)}(t,s,K,T).
\end{aligned}
$$

Remark 2. By using the relation

$$
\begin{aligned}
&\max\{c^{(m)}(T_0,S(T_0),K,T),\ p^{(m)}(T_0,S(T_0),K,T)\} \\
&= (p^{(m)}(T_0,S(T_0),K,T) - c^{(m)}(T_0,S(T_0),K,T))^+ + c^{(m)}(T_0,S(T_0),K,T),
\end{aligned}
$$

we can obtain another expression for $\mathrm{CH}^{(m)}(t,s,T_0,K,T)$:

$$
\begin{aligned}
&\mathrm{CH}^{(m)}(t,s,T_0,K,T) \\
&= e^{(m-1)(r+\frac{m\sigma^2}{2})(T-T_0)} p^{(m)}(t,s,e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K,T_0) + c^{(m)}(t,s,K,T).
\end{aligned}
$$

This can also be obtained from (2) by applying the put-call parity:

$$
\begin{aligned}
&c^{(m)}(t,s,e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K,T_0) - p^{(m)}(t,s,e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K,T_0) \\
&= (S(t))^m e^{(m-1)(r+\frac{m\sigma^2}{2})(T_0-t)} - e^{-r(T_0-t)}(e^{-(r+\frac{m-1}{2}\sigma^2)(T-T_0)}K)^m, \\
&c^{(m)}(t,s,K,T) - p^{(m)}(t,s,K,T) \\
&= (S(t))^m e^{(m-1)(r+\frac{m\sigma^2}{2})(T-t)} - e^{-r(T-t)}K^m.
\end{aligned}
$$

To illustrate Theorem 3.2, we provide a numerical example.

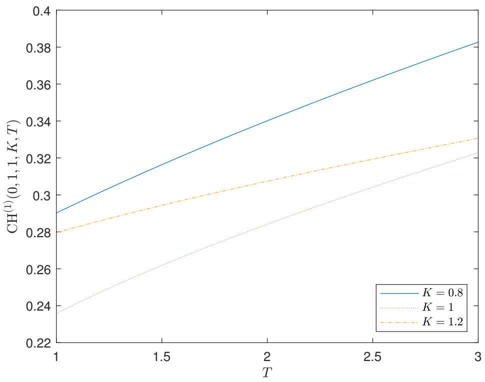
Example 1. Consider the Black-Scholes model with parameters: $r = 0.05$, $\sigma = 0.3$, $s = 1$, $T_0 = 1$.

Figure 3 shows the prices at 0, $\mathrm{CH}^{(m)}(0,1,1,K,T)$ of chooser power options that choose at $T_0 = 1$ between the mth power call and the mth power put options with strike price K and maturity T for Example 1 with T varying, when $K = 0.8$, 1, 1.2 and $m = 1$, 1.5, 2. It is noted that when $m = 1$, $\mathrm{CH}^{(m)}(0,1,1,K,T)$ corresponds to the price of a chooser option on the vanilla call and put options discussed in Section 2.

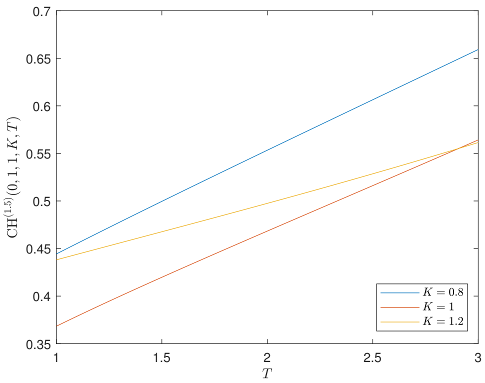
4. Chooser options on exchange options

Suppose there are two assets called the first and the second assets. Under a risk-neutral probability $\mathbb{P}$, the prices at t of the assets $S_i(t)$, $i = 1, 2$, satisfy the following equations:

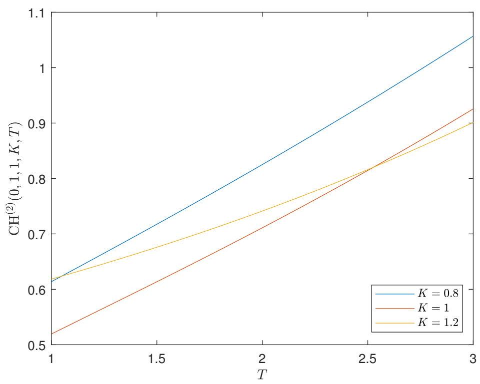
$$
\begin{aligned}
dS_1(t) &= S_1(t)(rdt + \sigma_1 dW_1(t)), \\
dS_2(t) &= S_2(t)(rdt + \sigma_2 dW_2(t)),
\end{aligned}
$$



(a) $m = 1$.



(b) $m = 1.5$.



(c) $m = 2$.

FIGURE 3. Prices at 0 of chooser power options for Example 1 with varying T.

FIGURE 4. Expiration time of a chooser exchange option.

where r is the risk-free interest rate, σ_1 and σ_2 are the volatilities of the asset prices and $W_1(t)$ and $W_2(t)$, $t \geq 0$, are Brownian motions under $\mathbb{P}$ with $dW_1(t)dW_2(t) = \rho dt$, $-1 < \rho < 1$.

We consider the exchange option on two underlying assets $S_1(t)$ and $S_2(t)$. The holder of the exchange option with maturity T has the right to exchange the second asset for the first asset at T. Therefore, the payoff of the exchange option at maturity T is

$$(S_1(T) - S_2(T))^+.$$

For $0 \leq t < T$, $s_1 > 0$ and $s_2 > 0$, let $\mathrm{ex}_{1,2}(t, s_1, s_2, T)$ be the price at t of the exchange option with maturity T, given $S_1(t) = s_1$ and $S_2(t) = s_2$. By the risk-neutral pricing, the price $\mathrm{ex}_{1,2}(t, s_1, s_2, T)$ is given by

$$\mathrm{ex}_{1,2}(t, s_1, s_2, T) = e^{-r(T-t)}\mathbb{E}[(S_1(T) - S_2(T))^+ \mid S_1(t) = s_1, S_2(t) = s_2].$$

The following theorem provides the explicit expression for $\mathrm{ex}_{1,2}(t, s_1, s_2, T)$:

Theorem 4.1 (Margrabe [6])**.** *Let $0 \leq t < T$. Given $S_1(t) = s_1$ and $S_2(t) = s_2$, the price at t of the exchange option with the right to exchange the second asset for the first asset at maturity T is given by*

$$\mathrm{ex}_{1,2}(t, s_1, s_2, T) = S_1(t)N(d_1) - S_2(t)N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S_1(t)}{S_2(t)} + \frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}},$$
$$d_2 = d_1 - \tilde{\sigma}\sqrt{T-t}$$

with $\tilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho$.

Now, we describe a chooser option on two exchange options: (i) the exchange options with the right to exchange the second asset for the first asset at maturity T, and alternatively, (ii) the exchange options with the right to exchange the first asset for the second asset at maturity T. If the maturity of the chooser option is T_0, then the holder of the chooser option can choose at T_0 between the two exchange options; see Figure 4. The prices at t of the two exchange options are denoted by $\mathrm{ex}_{1,2}(t, s_1, s_2, T)$ and $\mathrm{ex}_{2,1}(t, s_1, s_2, T)$, respectively. Let $0 \leq t < T_0 < T$. Note that the prices at T_0 of the two exchange options are given by $\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T)$ and $\mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)$, respectively. The payoff at maturity T_0 of the chooser option that chooses at T_0 between the two exchange options is given by

$$\max\{\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T),\ \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)\}.$$

Therefore, given $S_1(t) = s_1$ and $S_2(t) = s_2$, the price at t, $\mathrm{CHEX}(t, s_1, s_2, T_0, T)$ of the chooser exchange option is given by

$$\begin{aligned}
&\mathrm{CHEX}(t, s_1, s_2, T_0, T)\\
&= e^{-r(T_0-t)}\mathbb{E}[\max\{\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T),\ \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)\}\\
&\qquad\qquad \mid S_1(t) = s_1, S_2(t) = s_2].
\end{aligned}$$

Note that

$$\begin{aligned}
&\max\{\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T),\ \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)\}\\
&= (\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T) - \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T))^+\\
&\quad + \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T).
\end{aligned}$$

Since

$$\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T) - \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T) = S_1(T_0) - S_2(T_0)$$

by the put-call parity, we have

$$\begin{aligned}
&\max\{\mathrm{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T),\ \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)\}\\
&= (S_1(T_0) - S_2(T_0))^+ + \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{CHEX}(t, s_1, s_2, T_0, T)\\
&= e^{-r(T_0-t)}\mathbb{E}[(S_1(T_0)-S_2(T_0))^+ + \mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T) \mid S_1(t)=s_1, S_2(t)=s_2]\\
&= e^{-r(T_0-t)}\mathbb{E}[(S_1(T_0) - S_2(T_0))^+ \mid S_1(t) = s_1, S_2(t) = s_2]\\
&\quad + e^{-r(T_0-t)}\mathbb{E}[\mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T) \mid S_1(t) = s_1, S_2(t) = s_2].
\end{aligned}$$

Since

$$e^{-r(T_0-t)}\mathbb{E}[(S_1(T_0)) - S_2(T_0))^+ \mid S_1(t) = s_1, S_2(t) = s_2] = \mathrm{ex}_{1,2}(t, s_1, s_2, T_0),$$

and

$$\begin{aligned}
&e^{-r(T_0-t)}\mathbb{E}[\mathrm{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T) \mid S_1(t) = s_1, S_2(t) = s_2]\\
&= e^{-r(T_0-t)}\mathbb{E}[e^{-r(T-T_0)}\mathbb{E}[(S_2(T)-S_1(T))^+ | S_1(T_0), S_2(T_0)] \mid S_1(t)=s_1, S_2(t)=s_2]\\
&= e^{-r(T-t)}\mathbb{E}[(S_2(T) - S_1(T))^+ \mid S_1(t) = s_1, S_2(t) = s_2]\\
&= \mathrm{ex}_{2,1}(t, s_1, s_2, T),
\end{aligned}$$

we have the following theorem.

Theorem 4.2. *Let $0 \le t < T_0 < T$. Given $S_1(t) = s_1$ and $S_2(t) = s_2$, the price at t of the chooser exchange option that chooses at T_0 between the two exchange options is given by*

$$\text{CHEX}(t, s_1, s_2, T_0, T) = \text{ex}_{1,2}(t, s_1, s_2, T_0) + \text{ex}_{2,1}(t, s_1, s_2, T). \tag{3}$$

Remark 3. By using the relation

$$\begin{aligned} &\max\{\text{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T),\ \text{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T)\} \\ &= (\text{ex}_{2,1}(T_0, S_1(T_0), S_2(T_0), T) - \text{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T))^+ \\ &\quad + \text{ex}_{1,2}(T_0, S_1(T_0), S_2(T_0), T), \end{aligned}$$

we can obtain another expression for $\text{CHEX}(t, s_1, s_2, T_0, T)$:

$$\text{CHEX}(t, s_1, s_2, T_0, T) = \text{ex}_{2,1}(t, s_1, s_2, T_0) + \text{ex}_{1,2}(t, s_1, s_2, T).$$

This can also be obtained from (3) by applying the put-call parity:

$$\begin{aligned} \text{ex}_{1,2}(t, s_1, s_2, T_0) - \text{ex}_{2,1}(t, s_1, s_2, T_0) &= S_1(t) - S_2(t), \\ \text{ex}_{1,2}(t, s_1, s_2, T) - \text{ex}_{2,1}(t, s_1, s_2, T) &= S_1(t) - S_2(t). \end{aligned}$$

To illustrate Theorem 4.2, we provide a numerical example.

Example 2. Consider the Black-Scholes model with two assets. We use the following parameter values: $s_1 = 1$, $\sigma_1 = 0.4$, $\sigma_2 = 0.2$, $\rho = 0.5$, $T_0 = 1$.

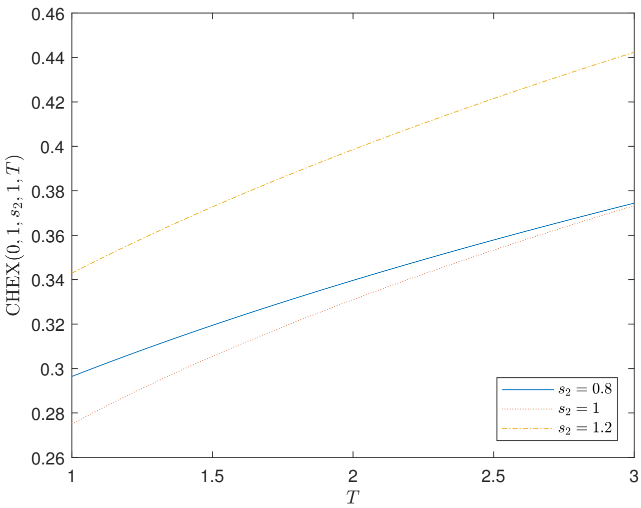


FIGURE 5. Prices at 0 of chooser exchange options for Example 2 with varying T.

Figure 5 shows the prices at 0, $\text{CHEX}(0, 1, s_2, 1, T)$ of chooser exchange options for Example 2 with T varying, when $s_2 = 0.8$, 1, 1.2.

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