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RIEMANNIAN SUBMERSIONS WHOSE TOTAL MANIFOLD ADMITS h-ALMOST RICCI-YAMABE SOLITON

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Abstract. In this paper, we study Riemannian submersions whose total manifold admits h-almost Ricci-Yamabe soliton. We characterize the fibers of the submersion and see under what conditions the fibers form h-almost Ricci-Yamabe soliton. Moreover, we find the necessary condition for the base manifold to be an h-almost Ricci-Yamabe soliton and Einstein manifold. Later, we compute scalar curvature of the total manifold and using this we find the necessary condition for h-almost Yamabe solition to be shrinking, expanding and steady. At the end, we give a non-trivial example.

1. Introduction

In the twentieth century, the Poincare conjecture was one of the famous unsolved problems of modern mathematics. Poincare asked whether a simply-connected closed 3-manifold is always a 3-sphere $\mathbb{S}^3$. After years of topological difficulties, William Thurston produced encouraging results in 1970s. Thurston's geometrization conjecture claims that every closed 3-manifold may be divided into pieces, and each piece admits one of the 8-geometric structures. This provides a connection between the geometry and topology of 3-manifolds, similar in concept to the case of surfaces. The Poincare conjecture in particular was a special example of Thurston's geometrization conjecture.

Inspired by Eells and Sampson's work [5] on the harmonic map flow, Hamilton [10] introduced the concept of Ricci flow in 1982. The Ricci flow is an evolution equation for metric on a Riemannian manifold given by

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric},$$

where $g = g(t)$ is the Riemannian metric and Ric denotes the Ricci tensor. A self-similar solution of the Ricci flow [10], which moves only by one parameter

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family of diffeomorphism and scaling is called a Ricci soliton [11]. The Ricci soliton is given by

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} - \lambda g = 0,$$

where $\mathcal{L}_V$ is the Lie derivative, g is the Riemannian metric, V is the vector field and λ is a scalar. The Ricci soliton is denoted by (g, V, λ) and is said to be shrinking, steady and expanding according to whether λ is positive, zero and negative, respectively.

In 1988, Hamilton introduced the notion of Yamabe flow on a smooth manifold. This geometric flow was used as a tool for constructing metrics of constant scalar curvature in a given conformal class of a Riemannian metric. The Yamabe flow on a smooth manifold (M, g) is defined as an evolution equation of a Riemannian metric $g = g(t)$ as

$$\frac{\partial g(t)}{\partial t} = -\tau g(t),$$

where τ denotes the scalar curvature of the manifold. It should be noted that in two dimensional case, the Ricci and Yamabe solitons coincide. However, in higher dimensions, Yamabe flow and Ricci flow does not agree as the former preserves the conformal class of the metric while as the latter does not. Like Ricci soliton, Yamabe soliton is self similar solution of Yamabe flow which moves by a one-parameter family diffeomorphism and scaling and is given by

$$\frac{1}{2}\mathcal{L}_V g + (\lambda - \tau)g = 0.$$

As a generalization of Ricci and Yamabe flow, Güler and Crâşmăreanu [8] introduced the concept of Ricci-Yamabe flow given by

$$\frac{\partial g(t)}{\partial t} = -2\alpha \mathrm{Ric} + \beta\tau g.$$

Ricci-Yamabe solitons are defined by the equation

$$\frac{1}{2}\mathcal{L}_V g + \alpha \mathrm{Ric} + (\lambda - \frac{\beta}{2}\tau)g = 0, \tag{1.1}$$

where $\lambda, \alpha, \beta \in \mathbb{R}$. If λ is a smooth function, then equation (1.1) is called almost Ricci-Yamabe soliton.

Gomes et al. [7] extended the concept of almost Ricci soliton to h-almost Ricci solitons on a complete Riemannian manifold by

$$\frac{h}{2}\mathcal{L}_V g + \mathrm{Ric} + \lambda g = 0,$$

where $h : M \longrightarrow \mathbb{R}$ is a smooth function. In particular, Ricci soliton is 1-almost Ricci soliton for $\lambda \in \mathbb{R}$. Zeng [20] introduced the concept of h-almost Yamabe soliton and obtained some rigidity results. Ghahremani-Gol [6] investigated h-almost Ricci solitons further and obtained some structure equations and an integral formula for the compact h-almost Ricci solitons. In [3], Cunha and

Siddiqi studied gradient h-almost Yamabe solitons. Recently, De et al. [4] extended the concept of Ricci-Yamabe solitons to h-almost Ricci-Yamabe solitons on a complete Riemannian manifold by

$$\frac{h}{2}\mathcal{L}_V g + \alpha \mathrm{Ric} + (\lambda - \frac{\beta}{2}\tau)g = 0, \tag{1.2}$$

where $h : M \longrightarrow \mathbb{R}$ is a smooth function.

An h-almost Ricci-Yamabe soliton is

- (i) h-almost Ricci soliton for $\alpha = 1$ and $\beta = 1$,
- (ii) h-almost Yamabe soliton for $\alpha = 0$ and $\beta = 1$,
- (iii) h-almost Einstein soliton for $\alpha = 1$ and $\beta = -1$.

On the other hand Riemannian submersions are very important tools in Riemannian geometry. In addition to having a vital role in Riemannian geometry, Riemannian submersions are very relevant to many fields of theoretical physics, including Yang-Mills theory [2], Kaluza-Klein theory [1], supergravity, and string theories. Riemannian submersions are also useful in explaining extensions of important aspects of theoretical particle physics in the presence of non-Abelian gauge theories. An evidence of this phenomena was given by Watson [17] who studied the relations between Riemannian submersions and instantons, the latter of which are critical functionals of the Yang-Mills action. Other applications in physics where Riemannian submersions are widely used are generalized nonlinear sigma models in curved spaces, the Dirac monopole, Einstein equations, among others. For further details see [12–16, 20] and references therein.

In 2021, Meriç et al. [14] established a link between Riemannian submersions and Ricci solitons. They studied Riemannian submersions whose total manifold admit a Ricci soliton. Under this setup, the authors investigated the geometry of fibers and base manifold. The harmonocity and biharmonocity was also studied. This research article quickly attracted attention of various mathematicians. Yadav et al. studied Riemannian maps [18] and Clairaut Riemannian [19] maps whose total manifolds admit Ricci solitons. Gupta et al. [9] investigated conformal Riemannian maps whose total manifold admits a Ricci soliton. Motivated by above studies, we will investigate Riemannian submersions whose total manifolds admit h-almost Ricci-Yamabe solitons.

2. Preliminaries

Let $\varphi : M \longrightarrow B$ be a smooth map from the Riemannian manifold M of dimension m onto the Riemannian manifold B of dimension n where $m > n$. Then φ is said to be a Riemannian submersion [15] if it satisfies the following conditions:

- (i) φ is of maximal rank.
- (ii) The differential map φ_* of φ preserves the lengths of horizontal vectors.

By a horizontal vector field X on M we mean a vector field which is orthogonal to the kernel of φ_* at each point p of M and by a vertical vector field V on M we mean a vector field which is tangent to the kernel of φ_* at each point $p \in M$. The Riemannian manifolds M^m and B^n are called *total manifold* and *base manifold*, respectively. For any fixed $b \in B$, $\varphi^{-1}(b)$ forms a closed submanifold of M of dimension $r = m - n$. Denote by $\mathscr{H}_p = \{\text{set of all horizontal vectors at } p\}$ and by $\mathscr{V}_p = \{\text{set of all vertical vectors at } p\}$. Thus a Riemannian submersion defines two complementary orthogonal distributions $\mathscr{H}$ and $\mathscr{V}$, called horizontal and vertical distribution, respectively, on M. Further the vertical distribution $\mathscr{V}$ is always integrable.

O' Neill defined two fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ of a Riemannian submersion. These are (1,2)-tensors and are defined by the following formulae:

$$\mathcal{T}_E F = \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F, \tag{2.1}$$

$$\mathcal{A}_E F = \mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F + \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F, \tag{2.2}$$

where ∇ denotes Riemannian connection on M, E and F are arbitrary vector fields on M and $\mathscr{V}$, $\mathscr{H}$ denote the projection morphisms on the distributions $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$, respectively. These tensors are called O'Neill's integrability tensors. For any $F \in \Gamma(TM)$, $\mathcal{T}_F$ and $\mathcal{A}_F$ are skew-symmetric operators on $\big(\Gamma(TM), g\big)$ and they reverse the horizontal and vertical distributions. It is easy to see that $\mathcal{T}$ is vertical, i.e., $\mathcal{T}_F = \mathcal{T}_{\mathscr{V}F}$ and $\mathcal{A}$ is horizontal, i.e., $\mathcal{A}_F = \mathcal{A}_{\mathscr{H}F}$.

The tensor field $\mathcal{T}$ and $\mathcal{A}$ also satisfy:

$$\mathcal{T}_V U = \mathcal{T}_U V \quad \forall\, U, V \in \ker\varphi_*, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathscr{V}[X, Y] \quad \forall\, X, Y \in (\ker\varphi_*)^\perp. \tag{2.4}$$

The above equations imply that $\mathcal{T}$ restricted over vertical distribution $\mathscr{V}$ is a symmetric operator and $\mathcal{A}$ restricted over horizontal distribution $\mathscr{H}$ is a skew-symmetric operator. Also, the operator $\mathcal{A}$ measures the obstruction of the horizontal distribution from being integrable.

Let $\varphi : M \longrightarrow B$ be a Riemannian submersion and denote the Levi-Civita connection of M and B by ∇ and ∇', respectively. If E, F are basic vector fields φ-related to E', F', then

(i) $g(E, F) = g'(E', F') \circ \varphi$,
(ii) $\mathscr{H}[E, F]$ is a basic vector field φ-related to $[E', F']$,
(iii) $\mathscr{H}\nabla_E F$ is the basic vector field φ-related to $\nabla'_{E'} F'$.
(iv) for any vector field U, $[U, E]$ is a vertical vector field.

From (2.1) and (2.2), we have following equations:

$$\nabla_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{2.5}$$

$$\nabla_U X = \mathscr{H}\nabla_U X + \mathcal{T}_U X, \tag{2.6}$$

$$\nabla_X U = \mathcal{A}_X U + \mathscr{V}\nabla_X U, \tag{2.7}$$

$$\nabla_X Y = \mathscr{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $U, V \in \Gamma(\ker\varphi_*)$, $X, Y \in \Gamma\big((\ker\varphi_*)^\perp\big)$ and $\hat{\nabla}_U V$ denotes $\mathscr{V}\nabla_U V$. On any fibre $\varphi^{-1}(q)$, $q \in B$, denote the induced metric by $\hat{g}$. Then $\hat{\nabla}$ denotes the Levi-Civita connection with respect to metric $\hat{g}$.

On fibers, $\mathcal{T}$ acts as second fundamental form of the submersion and restricted to vertical vector fields, it could be easily seen that $\mathcal{T} = 0$ is equivalent to the condition that the fibers are totally geodesic. A Riemannian submersion is said to have totally geodesic fibers if $\mathcal{T}$ vanishes identically. Let $U_1, U_2, \dots, U_r$ be an orthonormal frame of $\ker\varphi_*$. Then the horizontal vector field

$$H = \frac{1}{r} N \tag{2.9}$$

is called the *mean curvature* vector field of the fiber, where

$$N = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j. \tag{2.10}$$

If $H = 0$, then the Riemmanian submersion is said to be minimal. A Riemannian submersion is said to have totally umbilical fibers if

$$\mathcal{T}_U V = g(U, V) H \tag{2.11}$$

for $U, V \in \ker\varphi_*$.

Lemma 2.1 ([14]). *Let $\varphi : M \longrightarrow B$ be a Riemannian submersion between Riemannian manifolds. Then the following are equivalent*:

(i) *the vertical distribution $\mathscr{V}$ is parallel.*
(ii) *the horizontal distribution $\mathscr{H}$ is parallel.*
(iii) *the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ vanish identically, that is, $\mathcal{T} \equiv 0$ and $\mathcal{A} \equiv 0$.*

Let R, R' and $\hat{R}$ denote the Riemannian curvature tensors of (M, g), (B, g') and any fiber of φ, respectively. Then we have

$$R(U, V, W, E) = \hat{R}(U, V, W, E) - g(T_U E, T_V W) + g(T_V E, T_U W), \tag{2.12}$$

$$\begin{aligned} R(X, Y, Z, F) = {} & R'(X', Y', Z', F') \circ \varphi + 2g(A_X Y, A_Z F) \\ & - g(A_Y Z, A_X F) + g(A_X Z, A_Y F), \end{aligned} \tag{2.13}$$

where $U, V, W, E \in \Gamma(\ker\varphi_*)$ and $X, Y, Z, F \in \Gamma\big((\ker\varphi_*)^\perp\big)$.

Denote the sectional curvatures of (M, g), (B, g') and of any fiber of φ by K, K' and $\hat{K}$, respectively. Let (U, V) and (X, Y) be an orthonormal basis of 2-plane in $\ker\varphi_*$ and $(\ker\varphi_*)^\perp$, respectively. Then

$$K(U, V) = \hat{K}(U, V) - \|\mathcal{T}_U V\|^2 + g(\mathcal{T}_U U, \mathcal{T}_V V), \tag{2.14}$$

$$K(X, Y) = K'(\varphi_* X, \varphi_* Y) + 3\|\mathcal{A}_X Y\|^2. \tag{2.15}$$

The Ricci tensor Ric on (M,g) is given by

$$\begin{aligned}\mathrm{Ric}(U,V) = {} & \hat{R}(U,V) + g(N,\mathcal{T}_U V)\\ & - \sum_{i=1}^{n}\big\{g\big((\nabla_{X_i}\mathcal{T})(U,V),X_i\big) - g(\mathcal{A}_{X_i}U,\mathcal{A}_{X_i}V)\big\},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathrm{Ric}(X,Y) = {} & \mathrm{Ric}'(X',Y')\circ\varphi - \frac{1}{2}\{g(\nabla_X N,Y) + g(\nabla_Y N,X)\}\\ & + 2\sum_{i=1}^{n} g(\mathcal{A}_X X_i,\mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}Y),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\mathrm{Ric}(U,X) = {} & -g(\nabla_U N,X) + \sum_{j=1}^{r} g\big((\nabla_{U_j}\mathcal{T})(U_j,U),X\big)\\ & - \sum_{i=1}^{n}\{g((\nabla_{X_i}\mathcal{A})(X_i,X),U) + 2g(\mathcal{A}_{X_i}X,\mathcal{T}_U X_i)\},\end{aligned} \tag{2.18}$$

where $\{U_i\}$ and $\{X_i\}$ are orthonormal bases of vertical and horizontal distribution, respectively, and $U,V \in \Gamma(\ker\varphi_*)$ and $X,Y \in \Gamma\big((\ker\varphi_*)^{\perp}\big)$.

Let τ, τ' and $\hat{\tau}$ denote the scalar curvatures of total manifold (M,g), base manifold (B,g') and any fiber of φ, respectively. These are related by the following expression:

$$\tau = \tau'\circ\varphi + \hat{\tau} - \|N\|^2 - \|\mathcal{A}\|^2 - \|\mathcal{T}\|^2 + 2\sum_{i=1}^{n} g(\nabla_{X_i}N,X_i). \tag{2.19}$$

Let's recall the notion of harmonic map between Riemannian manifolds. Let (M,g) and (B,g') be Riemannian manifolds and suppose that $\varphi : M \longrightarrow B$ is a smooth map between them. Then the differential φ_* of φ can be viewed as a section of the bundle $\mathrm{Hom}(TM,\varphi^{-1}TB) \longrightarrow M$, where $\varphi^{-1}TB$ is the pullback bundle which has fibers $(\varphi^{-1}TB)_p = T_{\varphi(p)}B$, $p \in M$. $\mathrm{Hom}(TM,\varphi^{-1}TB)$ has a connection ∇ induced from Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X\varphi_*(Y) - \varphi_*(\nabla^M_X Y) \tag{2.20}$$

for $X,Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M,g) \longrightarrow (B,g')$ is said to be harmonic if $\mathrm{trace}(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tilde{\tau}(\varphi)$ of $\Gamma(\varphi^{-1}TB)$ defined by

$$\tilde{\tau}(\varphi) = \mathrm{div}\varphi_* = \sum_{i=1}^{m}(\nabla\varphi_*)(e_i,e_i), \tag{2.21}$$

where $\{e_1,e_2,\ldots,e_m\}$ is the orthonormal frame on M. Then it follows that φ is harmonic if and only if $\tilde{\tau}(\varphi) = 0$.

The divergence of a horizontal vector field X is denoted by $\check{\delta}(X)$ and is given by

$$\check{\delta}(X) = \sum_{j=r+1}^{n} g(\nabla_{X_j} X, X_j). \tag{2.22}$$

Using (2.11) in (2.22), we get

$$\check{\delta}(N) = \sum_{i=1}^{r} \sum_{j=r+1}^{n} g\big((\nabla_{X_j}\mathcal{T})_{U_i} U_i, X_j\big), \tag{2.23}$$

where $\{U_i\}$ and $\{X_j\}$ are orthonormal bases of $\ker\varphi_*$ and $(\ker\varphi_*)^{\perp}$, respectively.

A Ricci flat manifold is a (pseudo-)Riemannian manifold whose Ricci tensor is zero at every point. In particular, scalar curvature τ of Ricci flat manifolds is also identically zero. Ricci flat manifolds are of particular interest in theoretical physics and differential geometry. In physics, they arise naturally in string theory and in the study of the geometry of space-time in general relativity. In differential geometry, Ricci flat manifolds are important because they have special geometric properties that allow for the development of new mathematical techniques and tools.

Examples of Ricci flat manifolds include flat Euclidean space, Calabi-Yau manifolds, and hyperkahler manifolds. The study of these manifolds has important applications in fields such as algebraic geometry, mathematical physics, and theoretical physics.

3. Main results

Theorem 3.1. *Let (M, g) be an h-almost Ricci-Yamabe soliton with vertical potential field V and $\varphi : M \longrightarrow B$ be a Riemannian submersion where (B, g') is a Ricci flat Riemannian manifold. If the submersion satisfies one of the conditions of Lemma 2.1, then the fibers of the submersion are h-almost Ricci-Yamabe solitons.*

Proof. Since (M, g) is an h-almost Ricci-Yamabe soliton, we have by (1.2)

$$\frac{h}{2}(\mathcal{L}_V g)(U, W) + \alpha \mathrm{Ric}(U, W) + (\lambda - \frac{\beta}{2}\tau) g(U, W) = 0 \tag{3.1}$$

for any $U, W \in \Gamma(\mathcal{V})$.

Using (2.16) and (2.19) in (3.1), we get

$$\begin{aligned}
&\frac{h}{2}[g(\nabla_U V, W) + g(U, \nabla_W V)] + \alpha\{\hat{\mathrm{Ric}}(U, W) + g(N, \mathcal{T}_U W) \\
&- \sum_{i=1}^{r}[g\big((\nabla_{X_i}\mathcal{T})(U, W), X_i\big) - g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W)]\} \\
&+ \big(\lambda - \frac{\beta}{2}(\hat{\tau} + \tau' \circ \varphi - \|N\|^2 - \|\mathcal{A}\|^2 - \|\mathcal{T}\|^2 + 2\check{\delta}(N))\big) g(U, W) = 0.
\end{aligned}$$

Now using Lemma 2.1 and (2.5) in the above equation, we immediately get

$$\frac{h}{2}[g(\hat{\nabla}_U V,W)+g(U,\hat{\nabla}_W V)]+\alpha\{\hat{\mathrm{Ric}}(U,W)\}+\big(\lambda-\frac{\beta}{2}(\hat{\tau}+\tau'\circ\varphi)\big)g(U,W)=0. \tag{3.2}$$

Since (B,g') is a Ricci flat manifold, that is $\tau'=0$, we get from (3.2)

$$\frac{h}{2}[g(\hat{\nabla}_U V,W)+g(U,\hat{\nabla}_W V)]+\alpha\{\hat{\mathrm{Ric}}(U,W)\}+\big(\lambda-\frac{\beta}{2}\hat{\tau}\big)g(U,W)=0,$$

which implies that each fiber is an h-almost Ricci-Yamabe soliton. □

Theorem 3.2. *Let (M,g) be an h-almost Ricci-Yamabe soliton with vertical potential field V and $\varphi : M \longrightarrow B$ be a Riemannian submersion with umbilical fibers. Suppose the horizontal distribution is integrable. Then each fiber is an h-almost Ricci-Yamabe soliton.*

Proof. For vertical vector fields U and W, we have by (1.2)

$$\frac{h}{2}(\mathcal{L}_V g)(U,W)+\alpha\mathrm{Ric}(U,W)+(\lambda-\frac{\beta}{2}\tau)g(U,W)=0.$$

Using (2.16) and (2.19) in the above equation we get

$$\begin{aligned}&\frac{h}{2}(\mathcal{L}_V g)(U,W)+\alpha\{\hat{\mathrm{Ric}}(U,W)+g(N,\mathcal{T}_U W)\\&-\sum_{i=1}^{r}[g\big((\nabla_{X_i}\mathcal{T})(U,W),X_i\big)-g(\mathcal{A}_{X_i}U,\mathcal{A}_{X_i}W)]\}\\&+\big(\lambda-\frac{\beta}{2}(\hat{\tau}+\tau'\circ\phi-\|N\|^2)-\|\mathcal{A}\|^2-\|\mathcal{T}\|^2+2\check{\delta}(N)\big)g(U,W)=0.\end{aligned}$$

Now using (2.9) and the fact that ∇ is a metric connection, we get from the above equation

$$\begin{aligned}&\frac{h}{2}(\mathcal{L}_V g)(U,W)+\alpha\{\hat{\mathrm{Ric}}(U,W)+r\sum_{i=1}^{r}g(\mathcal{T}_{U_i}U_i,\mathcal{T}_U W)\\&-\sum_{i=1}^{r}(\nabla_{X_i}g)\big(\mathcal{T}_U W,X_i\big)-\sum_{i=1}^{r}g(\mathcal{T}_U W,\nabla_{X_i}X_i)+\sum_{i=1}^{r}g(\mathcal{A}_{X_i}U,\mathcal{A}_{X_i}W)\}\\&+\big(\lambda-\frac{\beta}{2}(\hat{\tau}+\tau'\circ\phi-\|N\|^2-\|\mathcal{A}\|^2-\|\mathcal{T}\|^2+2\check{\delta}(N))\big)g(U,W)=0.\end{aligned} \tag{3.3}$$

Using (2.9), (2.22) and the fact that fibers of φ are umblical in (3.3), we get

$$\begin{aligned}&\frac{h}{2}(\mathcal{L}_V g)(U,W)+\alpha\hat{\mathrm{Ric}}(U,W)+\alpha\Big[r\|H\|^2 g(U,W)+\check{\delta}(H)g(U,W)\Big]\\&+\big(\lambda-\frac{\beta}{2}(\hat{\tau}+\tau'\circ\varphi-\|N\|^2-\|\mathcal{T}\|^2+2\check{\delta}(N))\big)g(U,W)=0.\end{aligned}$$

Upon rearranging, we get from the above equation

$$\frac{h}{2}[g(\hat{\nabla}_U V,W)+g(U,\hat{\nabla}_W V)]+\alpha\hat{\mathrm{Ric}}(U,W)+(\Lambda-\frac{\beta}{2}\hat{\tau})g(U,W)=0,$$

where $\Lambda = \alpha(r\|H\|^2 + \check{\delta}(H)) + \lambda + \frac{\beta}{2}\big(\|\mathcal{T}\|^2 + \|N\|^2 - 2\hat{\delta}(N)\big)$.

Thus the fibers form h-almost Ricci-Yamabe soliton. □

Remark. If the fibers of the Riemannian submersions are minimal and (B, g') is a Ricci flat manifold, then $\mathcal{T}, N, H$ are identically equal to zero. Hence we have from the above equation,

$$\frac{h}{2}[g(\hat{\nabla}_U V, W) + g(U, \hat{\nabla}_W V)] + \alpha \hat{\mathrm{Ric}}(U, W) + (\lambda - \frac{\beta}{2}\hat{\tau})g(U, W) = 0.$$

Theorem 3.3. *Consider an h-almost Ricci-Yamabe soliton (M,g) with horizontal potential field V and let $\varphi : M \longrightarrow B$ be a Riemannian submersion with Ricci flat fibers. If one of the conditions of Lemma 2.1 is satisfied, then the base manifold (B, g') is an h-almost Ricci-Yamabe soliton.*

Proof. For X, $Y \in \Gamma\big((\ker\varphi_*)^\perp\big)$, we have by (1.2)

$$\frac{h}{2}(\mathcal{L}_V g)(X, Y) + \alpha \mathrm{Ric}(X, Y) + (\lambda - \frac{\beta}{2}\tau)g(X, Y) = 0.$$

Using (2.17) and (2.19) we get from the above equation

$$\begin{aligned}
&\frac{h}{2}[g(\nabla_X V, Y) + g(X, \nabla_Y V)] \\
&+ \alpha\{\mathrm{Ric}'(X, Y)\circ\varphi - \frac{1}{2}\{g(\nabla_X N, Y) + g(\nabla_Y N, X)\} \\
&+ 2\sum_{i=1}^{n} g\big(\mathcal{A}_X X_i, \mathcal{A}_Y X_i\big) + \sum g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y)\} \\
&+ \big(\lambda - \frac{\beta}{2}(\hat{\tau} + \tau'\circ\varphi - \|N\|^2) - \|\mathcal{A}\|^2 - \|\mathcal{T}\|^2 + 2\check{\delta}(N)\big)g(U, W) = 0.
\end{aligned} \tag{3.4}$$

Now using Lemma 2.1 and the fact that the fibers of φ are Ricci flat, that is $\hat{\tau} = 0$, we get from (3.4)

$$\begin{aligned}
&\frac{h}{2}[g\big(\mathcal{H}(\nabla_X V), Y\big) + g\big(X, \mathcal{H}(\nabla_Y V)\big)] \\
&+ \alpha \mathrm{Ric}'(X', Y')\circ\varphi + \big(\lambda - \frac{\beta}{2}(\tau'\circ\varphi)\big)g(X, Y) = 0.
\end{aligned} \tag{3.5}$$

Since $\mathcal{H}(\nabla_X V)$ and $\mathcal{H}(\nabla_Y V)$ are basic vector fields φ-related to $\nabla'_{X'}V'$ and $\nabla'_{Y'}V'$, respectively, we get from (3.5)

$$\begin{aligned}
&\frac{h}{2}[g'\big(\nabla'_{X'}V', Y'\big) + g'\big(X', \nabla'_{Y'}V'\big)]\circ\varphi \\
&+ \alpha \mathrm{Ric}'(X', Y')\circ\phi + \big(\lambda - \frac{\beta}{2}(\tau'\circ\varphi)\big)g'(X', Y')\circ\varphi = 0,
\end{aligned}$$

which implies

$$\frac{h}{2}(\mathcal{L}_{V'}g')(X', Y') + \alpha \mathrm{Ric}'(X', Y') + (\lambda - \frac{\beta}{2}\tau')g'(X', Y') = 0.$$

Hence (B, g') is an h-almost Ricci-Yambe soliton. □

Theorem 3.4. *Let (M,g) be an h-almost Ricci soliton with vertical potential field V and $\varphi : M \longrightarrow B$ be a Riemannian submersion. If one of the conditions of Lemma 2.1 is satisfied, then (B,g') is an Einstein manifold.*

Proof. Since (M,g) is an h-almost Ricci soliton, we have

$$\frac{h}{2}(\mathcal{L}_V g)(X,Y) + \alpha \mathrm{Ric}(X,Y) + \lambda g(X,Y) = 0.$$

Using (2.17) in the above equation, we get

$$\begin{aligned}&\frac{h}{2}[g(\nabla_X V, Y) + g(X, \nabla_Y V)]\\&+ \alpha\{\mathrm{Ric}'(X',Y')\circ\varphi - \frac{1}{2}\{g(\nabla_X N, Y) + g(\nabla_Y N, X)\}\\&+ 2\sum_{i=1}^{n} g\big(\mathcal{A}_X X_i, \mathcal{A}_Y X_i\big) + \sum g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y)\} + \lambda g(X,Y) = 0.\end{aligned}$$

Using Lemma 2.1, we get

$$\frac{h}{2}[g(\nabla_X V, Y) + g(X, \nabla_Y V)] + \alpha \mathrm{Ric}'(X',Y')\circ\varphi + \lambda g(X,Y) = 0.$$

Since V is vertical, using (2.7) we get

$$\frac{h}{2}[g(\mathcal{A}_X V, Y) + g(X, \mathcal{A}_Y V)] + \alpha \mathrm{Ric}'(X',Y')\circ\varphi + \lambda g'(X',Y')\circ\varphi = 0.$$

Again using Lemma 2.1, we get

$$\alpha \mathrm{Ric}'(X',Y')\circ\varphi + \lambda g'(X',Y')\circ\varphi = 0$$

which implies (B,g') is an Einstein manifold. ☐

Theorem 3.5. *Let $(M,g,h,V,\alpha,\beta,\lambda)$ be an h-almost Ricci-Yamabe soliton with $\beta \in \mathbb{R}$ admitting a Riemannian submersion $\varphi : M \longrightarrow B$. If one of the conditions of Lemma 2.1 is satisfied, then scalar curvature of (M,g) is given by $\tau = \frac{2\lambda m}{\beta m - 2\alpha}$.*

Proof. Since (M,g) is an h-almost Ricci-Yamabe soliton, we have

$$\frac{h}{2}(\mathcal{L}_V g)(E,F) + \alpha \mathrm{Ric}(E,F) + (\lambda - \frac{\beta}{2}\tau) g(E,F) = 0,$$

where $E, F \in \Gamma(TM)$. Taking trace of above equation, we get

$$\begin{aligned}&h\Big[\sum_{i=1}^{r} g(\nabla_{U_i}U_i, V) + \sum_{j=m+1}^{m} g(\nabla_{X_i}X_i, V)\Big]\\&+ \alpha\Big[\sum_{i=1}^{r} \hat{\mathrm{Ric}}(U_i,U_i) + \sum_{j=r+1}^{m} \mathrm{Ric}'(X_i',X_i')\circ\varphi\Big]\\&+ (\lambda - \frac{\beta}{2}\tau)\Big(\sum_{i=1}^{r} g(U_i,U_i) + \sum_{j=r+1}^{m} g(X_j,X_j)\Big) = 0,\end{aligned} \tag{3.6}$$

where $\{U_i\}_{1\leq i\leq r}$ and $\{X_j\}_{r+1\leq j\leq m}$ are orthonormal bases of vertical and horizontal distributions of φ, respectively.

Since φ satisfies Lemma 2.1, using (2.16), (2.17) and (2.18) in (3.6), we get

(3.7)
$$h\Big[\sum_{i=1}^{r} U_i\big(g(U_i,V)\big)+\sum_{j=m+1}^{m} X_j\big(g(X_j,V)\big)\Big]+\alpha\big(\hat{\tau}+\tau'\circ\varphi\big)+(\lambda-\frac{\beta}{2}\tau)m=0.$$

Now using fact that ∇ is a metric connection, we immediately get

$$\alpha\tau+(\lambda-\frac{\beta}{2}\tau)m=0,$$

which implies

$$\tau=\frac{2\lambda m}{\beta m-2\alpha}. \qquad \square$$

Corollary 3.6. *Let $(M,g,h,V,\alpha,\beta,\lambda)$ be an h-almost Ricci-Yamabe soliton with $\lambda,\beta\in\mathbb{R}$ admitting a Riemannian submerion $\varphi: M\longrightarrow B$. If one of the conditions of Lemma 2.1 is satisfied, then (M,g) has constant scalar curvature $\tau=\frac{2\lambda m}{\beta m-2\alpha}$.*

Corollary 3.7. *Let $(M,g,h,V,\alpha,\beta,\lambda)$ be an h-almost Yamabe soliton with $\lambda\in\mathbb{R}$, $\beta\in\mathbb{R}^+$ and $\varphi: M\longrightarrow B$ be a Riemannian submersion. If one of the conditions of Lemma 2.1 is satisfied, then*

(1) *(M,g) is shrinking $\Longleftrightarrow$ τ is positive.*
(2) *(M,g) is steady $\Longleftrightarrow$ τ vanishes.*
(3) *(M,g) is expanding $\Longleftrightarrow$ τ is negative.*

4. Example

Example. Consider Riemannian manifolds $M=(\mathbb{R}^4,g)$ and $N=(\mathbb{R}^2,g')$, where

$$g=e^{-2y_2}dy_1^2+dy_2^2+e^{-2y_2}dy_3^2+dy_4^2,$$
$$g'=dy_1^2+dy_2^2.$$

Define a map $\varphi:\mathbb{R}^4\longrightarrow\mathbb{R}^2$ by

$$\varphi(y_1,y_2,y_3,y_4)=\big(\frac{y_1+y_4}{\sqrt{2}},\frac{y_2+y_3}{\sqrt{2}}\big).$$

Then by direct calculations, we have

$$\ker\varphi_*=\mathrm{Span}\{U_1=\partial y_1-\partial y_4, U_2=\partial y_2-\partial y_3\},$$
$$\big(\ker\varphi_*\big)^\perp=\mathrm{Span}\{X_1=\partial y_1+\partial y_4, X_2=\partial y_2+\partial y_3\},$$

where $\{e_1=e^{y_2}\partial y_1, e_2=\partial y_2, e_3=e^{y_2}\partial y_3, e_4=\partial y_4\}$ is an orthonormal basis of $T_p\mathbb{R}^4$ for any $p\in\mathbb{R}^4$.

Then it is easy to see that φ is a Riemannian submersion.

Now, for the metric g, we have

$$[g]_{ij} = \begin{pmatrix} e^{-2y_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-2y_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [g]^{ij} = \begin{pmatrix} e^{2y_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2y_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The non-zero Christofell symbols of g are given by

$$\Gamma^2_{11} = e^{-2y_2},\ \Gamma^1_{12} = -1 = \Gamma^1_{12},\ \Gamma^3_{23} = -1 = \Gamma^3_{32},\ \Gamma^2_{33} = e^{-2y_2}. \tag{4.1}$$

Using (4.1), we have

$$\begin{aligned}
&\nabla_{U_1} U_1 = e_2,\ \ \nabla_{U_1} U_2 = -e_1,\ \ \nabla_{U_1} X_1 = e_2,\ \ \nabla_{U_1} X_2 = -e_1,\\
&\nabla_{U_2} U_1 = 0,\ \ \nabla_{U_2} U_2 = e_2 + e_3,\ \ \nabla_{U_2} X_1 = 0,\ \ \nabla_{U_2} X_2 = e_3 - e_2,\\
&\nabla_{X_1} U_1 = e_2,\ \ \nabla_{X_1} U_2 = -e_1,\ \ \nabla_{X_1} X_1 = e_2,\ \ \nabla_{X_1} X_2 = -e_1,\\
&\nabla_{X_2} U_1 = 0,\ \ \nabla_{X_2} U_2 = -e_3 - e_2,\ \ \nabla_{X_2} X_1 = 0,\ \ \nabla_{X_2} X_2 = -e_3 + e_2,
\end{aligned} \tag{4.2}$$

where ∇ is a Levi-Civita connection on M.

Using (4.2), we get

$$\begin{aligned}
&R(U_1, U_2)U_1 = U_2,\ \ R(U_1, U_2)U_1 = X_1,\ \ R(U_1, U_2)X_2 = 0,\\
&R(U_1, U_2)U_2 = -2e_1,\ \ R(U_1, X_1)U_1 = 0,\ \ R(U_1, X_1)U_1 = 0,\\
&R(U_1, X_1)X_1 = 0,\ \ R(U_1, X_1)X_2 = 0,\ \ R(U_1, X_2)U_1 = X_2,\\
&R(U_1, X_2)U_2 = 0,\ \ R(U_1, X_2)X_1 = X_2,\ \ R(U_1, X_2)X_2 = -2e_1,\\
&R(U_2, X_1)U_1 = -U_2,\ \ R(U_2, X_1)U_2 = 2e_1,\ \ R(U_2, X_1)X_1 = -U_2,\\
&R(U_2, X_1)X_2 = 0,\ \ R(U_2, X_2)U_1 = 0,\ \ R(U_2, X_2)U_2 = 2X_2,\\
&R(U_2, X_2)X_1 = 0,\ \ R(U_2, X_2)X_2 = -2U_2, R(X_1, X_2)U_1 = X_2,\\
&R(X_1, X_2)U_2 = 0,\ \ R(X_1, X_2)X_1 = X_2,\ \ R(X_1, X_2)X_2 = -2e_1,
\end{aligned} \tag{4.3}$$

where R is a curvature endomorphism on M.

Now using (4.3) and the definition $\mathrm{Ric}(X, Y) = \mathrm{trace}(Z \longrightarrow R(Z, X)Y)$, the non-zero components of the Ricci tensor Ric are given by

$$\begin{aligned}
&\mathrm{Ric}(U_1, U_1) = -4,\ \mathrm{Ric}(U_1, X_1) = -4 = \mathrm{Ric}(X_1, U_1),\ \mathrm{Ric}(U_2, U_2) = -8,\\
&\mathrm{Ric}(X_1, X_1) = -4,\ \mathrm{Ric}(X_2, X_2) = -8.
\end{aligned} \tag{4.4}$$

Therefore the scalar curvature $\tau = -24$.

For any $E_i \in \ker\varphi_*$, we can write $E_i = a_i U_1 + b_i U_2$. Therefore

$$g(E_2, E_3) = 2(a_2 a_3 + b_2 b_3), \tag{4.5}$$

$$(\mathcal{L}_{E_1} g)(E_2, E_3) = -2a_2 a_3 b_1 + a_1 a_2 b_3 + a_1 a_3 b_2, \tag{4.6}$$

$$\mathrm{Ric}(E_2, E_3) = -4a_2 a_3 - 8b_2 b_3. \tag{4.7}$$

Now substituting (4.5), (4.6) and (4.7) in (1.2) and simplifying we get

$$\lambda = \frac{h(2a_2 a_3 b_1 - a_1 a_2 b_3 - a_1 a_3 b_2) + 2\alpha(a_2 a_3 + 4b_2 b_3)}{4(a_1 a_2 + b_1 b_2)} - 12\beta.$$

Thus fibers of the Riemannian submersion admit h-almost Ricci-Yamabe soliton if λ satisfies the above equation.

5. Conclusions

The concept of Ricci-Yamabe solitons was introduced by Güler and Crâşmăreanu [8] as a generalization of Ricci and Yamabe solitons. On a complete Riemannian manifold, Gomes et al. [7] extended the notion of almost Ricci soliton to h-almost Ricci soliton. De et al. [4] extended the notion of almost Ricci-Yamabe soliton to h-almost Ricci-Yamabe soliton in para-Kenmotsu manifolds. In this paper, we connected the theory of Riemannian submersions and h-almost Ricci-Yamabe solitons by studying the behaviour of fibers and base manifold under different conditions. We also give a non-trivial example at the end.

In the near future, we or possibly other authors will investigate Riemannian submersions from different space forms admitting h-almost Ricci-Yamabe solitons.

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