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A NEW CRITERION FOR MOMENT INFINITELY DIVISIBLE WEIGHTED SHIFTS

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Abstract. In this paper we present the weighted shift operators having the property of moment infinite divisibility. We first review the monotone theory and conditional positive definiteness. Next, we study the infinite divisibility of sequences. A sequence of real numbers γ is said to be infinitely divisible if for any $p > 0$, the sequence $\gamma^p = \{\gamma_n^p\}_{n=0}^{\infty}$ is positive definite. For sequences $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers, we consider the weighted shift operators W_α. It is also known that W_α is moment infinitely divisible if and only if the sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}\}_{n=0}^{\infty}$ of W_α are infinitely divisible. Here γ is the moment sequence associated with α. We use conditional positive definiteness to establish a new criterion for moment infinite divisibility of W_α, which only requires infinite divisibility of the sequence $\{\gamma_n\}_{n=0}^{\infty}$. Finally, we consider some examples and properties of weighted shift operators having the property of $(k,0)$-CPD; that is, the moment matrix $M_\gamma(n,k)$ is CPD for any $n \geq 0$.

1. Introduction

Let $\mathbb{H}$ be a complex Hilbert space, and $\mathbf{B}(\mathbb{H})$ be the set of bounded linear operators on $\mathbb{H}$. Recall that an operator $T \in \mathbf{B}(\mathbb{H})$ is said to be *normal* (respectively, *hyponormal*) if $T^*T = TT^*$ (respectively, $T^*T \geq TT^*$), and *subnormal* if it has a normal extension, i.e., $T = N|_{\mathbb{H}}$, where N is a normal operator on some Hilbert space $\mathbb{K} \supseteq \mathbb{H}$. On the other hand, the Bram-Halmos criterion for subnormality states that an operator $T \in \mathbf{B}(\mathbb{H})$ is subnormal if and only if $\sum_{i,j}(T^i f_j, T^j f_i) \geq 0$ for any finite collections $f_0, \ldots, f_k \in \mathbb{H}$ (see [6]). It is easily seen that this is equivalent to the following positivity matrices

$$(1)\qquad (T^{*j}T^i)_{i,j=0}^k = \begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0, \ \ \forall\, k \geq 1.$$

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Note that the positivity condition (1) for $k = 1$ is equivalent to the hyponormality, and subnormality requires the positivity of (1) for all $k \geq 1$.

Let$[A, B] := AB - BA$ denote the commutator of two operators, A and B. An operator $T \in B(\mathbb{H})$ is said to be *k-hyponormal* whenever the $k \times k$ operator matrix $([T^{*j}, T^i])_{i,j=1}^k \geq 0$. An application of Choleski algorithm shows that the positivity of $k \times k$ matrix $([T^{*j}, T^i])_{i,j=1}^k$ is equivalent to the positivity of the $(k+1) \times (k+1)$ matrix $(T^{*j}T^i)_{i,j=0}^k$ in (1). The Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$.

An operator $T \in \mathbf{B}(\mathbb{H})$ is said to be Embry k-hyponormal whenever the $k \times k$ operator matrix $E_k(T) := ([T^{*i+j}, T^{i+j}])_{i,j=0}^k \geq 0$. It is easily seen that $E_k(T) = D_k(T)^*(T^{*j}T^i)_{i,j=0}^k D_k(T)$, where $D_k(T) = \mathrm{diag}(\mathrm{I}, \mathrm{T}, \ldots, \mathrm{T^k})$. Therefore, k-hyponormality implies Embry k-hyponormality. Moreover, k-hyponormality is equivalent to Embry k-hyponormality for weighted shift operators but it is not true in general (see [15]). Let W_α be a unilateral weighted shift with the weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$. The sequence $\gamma = \{\gamma_n\}_{n=0}^\infty$ is the moment sequence of W_α defined by

$$\begin{cases} \gamma_0 := 1, \\ \gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2. \end{cases}$$

Note that the positivity of $(W_\alpha^{*j} W_\alpha^i)_{i,j=0}^k \geq 0$ is equivalent to the positivity of $([W_\alpha^{*i+j}, W_\alpha^{i+j}])_{i,j=0}^k$. Therefore, k-hyponormality of a weighted shift equivalent to the positivity of the Hankel moment matrix $M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^k$ for each $n = 0, 1, \ldots$ (see [7]).

Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers. Recall that the sequence γ is said to be *positive definite* (PD) if for all finite sequences $z_0, z_1, \ldots, z_k \in \mathbb{C}$,

$$\sum_{i,j=0}^k \gamma_{ij} z_i \bar{z_j} \geq 0, \tag{2}$$

i.e., the $(k+1) \times (k+1)$ Hankel matrix $(\gamma_{i+j})_{i,j=0}^k$ is positive semidefinite for any $k \geq 1$. Note that a sequence of real numbers $\gamma = \{\gamma_n\}_{n=0}^\infty$ is positive definite if (2) holds for all finite real sequences $z_0, z_1, \ldots, z_k \in \mathbb{R}$. Also, recall that a sequence of real numbers $\{\gamma_n\}_{n=0}^\infty$ is said to be a *Stieltjes sequence* if there exists a finite Borel measure μ on $\mathbb{R}_+$ called a representing measure of $\{\gamma_n\}_{n=0}^\infty$, such that

$$\gamma_n = \int_{\mathbb{R}_+} x^n d\mu(x), \qquad n = 0, 1, \ldots.$$

By Stieltjes' Theorem, a sequence $\{\gamma_n\}_{n=0}^\infty \subset \mathbb{R}$ is a Stieltjes moment sequence if and only if the sequences $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are PD (see [5]). We also recall that a weighted shift W_α is subnormal if and only if it has a Berger measure, meaning a probability measure μ is supported on $\left[0, \|W_\alpha\|^2\right]$ such that for each $n \geq 0$, $\gamma_n = \int_0^{\|W_\alpha\|^2} t^n d\mu(t)$ (see [6]). It follows that a weighted shift W_α is subnormal if and only if the moment sequence $\{\gamma_n\}_{n=0}^\infty$ of W_α is a Stieltjes moment sequence. By using weighted shifts, an interesting characterization of subnormality that can be adapted to the context of not necessarily injective operators and it is stated that $T \in \mathbf{B}(\mathbb{H})$ is subnormal if and only if the sequence $\{\|T^n h\|\}_{n=0}^\infty$ is a positive definite sequence for every $h \in \mathbb{H}$. This is equivalent to the fact that the sequence $\{\|T^n h\|\}_{n=0}^\infty$ is a Stieltjes moment for every $h \in \mathbb{H}$ (see [13]). Moreover, Agler's characterization is another approach to subnormality based on the notion of n-contractivity. For $n \geq 1$, an operator $T \in \mathbf{B}(\mathbb{H})$ is said to be *n-contractive* if $A_n(T) := \sum_{i=0}^n (-1)^i \binom{n}{i} T^{*i} T^i \geq 0$. Agler's characterization (see [1]) states that a contraction operator $T \in \mathbf{B}(\mathbb{H})$ is subnormal if and only if T is n-contractive for all positive integers n. It is well known that for a weighted shift W_α it suffices to test this condition on basis vectors and that a weighted shift is n-contractive if and only if $\sum_{i=0}^n (-1)^i \binom{n}{i} \gamma_{k+i} \geq 0$, $k = 0, 1, \ldots$, where $\{\gamma_k\}_{k=0}^\infty$ is the moment of W_α. This relates to the monotone theory that we will mention in Section 2.1.

In Section 2.2, we briefly review some basic properties of infinitely divisible matrices that raise up for moment infinitely divisible weighted shifts in Section 2.4. Section 2.3 presents some important properties of CPD matrices and their proofs. Also Section 2.3 contains a brief summary of CPD operators. Then we use some properties of CPD matrices and CPD operators to establish a relation between subnormality and positive definiteness of the moment sequence for a contractive weighted shift that will be given in Section 3.

Lemma 1.1. *Let W_α be a weighted shift with positive weights, $\gamma = \{\gamma_n\}_{n=0}^\infty$ be the moment sequence of W_α. Assume that W_α is contractive, then the following conditions are equivalent.*

(i) *W_α is subnormal.*
(ii) *$\{\gamma_{n+j}\}_{n=0}^\infty$ is PD for any $j \geq 0$.*
(iii) *$\{\gamma_n\}_{n=0}^\infty$ is PD.*

We are now ready to state the main results of the present paper. We show that the above equivalence still holds without the contractivity condition.

Theorem 1.2. *Let W_α be a weighted shift with positive weights, $\{\gamma_n\}_{n=0}^\infty$ be the moment sequence of W_α. Then the following statements are equivalent.*

(i) *W_α is subnormal.*
(ii) *$\{\gamma_{n+j}\}_{n=0}^\infty$ is PD for any $j \geq 0$.*
(iii) *$\{\gamma_n\}_{n=0}^\infty$ is PD.*

Remark 1.3. As mentioned, a weighted shift W_α is subnormal if and only if the moment sequence $\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment, equivalently, the sequences $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are PD (by Stieltjes' Theorem). Theorem 1.2 here gives a weaker condition for the subnormality of a weighted shift W_α. That is, we only require the sequence $\{\gamma_n\}_{n=0}^\infty$ is PD to get subnormality of W_α.

As an immediate consequence of Theorem 1.2, we obtain a result including a new sufficient condition for moment infinitely divisibility of any weighted shift (without the contractivity condition). Recall that a weighted shift W_α with positive weight sequence $\{\alpha_n\}_{n=0}^\infty$ is called *moment infinitely divisible* (MID) if for all $p > 0$, Schur power $W_\alpha^{\circ p}$ is subnormal. We define a sequence of real numbers $\gamma = \{\gamma_n\}_{n=0}^\infty$ is *infinitely divisible* (ID) if for any $p > 0$, the sequence $\gamma^p = \{\gamma_n^p\}_{n=0}^\infty$ is positive definite. It is known that W_α is MID if and only if $\{\gamma_n\}_{n=0}^\infty$ and $\{\gamma_{n+1}\}_{n=0}^\infty$ are ID (see [4]) By Theorem 1.2, we also only require the sequence $\{\gamma_n\}_{n=0}^\infty$ to be ID in order for W_α to be MID. This is stated as follows.

Corollary 1.4. *Let W_α be a weighted shift with positive weight and $\gamma = \{\gamma_n\}_{n=0}^\infty$ be the moment sequence of W_α. The following statements are equivalent.*

(i) *W_α is MID.*
(ii) *$\{\gamma_{n+j}\}_{n=0}^\infty$ is ID for every $j = 0, 1, \ldots$.*
(iii) *$\{\gamma_n\}_{n=0}^\infty$ is ID.*

In Section 4, we consider weighted shifts W_α having the property of $(k, 0)$-CPD; that is, the moment matrix $M_\gamma(n, k) := (\gamma_{n+i+j})_{i,j=0}^k$ is CPD. Also, there are some examples and properties of $(k, 0)$-CPD weighted shifts.

2. Notation and preliminaries

2.1. Monotone theory

In this section, we briefly recall the monotone (alternating) theory. We first begin with the monotone theory of function.

Definition. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be *completely monotone* (respectively, *completely alternating*) if its derivatives alternate in sign, i.e., $(-1)^k f^{(k)} \geq 0$ (respectively, $(-1)^k f^{(k)} \leq 0$) for all $k = 0, 1, \ldots$.

Note that a function f is completely monotone if and only if $-f$ is completely alternating. Bernstein's theorem states that the function f is completely monotone if and only if $f = \mathcal{L}(\mu)$ for some positive measure μ, where $\mathcal{L}$ denotes the Laplace transform. The following lemma is quite useful in checking some completely monotone functions.

Lemma 2.1 ([8]). (i) *If f_1, f_2 are completely monotone functions and c is a any positive number, then $cf_1, f_1 + f_2$, and $f_1 f_2$ are completely monotone as well.*
(ii) *If $f_1, \ldots, f_n$ are completely monotone then so is any convex combination $\sum_{j=1}^n a_j f_j$.*
(iii) *If f is completely monotone and g is a Bernstein function, then $f \circ g$ is completely monotone. (Note that nonnegative functions whose derivative is completely monotone are called Bernstein functions).*

(iv) *If* $g : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ *has all its derivatives non-negative on* $\mathbb{R}_+$, $g(1) = 1$, *and* f *is completely monotone, then the composition* $g \circ f$ *is completely monotone.*

(v) *If* $n, m \in \mathbb{N}$, *and* f *is completely monotone, then so is* g *defined by* $g(x) = \frac{f(nx+m)}{f(m)}$.

There is a well known connection between completely monotone and completely alternating functions, which can be stated as follows.

Lemma 2.2 ([5, Proposition 6.10]). *The function* $f : \mathbb{R}_+ \to \mathbb{R}_-$ *is completely alternating if and only if the function* $g_t : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $g_t(x) := e^{-tf(x)}$ *is completely monotone for all* $t > 0$.

For each $n \in \mathbb{N}$, a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *n-monotone* if $(-1)^k f^{(k)} \geq 0$ for $k = 0, 1, \ldots, n$. Similarly, a function $f : \mathbb{R}_+ \to \mathbb{R}_-$ is said to be *n-alternating*, $n \in \mathbb{N}$ if $(-1)^k f^{(k)} \leq 0$ for $k = 0, 1, \ldots, n$.

Remark 2.3. (i) A function f is $1-$monotone (respectively, $1-$alternating) which means f is a nonnegative decreasing function (respectively, nonpositive increasing function).

(ii) A function f is completely monotone (respectively, completely alternating) if and only if f is n-monotone (respectively, n-alternating) for any $n \in \mathbb{N}$.

We next discuss the monotone theory of sequences. We first introduce the *forward difference operator* ∇. Given a sequence $\varphi = \{a_k\}_{k=0}^{\infty}$,

$$\begin{aligned} \varphi :&\mathbb{N} \to \mathbb{R} \\ &k \mapsto \varphi(k) = a_k, \end{aligned}$$

the forward difference operator ∇ acting on φ is defined by

$$\begin{aligned} \nabla\varphi :&\mathbb{N} \to \mathbb{R} \\ &k \mapsto (\nabla\varphi)(k), \end{aligned}$$

where $(\nabla\varphi)(k) = \varphi(k) - \varphi(k+1) = a_k - a_{k+1}$. For each $n \in \mathbb{N}$, the iterated forward difference operator ∇^n is defined by

$$\begin{aligned} \nabla^0\varphi &:= \varphi \\ \nabla^n\varphi &:= \nabla(\nabla^{n-1}\varphi). \end{aligned}$$

For instance,

$$\begin{aligned} (\nabla\varphi)(k) &= \varphi(k) - \varphi(k+1), \\ (\nabla^2\varphi)(k) &= (\nabla\varphi)(k) - (\nabla\varphi)(k+1) = \varphi(k) - 2\varphi(k+1) + \varphi(k+2), \\ &\vdots \\ (\nabla^n\varphi)(k) &= \sum_{i=1}^{n}(-1)^i \binom{n}{i}\varphi(k+i), \end{aligned}$$

and so on.

Definition (see [4]). (i) A sequence φ is said to be *n-monotone* (respectively, *n-alternating*) if $(\nabla^n\varphi)(k) \geq 0$ (respectively, $(\nabla^n\varphi)(k) \leq 0$) for all $k = 0, 1, \ldots$.

(ii) A sequence φ is *n-hypermonotone* (respectively, *n-hyperalternating*) if it is j-monotone (respectively, j-alternating) for all $j = 1, \ldots, n$.

(iii) A sequence φ is said to be *completely monotone* (respectively, *completely alternating*) if it is n-monotone (respectively, n-alternating) for all $n = 1, 2, \ldots$.

Definition (see [4]). Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers.

(i) $\{a_k\}_{k=0}^{\infty}$ is *n-log monotone* (respectively, *n-log alternating*) if $\{\log a_k\}_{k=0}^{\infty}$ is n-monotone (respectively, n-alternating).

(ii) $\{a_k\}_{k=0}^{\infty}$ is *n-log hypermonotone* (respectively, *n-log hyperalternating*) if $\{\log a_k\}_{k=0}^{\infty}$ is n-hypermonotone (respectively, n-hyperalternating).

(iii) $\{a_k\}_{k=0}^{\infty}$ is *completely log monotone* (respectively, *completely log alternating*) if $\{\log a_k\}_{k=0}^{\infty}$ is completely monotone (respectively, completely alternating).

Note that $\{a_k\}_{k=0}^{\infty}$ is n-monotone (respectively, n-hypermonotone, n-log monotone,...) if and only if $\{-a_k\}_{k=0}^{\infty}$ is a n-alternating (respectively, n-hyperalternating, n-log alternating,...).

Definition. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence. We say that it is interpolated by $f : \mathbb{R}_+ \to \mathbb{R}$ if $a_k = f(k)$ for all $k = 0, 1, \ldots$.

Lemma 2.4. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function and $\{a_n\}_{n=0}^{\infty}$ be a sequence interpolated by f.*

(i) *If f is n-monotone then $\{a_n\}_{n=0}^{\infty}$ is n-hypermonotone, so $\{a_n\}_{n=0}^{\infty}$ is n-monotone.*

(ii) *If f is n-alternating then $\{a_n\}_{n=0}^{\infty}$ is n-hyperalternating, so $\{a_n\}_{n=0}^{\infty}$ is n-alternating.*

(iii) *If f is completely monotone (respectively, completely alternating) then $\{a_n\}_{n=0}^{\infty}$ is completely monotone (respectively, completely alternating).*

(iv) *If f is n-log monotone then $\{a_n\}_{n=0}^{\infty}$ is n-log hypermonotone (so is n-log monotone).*

(v) *If f is n-log alternating then $\{a_n\}_{n=0}^{\infty}$ is n-log hyperalternating (so is n-log alternating).*

(vi) *If f is completely log monotone (respectively, completely log alternating) then $\{a_n\}_{n=0}^{\infty}$ is completely log monotone (respectively, completely log alternating).*

Proof. We prove only the first assertion, as the other proofs are similar. Assume that f is n-monotone, i.e., $(-1)^k f^{(k)} \geq 0$ for any $k = 0, 1, 2, \ldots, n$. Since $f'(x) \leq 0$ $\forall x \in \mathbb{R}_+$, we have that $f(x) - f(x+1) \geq 0$, $\forall\, x \in \mathbb{R}_+$. Therefore we can construct a positive function $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g_1(x) = f(x) - f(x+1) \geq 0, \forall\, x \in \mathbb{R}_+.$$

Moreover, since $f''(x) \geq 0\ \forall x \in \mathbb{R}_+$, so

$${g_1}'(x) = f'(x) - f'(x+1) \leq 0, \forall\, x \in \mathbb{R}_+.$$

This means g_1 is decreasing on $\mathbb{R}_+$, and so $g_1(x) - g_1(x+1) \geq 0, \forall\, x \in \mathbb{R}_+$. Similarly, we get a positive function $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g_2(x) = g_1(x) - g_1(x+1) \geq 0, \forall\, x \in \mathbb{R}_+.$$

Repeating this process, we get n positive functions $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$g_j(x) = g_{j-1}(x) - g_{j-1}(x+1) \geq 0, \forall\, x \in \mathbb{R}^+,\ j = 1, \dots, n.$$

For each $k \geq 0$, we get

$$\begin{aligned} \nabla a_k &= a_k - a_{k+1} = f(k) - f(k+1) = g_1(k) \geq 0, \\ \nabla^2 a_k &= \nabla a_k - \nabla a_{k+1} = g_1(k) - g_1(k+1) = g_2(k) \geq 0, \\ &\vdots \\ \nabla^n a_k &= \nabla^{n-1} a_k - \nabla^{n-1} a_{k+1} = g_{n-1}(k) - g_{n-1}(k+1) = g_n(k) \geq 0, \end{aligned}$$

so that $\{a_k\}_{k=0}^{\infty}$ is n-hypermonotone. This completes the proof. □

We observe that if a sequence $\{a_k\}_{k=0}^{\infty}$ is n-hypermonotone then the sequence $\{a_k\}_{k=0}^{\infty}$ is n-monotone, but the converse is not true as we see in the following example.

Example 2.5. The Dirichlet shift $D = \text{shift}\left\{\sqrt{\frac{n+2}{n+1}}\right\}_{n=0}^{\infty}$ has the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$, where $\gamma_0 = 1$ and $\gamma_n = \frac{2}{1}\frac{3}{2}\cdots\frac{n+1}{n} = n+1$. Since

$$\nabla^2 \gamma_k = \gamma_k - 2\gamma_{k+1} + \gamma_{k+2} = 0,$$

and

$$\nabla \gamma_k = \gamma_k - \gamma_{k+1} = -1 < 0,$$

it is clear that this moment sequence γ is 2-monotone but not 1-monotone.

By Lemma 2.4, we see that all properties of monotone functions theory still remain valid for monotone sequences theory. As a consequence of Lemma 2.4, we get the following result.

Lemma 2.6. *The sequence $\{a_k\}_{k=0}^{\infty}$ is completely alternating if and only if the sequence $\{e^{ta_k}\}_{k=0}^{\infty}$ is completely monotone for all $t > 0$.*

Lemma 2.7 ([4, Proposition 2.4])**.** *Suppose $n \in \mathbb{N}$ and $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers, and in addition, $\lim_{k\to\infty} a_k$ exists. Then, if $\{a_k\}_{k=0}^{\infty}$ is n-monotone* (*respectively, n-alternating*), *it is n-hypermonotone* (*respectively, n-hyperalternating*)*. Moreover, in addition assume that $\lim_{k\to\infty} a_k \neq 0$ then, if*

$\{a_k\}_{k=0}^{\infty}$ *is n-log monotone (respectively, n-log alternating), it is n-log hypermonotone (respectively, n-log hyperalternating).*

2.2. Infinitely divisible (ID) matrices

Let $A = (a_{ij})_{i,j=1}^{k}$ and $B = (b_{ij})_{i,j=1}^{k}$ be two $k \times k$ matrices. Recall that the Schur product (or the Hadamard product) of A and B is the matrix $A \circ B = (a_{ij}b_{ij})_{i,j=1}^{k}$. The following theorem is the most interesting theorem about the Schur product.

Theorem 2.8 (see [16]). *If $A = (a_{ij})_{i,j=1}^{k}$ and $B = (b_{ij})_{i,j=1}^{k}$ are two $k \times k$ positive semidefinite matrices then the Schur product matrix $A \circ B = (a_{ij}b_{ij})_{i,j=1}^{k}$ is positive semidefinite.*

Remark 2.9. We can easily see that if $A = (a_{ij})_{i,j=1}^{k}$ is a $k \times k$ positive semidefinite matrix, then p-Schur power $A^{\circ p} = (a_{ij}^{p})_{i,j=1}^{k}$ is positive semidefinite for each nonnegative integer $p \in \mathbb{Z}$. However this is no longer true for positive real numbers $p \in \mathbb{R}^{+}$. For example, we consider the Hermitian 3×3 matrix A defined by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The p-Schur power matrix $A^{\circ p}$ is

$$A^{\circ p} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2^p & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that all eigenvalues of Hermitian matrix A are $\{0, 1, 3\}$, so $A \geq 0$. Also, a simple computation gives eigenvalues of p-Schur power matrix $A^{\circ p}$,

$$\{1, 2^p + 1 - \sqrt{(2^p - 1)^2 + 8}, 2^p + 1 + \sqrt{(2^p - 1)^2 + 8}\}.$$

For the eigenvalue $\lambda = 2^p + 1 - \sqrt{(2^p - 1)^2 + 8}$, it is obvious that $\lambda \geq 0$ if and only if $p \geq 1$. Therefore p-Schur power $A^{\circ p}$ is not positive semidefnite if $0 < p < 1$.

Theorem 2.10 ([10]). *If $A = (a_{ij})_{i,j=1}^{k}$ is a $k \times k$ positive semidefinite matrix with $a_{ij} \geq 0$ for all i and j, then the p-Schur power matrix $A^{\circ p}$ is positive semidefinite for each real numbers $p \geq k - 2$. Moreover, if $p < k - 2$, we can construct a positive semidefinite matrix A for which p-Schur power matrix $A^{\circ p}$ is not positive semidefinite.*

Definition. Suppose that $A = (a_{ij})_{i,j=1}^{k}$ is a $k \times k$ positive semidefinite matrix and $a_{ij} \geq 0$ for all i and j. We say that A is *infinitely divisible* if the p-Schur power matrix $A^{\circ p}$ is positive semidefinite for every nonnegative p.

Remark 2.11. If the $\frac{1}{m}$-Schur power matrix $A^{\circ \frac{1}{m}} \geq 0$ for all $m = 1, 2, \ldots$. then p-Schur power $A^{\circ p} \geq 0$ for all positive rational numbers r. Therefore, a $k \times k$

positive semidefinite matrix $A = (a_{ij})_{i,j=1}^k$ with entries $a_{ij} \geq 0$ for all i and j is infinitely divisible if and only if for each positive integer m, there exists a positive semidefinite matrix B such that $A = B^{\circ m}$.

Lemma 2.12. *Every* 2×2 *positive semidefinite matrix with nonnegative entries is infinitely divisible.*

Proof. Assume that A is a 2×2 positive semidefinite matrix with nonnegative entries,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Note that $A \geq 0$ if and only if $a, c \geq 0$ and $\det A = ac - b^2 \geq 0$. For any $p > 0$, the p-Schur power matrix $A^{\circ p}$ is

$$A^{\circ p} = \begin{pmatrix} a^p & b^p \\ b^p & c^p \end{pmatrix}.$$

Since $a, c \geq 0$ and $\det A = ac - b^2 \geq 0$, so a^p and c^p are nonnegative numbers and $\det A^{\circ p} = (ac)^p - b^{2p} \geq 0$ for any $p > 0$. That completes the proof. □

2.3. Conditionally positive definiteness (CPD)

The class of positive semidefinite matrices is important and is also well studied. Now we consider a class of Hermitian matrices with exactly one eigenvalue of one sign and the remaining eigenvalues of the other sign.

Definition. A Hermitian $k\times k$ complex matrix $A = (a_{ij})_{i,j=1}^k$ is said to be *conditionally positive definite*, abbreviated CPD (respectively, *conditionally negative definite*, abbreviated CND) if $v^*Av (= \langle Av, v\rangle) = \sum a_{ij} v_i \bar{v_j} \geq 0$ (respectively, ≤ 0) for any $v = (v_i)_{i=1}^k \in \mathbb{C}^k$ such that $\sum v_i = 0$. An infinite (scalar) matrix is CPD (respectively, CND) if all of its principal minors of finite size are CPD (respectively, CND).

Remark 2.13.
(i) A Hermitian $k \times k$ real matrix $A = (a_{ij})_{i,j=1}^k$ is CPD (respectively, CND) if and only if it satisfies the CPD positivity condition (respectively, CND negativity condition) for vectors with real coordinates adding up to zero.
(ii) $A \geq 0$ (respectively, CPD) if and only if $-A \leq 0$ (respectively, CND).

Strictly speaking, we should use the term "semidefinite" instead of "definite" in the definition above, but we continue to use "definite" for convenience. It is obvious that any positive (respectively, negative) semidefinite matrix is CPD (respectively, CND). Note, however, that the converse of this statement need not necessarily hold. For example, we consider the following matrix on $\mathbb{C}^2$

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Clearly, A is not positive semidefinite since $\det(A) = -1 < 0$. However, A is conditionally positive definite. Indeed, for $x = (x_1, x_2) \in \mathbb{C}^2$ such that $x_1 + x_2 = 0$. We put $x_1 = t$ where $t \in \mathbb{C}$, thus $x_2 = -t$. Then

$$Ax = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} = x$$

and so,

$$\langle Ax, x \rangle = \langle x, x \rangle = \|x\|^2 \geq 0.$$

Remark 2.14. Observe that positive definite matrices are preserved under the unitary equivalence. However, this is no longer true for CPD matrices. For example, we consider

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} -\frac{24}{25} & \frac{7}{25} \\ \frac{7}{25} & \frac{24}{25} \end{pmatrix}.$$

Observe that A and B are unitary equivalent under the unitary matrix C, where

$$C = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix},$$

and hence A is CPD. However, B is not CPD. We can use the following lemma to easily prove it.

Lemma 2.15 ([11, Excerise 8, p. 457])**.** *Let $A = (a_{ij})_{i,j=1}^k$ be a Hermitian $k \times k$ complex matrix. Then A is CPD if and only if $B := \big(a_{i,j} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1}\big)_{i,j=1}^{k-1} \geq 0$.*

Proof. For $x = (x_i)_{i=1}^k \in \mathbb{C}^k$ such that $x_1 + x_2 + \cdots + x_n = 0$, we put

$$\begin{aligned} x_1 :=& t_1, \\ x_j :=& -t_{j-1} + t_j, \forall j \in \{2, \ldots, k-1\} \\ x_k :=& -t_{k-1}, \end{aligned}$$

where $t_1, t_2, ..., t_{k-1} \in \mathbb{C}$. Then the vector x can be written as follows

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} t_2 + \cdots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix} t_{k-1}.$$

Observe that

$$Ax = \begin{pmatrix} a_{11} - a_{12} \\ a_{21} - a_{22} \\ a_{31} - a_{32} \\ \vdots \\ a_{k-1,1} - a_{k-1,2} \\ a_{k1} - a_{k2} \end{pmatrix} t_1 + \begin{pmatrix} a_{12} - a_{13} \\ a_{22} - a_{23} \\ a_{32} - a_{33} \\ \vdots \\ a_{k-1,2} - a_{k-1,3} \\ a_{k2} - a_{k3} \end{pmatrix} t_2 + \dots$$

$$+ \begin{pmatrix} a_{1,k-1} - a_{1,k} \\ a_{2,k-1} - a_{2,k} \\ a_{3,k-1} - a_{3,k} \\ \vdots \\ a_{k-1,k-1} - a_{k-1,k} \\ a_{k,k-1} - a_{k,k} \end{pmatrix} t_{k-1}.$$

A simple computation gives

$$\langle Ax, x\rangle = \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(a_{i,j} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1})\langle t_j, t_i\rangle.$$

Let B be a $(k-1)\times(k-1)$ matrix defined by

$$B := \big(a_{i,j} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1}\big)_{i,j=1}^{k-1}.$$

Now write $t = (t_i)_{i=1}^{k-1}$, then

$$Bt = \begin{pmatrix} \sum_{j=1}^{k}(a_{1,j} - a_{1,j+1} - a_{2,j} + a_{2,j+1})t_j \\ \sum_{j=1}^{k}(a_{2,j} - a_{2,j+1} - a_{3,j} + a_{3,j+1})t_j \\ \vdots \\ \sum_{j=1}^{k}(a_{k-1,j} - a_{k-1,j+1} - a_{k,j} + a_{k,j+1})t_j \end{pmatrix},$$

yielding that

$$\langle Bt, t\rangle = \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(a_{i,j} - a_{i,j+1} - a_{i+1,j} + a_{i+1,j+1})\langle t_j, t_i\rangle.$$

Therefore $\langle Ax, x\rangle = \langle Bt, t\rangle$ for all $t = (t_i)_1^{k-1} \in \mathbb{C}^{k-1}$, where $x = (t_1, -t_1 + t_2, \dots, -t_{k-2} + t_{k-1}, t_{k-1})^{\mathrm{t}} \in \mathbb{C}^k$. This is immediate that $\langle Ax, x\rangle \geq 0$ for all $x = (x_i)_{i=1}^k \in \mathbb{C}^k$ such that $\sum_{i=1}^k x_i = 0$ if and only if $\langle Bt, t\rangle \geq 0$ for all $t = (t_i)_{i=1}^{k-1} \in \mathbb{C}^{k-1}$, and the result is proved. □

Lemma 2.16 ([2, Lemma 4.1.4])**.** (i) *If a Hermitian matrix $A = (a_{ij})_{i,j=1}^k$ is CPD, then A has at most one negative eigenvalue.*

(ii) *If a Hermitian matrix* $A = (a_{ij})_{i,j=1}^k$ *is CND, then* A *has at most one positive eigenvalue.*

Corollary 2.17 ([2, Corollary 4.1.5]). (i) *If a Hermitian* $k \times k$ *nonzero matrix* $A = (a_{ij})_{i,j=1}^k$ *with non-positive entries* ($a_{ij} \le 0$) *is CPD, then* A *has exactly one negative eigenvalue.*

(ii) *If a Hermitian* $k \times k$ *nonzero matrix* $A = (a_{ij})_{i,j=1}^k$ *with non-negative entries* ($a_{ij} \ge 0$) *is CND, then* A *has exactly one positive eigenvalue.*

The following results will answer the question about the necessary and sufficient condition of CPD matrix A that has exactly one negative eigenvalue.

Corollary 2.18. (i) *Let* $A = (a_{ij})_{i,j=1}^k$ *is a nonzero CPD matrix. Then* A *has exactly one negative eigenvalue if and only if* A *is non-positive semidefinite.*

(ii) *Let* $A = (a_{ij})_{i,j=1}^k$ *is a nonzero CND matrix. Then* A *has exactly one positive eigenvalue if and only if* A *is non-negative semidefinite.*

Proof. We prove only the first assertion as the second assertion is implied from the first assertion for $-A$. Note that A is non-positive semidefinite, i.e., there is a vector $v \neq 0$ such that $\langle Av, v\rangle < 0$. First, suppose that A has exactly one negative eigenvalue $\lambda < 0$ and v is an eigenvector of A corresponding to λ. This implies

$$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda \|v\|^2 < 0.$$

Conversely, suppose that $A = (a_{ij})_{i,j=1}^k$ is a non-positive semidefinite matrix, i.e., there is a nonzero vector v such that $\langle Av, v\rangle < 0$. Let $\lambda_1, \dots, \lambda_k$ be eigenvalues of A. Without lost of generally, we may suppose that $\lambda_1, \dots, \lambda_m$ are distinct ($m \le k$). Let V_i be an eigenspace of A corresponding to λ_i, $i = 1, \dots, m$. Therefore we may write $\mathbb{C}^k = \bigoplus_{i=1}^m V_i$. For each $i \in \{1, \dots, m\}$, there exists unique vector $v_i \in V_i$, such that the above nonzero vector $v \in \mathbb{C}^k$ can be written by $v = \sum_{i=1}^m v_i$. This implies

$$\langle Av, v\rangle = \left\langle \sum_{i=1}^m Av_i, \sum_{j=1}^m v_j \right\rangle = \left\langle \sum_{i=1}^m \lambda_i v_i, \sum_{j=1}^m v_j \right\rangle = \sum_{i=1}^m \sum_{j=1}^m \lambda_i \langle v_i, v_j\rangle .$$

Since $\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct, $\langle v_i, v_j\rangle = 0$ for all $i \neq j$. Thus

$$\langle Av, v\rangle = \sum_{i=1}^m \lambda_i \langle v_i, v_i\rangle = \sum_{i=1}^m \lambda_i \|v_i\|^2.$$

Note that $\langle Av, v\rangle < 0$, so there exists $i \in \{1, \dots, m\}$ such that $\lambda_i \|v_i\|^2 < 0$, which says that $\lambda_i < 0$. Since A is CPD, by Lemma 2.16, A has at most one negative eigenvalue. It follows that A has exactly one negative eigenvalue, and the proof is complete. □

Lemma 2.19 ([2, Theorem 4.1.3]). *Let* $A = (a_{ij})_{i,j=1}^k$ *be a symmetric* $k \times k$ *matrix. The following conditions are equivalent.*

(i) *A is CPD.*
(ii) *There exits real number $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the matrix $C = (a_{ij} - \alpha_i - \alpha_j)_{i,j=1}^k$ is positive semidefinite.*
(iii) *For any $p > 0$, the $k \times k$ matrix $e^{pA} = (a^{pa_{ij}})_{i,j=1}^k$ is positive semidefinite.*

Proposition 2.20 ([11, Theorem 6.3.13]). *Let $A = (a_{ij})_{i,j=1}^k$ be a $k \times k$ symmetric matrix with positive entries. Then A is infinitely divisible if and only if $\log A = (\log a_{ij})_{i,j=1}^k$ is CPD.*

Proof. This is immediate from Lemma 2.19 (i) $\Leftrightarrow$ (iii). □

Lemma 2.21 ([2, Theorem 4.4.2]). *Let $A = (a_{ij})_{i,j=1}^k$ be a $k \times k$ CND matrix with postive entries and let $F : (0, \infty) \to \mathbb{R}$ be a completely monotone function. Then the matrix $F(A) = F(a_{ij})_{i,j=1}^k$ is positive semidefinite.*

Corollary 2.22 ([11, Exercise 11, p.458]). *Let $A = (a_{ij})_{i,j=1}^k$ be a $k \times k$ matrix such that $a_{ij} > 0$ for all $i, j = 1, 2, \ldots, k$ and $-A = (-a_{ij})_{i,j=1}^k$ be CPD. Then the $k \times k$ matrix of reciprocals $A^{-1} = ({a_{ij}}^{-1})_{i,j=1}^k$ (not inverse matrix) is infinitely divisible.*

Proof. For each $p > 0$, we choose a function $f_p : (0, \infty) \to \mathbb{R}$ defined by

$$f_p(x) := x^{-p}, \ x > 0.$$

It is easily seen that f_p is completely monotone and $-A$ is CPD, i.e., A is CND with entries $a_{ij} > 0$. By Lemma 2.21, $f(A) = A^{-p} = (a_{ij}^{-p})_{i,j=1}^k$ is positive definite. Therefore, $A^{-p} = (a_{ij}^{-p})_{i,j=1}^k \geq 0$ for all $p > 0$. □

Definition. Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers. The sequence γ is said to be *conditionally positive definite* (CPD) if for all finite sequences $z_0, z_1, \ldots, z_k \in \mathbb{C}$ such that $\sum_{i=0}^k z_i = 0$,

$$\sum_{i,j=0}^{k} \gamma_{ij} z_i \bar{z_j} \geq 0, \tag{3}$$

that is, for any $k \geq 1$, the $(k+1) \times (k+1)$ Hankel matrix $(\gamma_{i+j})_{i,j=0}^k$

$$(\gamma_{i+j})_{i,j=0}^k = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix} \text{ is CPD.}$$

Remark 2.23. (i) If a sequence of real numbers $\{\gamma_n\}_{n=0}^\infty$ is positive definite then it is CPD. Note, however, that the converse of this statement need not necessarily hold.
(ii) A sequence of real numbers $\{\gamma_n\}_{n=0}^\infty$ is CPD if and only if (3) holds for all finite sequences $z_0, z_1, \ldots, z_k \in \mathbb{R}$ such that $\sum_{i=0}^k z_i = 0$.

(iii) It follows from the definition that if $\{\gamma_n\}_{n=0}^{\infty}$ is CPD, then so is the sequence $\{\gamma_{n+2j}\}_{n=0}^{\infty}$ for every $j \in \mathbb{Z}_+$. However, it may happen that $\{\gamma_n\}_{n=0}^{\infty}$ is CPD but $\{\gamma_{n+j}\}_{n=0}^{\infty}$ is not (e.g., $\gamma_n = (-1)^n$).

The following fundamental characterization of conditional positive definiteness in terms of positive definiteness is essentially due to Schoenberg.

Lemma 2.24 ([5, Theorem 3.2.2])**.** *If $\{\gamma_n\}_{n=0}^{\infty}$ is a sequence of real numbers, then the following conditions are equivalent.*

(i) *$\{\gamma_n\}_{n=0}^{\infty}$ is CPD.*
(ii) *$\{e^{p\gamma_n}\}_{n=0}^{\infty}$ is PD for every positive real number p.*

We say that a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of real numbers is of exponential growth if $\limsup_{n\to\infty} |\gamma_n|^{1/n} < \infty$, or equivalently if and only if there exist $\alpha, \theta \in \mathbb{R}_+$ such that

$$|\gamma_n|^{1/n} \leq \alpha\theta^n, \qquad n \in \mathbb{Z}_+.$$

Let W_α be a weighted shift with positive weight. Then the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is a sequence of exponential growth. Indeed,

$$0 < \gamma_n = \alpha_0^2\alpha_1^2 \cdots \alpha_{n-1}^2 \leq \|W_\alpha\|^{2n},$$

thus

$$\gamma_n^{\frac{1}{n}} \leq \|W_\alpha\|^2,$$

so $\limsup_{n\to\infty} |\gamma_n|^{1/n} \leq \|W_\alpha\|^2 < \infty$.

Definition. An operator $T \in \mathbf{B}(\mathbb{H})$ is said to be *conditionally positive definite*, abbreviated CPD, if the sequence $\{\|T^n h\|^2\}_{n=0}^{\infty}$ is CPD for every $h \in \mathbb{H}$.

Remark 2.25. Let W_α be a weighted shift with positive weight, $\{\gamma_n\}_{n=0}^{\infty}$ be the moment sequence of W_α. By [12, Theorem 3.1], the operator W_α is CPD if and only if the moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is a CPD sequence of exponential growth.

2.4. Moment infinitely divisible (MID) weighted shifts

It is known that if we assume that W_α is contractive with the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$, then the following conditions are equivalent.

(i) W_α is subnormal.
(ii) The Hankel matrix $M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^{k} \geq 0$ for every $n \geq 0, k \geq 1$.
(iii) The moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is a Stieltjes moment sequence.
(iv) The sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}\}_{n=0}^{\infty}$ are PD.
(v) The moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is completely monotone.

Note that if $M_\gamma(n,k) \geq 0$ then $M_\gamma(n,k)$ is CPD. Moreover, if $M_\gamma(n,k)$ is CPD, then W_α is $2k$-contractive, i.e. $\{\gamma_n\}_{n=0}^{\infty}$ is $2k$-monotone. Since $\{\gamma_n\}_{n=0}^{\infty}$ is the moment sequence of contractive weighted shift W_α, $\{\gamma_n\}_{n=0}^{\infty}$ is $2k$-hypermonotone and this holds for every $k \geq 1$ ([4, Corolally 2.5]). Therefore the above condition (ii) implies that $\{\gamma_n\}_{n=0}^{\infty}$ is $2k$-hypermonotone for every $k \geq 1$, which means $\{\gamma_n\}_{n=0}^{\infty}$ is completely monotone. We get the following result.

Theorem 2.26. *Assume that W_α is contractive with the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$. The following conditions are equivalent.*

(i) *W_α is subnormal.*
(ii) *$M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^{k} \geq 0$ for every $n \geq 0, k \geq 1$.*
(iii) *The moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is completely monotone.*
(iv) *The sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}\}_{n=0}^{\infty}$ are PD.*
(v) *The sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}\}_{n=0}^{\infty}$ are CPD.*

Definition. A weighted shift W_α with (positive) weight sequence $\{\alpha_n\}_{n=0}^{\infty}$ is *moment infinitely divisible* (MID) if, for every $p > 0$, the p-Schur power $W_\alpha^{\circ p}$ is subnormal.

By Theorem 2.26, if W_α is a contractive weighted shift then, the following conditions are equivalent.

(i) W_α is MID, i.e., for every $p > 0$, the p-Schur power $W_\alpha^{\circ p}$ is subnormal.
(ii) $M_\gamma(n,k)$ is ID for any $n \geq 0$, $k \geq 1$, i.e., for every $p > 0$, the p-Schur power $M_\gamma^p(n,k) \geq 0$ for any $n \geq 0$, $k \geq 1$.
(iii) For any $p > 0$, the sequence $\gamma^p = \{\gamma_n^p\}_{n=0}^{\infty}$ is complete monotone.
(iv) For any $p > 0$, the sequences $\{\gamma_n^p\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}^p\}_{n=0}^{\infty}$ are PD.
(v) The sequences $\{\gamma_n^p\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}^p\}_{n=0}^{\infty}$ are CPD.

Since $M_\gamma(n,k)$ is a symmetric matrix with positive entries, by Proposition 2.20, we get

$$M_\gamma(n,k) \text{ is ID} \iff \log M_\gamma(n,k) \text{ is CPD.}$$

Similarly, we also get

$$M_\gamma(0,\infty) \text{ and } M_\gamma(1,\infty) \text{ are ID} \iff \log M_\gamma(0,\infty) \text{ and } \log M_\gamma(1,\infty) \text{ are CPD.}$$

Moreover, by Lemma 2.6 we get the following results.

Theorem 2.27 ([3, Theorem 3.1])**.** *Assume that W_α is contractive with the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$. The following conditions are equivalent.*

(i) *For any $p > 0$, the moment sequence $\gamma^p = \{\gamma_n^p\}_{n=0}^{\infty}$ is complete monotone, i.e., W_α is MID.*
(ii) *The sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is log completely alternating.*
(iii) *The moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is log completely monotone.*

In [4], C. Benhida, R. E. Curto, and G. R. Exner established some conditions that are equivalent to contractive MID weighted shift. This result was extended from the above mentioned theorem.

Theorem 2.28 ([4, Theorem 3.4])**.** *Assume that W_α is contractive with the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$. Then the following statements are equivalent.*

(i) *W_α is MID.*

(ii) $\log M_\gamma(0,\infty)$ *and* $\log M_\gamma(1,\infty)$ *are CPD.*
(iii) $\log M_\gamma(n,k)$ *is CPD for any* $n \geq 0$, $k \geq 1$.
(iv) $\forall p > 0, M_\gamma^p(0,\infty) \geq 0$ *and* $M_\gamma^p(1,\infty) \geq 0$.
(v) $\forall p > 0$, $M_\gamma^p(0,\infty)$ *and* $M_\gamma^p(1,\infty)$ *are CPD.*
(vi) *The moment sequence* $\gamma = \{\gamma_n\}_{n=0}^\infty$ *is log completely monotone.*
(vii) *The sequence* $\alpha = \{\alpha_n\}_{n=0}^\infty$ *is log completely alternating.*

In view of the results in Theorem 2.28, we plan to extend the notion of moment infinite divisibility to arbitrary operators $T \in \mathbf{B}(\mathbb{H})$.

Question 2.29. (see [4, Problem 3.5]) *How to extend the notion of moment infinite divisibility for weighted shift to arbitrary operators on Hilbert space?*

There are some ideas to extend the notion of moment infinite divisibility for weighted shifts to arbitrary operators on Hilbert space. In [4, Problem 3.5], one can try to do so by using the polar decomposition factor. That is we consider the subnormality of $V|T|^p$ for all $p > 0$ as the proper analog of moment infinite divisibility. The polar decomposition factor of a weighted shift W_α is $W_\alpha = UD_\alpha$, where U is the unilateral shift and $D_\alpha = \mathrm{diag}\{\alpha_\mathrm{n}\}$. Also, for any $p > 0$, p-Schur power $W_\alpha^{\circ p}$ has the polar decomposition factor $W_\alpha^{\circ p} = UD_\alpha^p$. However, finding the polar decomposition factor of an arbitrary operator T is not easy. In Section 3, we define infinitely divisible sequences. Should we consider the infinite divisibility of the sequence $\{\|T^n h\|^2\}_{n=0}^\infty$ for any $h \in \mathbb{H}$ as the proper analog of moment infinite divisibility for an arbitrary operator $T \in \mathbf{B}(\mathbb{H})$?

3. A bridge between ID moment sequences and MID weighted shifts

Definition. Let $\gamma = \{\gamma_n\}_{n=0}^\infty$ be a sequence of real numbers. The sequence γ is said to be *infinitely divisible* (ID) if for any $p > 0$, the sequence $\gamma^p = \{\gamma_n^p\}_{n=0}^\infty$ is positive definite.

Lemma 3.1. *Let* $a = \{a_n\}_{n=0}^\infty$ *and* $b = \{b_n\}_{n=0}^\infty$ *be ID sequences. Then*
(i) *The sequence* $\{\lambda a_n\}_{n=0}^\infty$ *is ID for any* $\lambda > 0$.
(ii) *The sequence* $\{a_n^r\}_{n=0}^\infty$ *is ID for any* $r > 0$.
(iii) *The Schur product sequence* $\{a_n b_n\}_{n=0}^\infty$ *is also ID.*
(iv) *If* $a_n \neq 1$ *for any* $n \geq 0$, *the sequence* $\frac{1}{1-a} = \left\{\frac{1}{1-a_n}\right\}_{n=0}^\infty$ *is ID.*

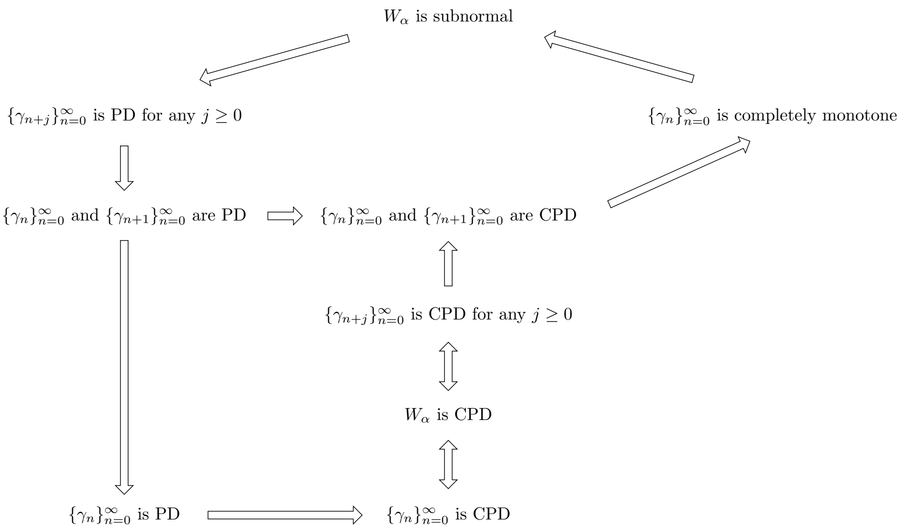
Proof. The statements (i), (ii), and (iii) are trivial. We need only to prove the statement (iv). Note that for any positive real number $r > 0$,

$$\frac{1}{(1-a)^r} = \sum_{m=0}^\infty \alpha_m a^m,$$

where $\alpha_0 = 1$ and $\alpha_m = r(r+1)\dots(r+m+1)/m!$.

Since $a = \{a_n\}_{n=0}^{\infty}$ is ID, this implies $a^m = \{a_n^m\}_{n=0}^{\infty}$ is ID for any $m \geq 0$. Note that $\alpha_m \geq 0$ for any m, then $\alpha_m a^m$ is ID. Therefore $\sum_{m=0}^{M} \alpha_m a^m$ is ID for any $M \geq 1$. Taking $M \to \infty$, we get the result. $\square$

We now consider a weighted shift W_α with the moment sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$. By Theorem 2.26, we know that the subnormality of a contractive weighted shift W_α is equivalent to the CPD of sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\gamma_{n+1}\}_{n=0}^{\infty}$. Moreover, W_α is CPD if and only if $\{\gamma_n\}_{n=0}^{\infty}$ is CPD (see Remark 2.25). Note that $W_\alpha^n e_0 = \gamma_n$ for every $n \geq 0$. This situation is well-captured by the following diagram.



Therefore we get the following result, which we restate from Section 1 for the reader's convenience.

Lemma 3.2. *Let W_α be a weighted shift with positive weight, $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be the moment sequence of W_α. Assume that W_α is contractive, then the following conditions are equivalent.*

(i) *W_α is subnormal.*
(ii) *$\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is PD.*

Note that a weighted shift W_α is subnormal if and only if $\{\gamma_{n+j}\}_{n=0}^{\infty}$ be PD for any $j \geq 0$. Observe that in the following theorem, we do not assume the contractivity condition.

Theorem 3.3. *Let W_α be a weighted shift with positive weight and $\{\gamma_n\}_{n=0}^{\infty}$ be the moment sequence of W_α. The statements are equivalent.*

(i) *W_α is subnormal.*

(ii) $\{\gamma_{n+j}\}_{n=0}^{\infty}$ *is PD for any* $j \geq 0$.
(iii) $\{\gamma_n\}_{n=0}^{\infty}$ *is PD.*

Proof. It is trivially that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). We need only to prove (iii) $\Longrightarrow$ (i). Assume that $\{\gamma_n\}_{n=0}^{\infty}$ is PD. Let $m := \|W_\alpha\| > 0$, and $W_\alpha^{'}$ be the weighted shift with the weight sequence $\alpha^{'} = \{\alpha_n^{'}\}_{n=0}^{\infty}$, where $\alpha_n^{'} = \alpha_n/m$ for each $n \geq 0$. The moment sequence $\gamma^{'} = \{\gamma_n^{'}\}_{n=0}^{\infty}$ of the weighted shift $W_\alpha^{'}$ satisfies

$$\begin{cases} \gamma_0^{'} = 1, \\ \gamma_n^{'} = \alpha_0^{'2} \dots \alpha_{n-1}^{'2} = \dfrac{\gamma_n}{m^n}. \end{cases}$$

For each $k \geq 1$,

$$\begin{aligned}(\gamma_{i+j}')_{i,j=0}^{k} &= \begin{pmatrix} \gamma_0' & \gamma_1' & \cdots & \gamma_k' \\ \gamma_1' & \gamma_2' & \cdots & \gamma_{k+1}' \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_k' & \gamma_{k+1}' & \cdots & \gamma_{2k}' \end{pmatrix} = \begin{pmatrix} 1 & \frac{\gamma_1}{m} & \cdots & \frac{\gamma_k}{m^k} \\ \frac{\gamma_1}{m} & \frac{\gamma_2}{m^2} & \cdots & \frac{\gamma_{k+1}}{m^{k+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\gamma_k}{m^k} & \frac{\gamma_{k+1}}{m^{k+1}} & \cdots & \frac{\gamma_{2k}}{m^{2k}} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix} \circ \begin{pmatrix} 1 & \frac{1}{m} & \cdots & \frac{1}{m^k} \\ \frac{1}{m} & \frac{1}{m^2} & \cdots & \frac{1}{m^{k+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m^k} & \frac{1}{m^{k+1}} & \cdots & \frac{1}{m^{2k}} \end{pmatrix},\end{aligned}$$

where $\circ$ denotes the Schur product. Now write

$$M_k := \begin{pmatrix} 1 & \frac{1}{m} & \cdots & \frac{1}{m^k} \\ \frac{1}{m} & \frac{1}{m^2} & \cdots & \frac{1}{m^{k+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m^k} & \frac{1}{m^{k+1}} & \cdots & \frac{1}{m^{2k}} \end{pmatrix},$$

and for any vector $x = (x_j)_{j=0}^{k} \in \mathbb{C}^{k+1}$, we have that

$$M_k x = \begin{pmatrix} \sum\limits_{j=0}^{k} \frac{1}{m^j} x_j \\ \sum\limits_{j=0}^{k} \frac{1}{m^{j+1}} x_j \\ \vdots \\ \sum\limits_{j=0}^{k} \frac{1}{m^{j+k}} x_j \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=0}^{k} \frac{1}{m^j} x_j \\ \frac{1}{m} \sum\limits_{j=0}^{k} \frac{1}{m^j} x_j \\ \vdots \\ \frac{1}{m^k} \sum\limits_{j=0}^{k} \frac{1}{m^j} x_j \end{pmatrix}.$$

It thus follows that

$$\langle M_k x, x\rangle = \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{x_0} + \frac{1}{m} \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{x_1} + \cdots + \frac{1}{m^k} \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{x_k}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{x_0} + \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{\frac{1}{m} x_1} + \cdots + \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{\frac{1}{m^k} x_k} \\
&= \sum_{j=0}^{k} \frac{1}{m^j} x_j \overline{\sum_{j=0}^{k} \frac{1}{m^j} x_j} \\
&= \left| \sum_{j=0}^{k} \frac{1}{m^j} x_j \right|^2 \geq 0.
\end{aligned}
$$

Hence $M_k \geq 0$ for any $k \geq 1$. Note that $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is PD, i.e.,

$$
(\gamma_{i+j})_{i,j=0}^{k} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix} \text{ is PD,} \qquad \forall\, k = 1, 2, \ldots.
$$

For each $k \geq 1$, the Hankel matrix $(\gamma'_{i+j})_{i,j=0}^{k}$ generated by Schur product between two positive matrices is also positive. By Theorem 2.26, the weighted shift $W_\alpha^{'}$ is subnormal. Since $W_\alpha^{'} = \frac{1}{m} W_\alpha$, the weighted shift W_α is also subnormal, and this completes the proof. □

Corollary 3.4. *Let W_α be a weighted shift with positive weight, $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ be the moment sequence of W_α. The following statements are equivalent.*

(i) *W_α is MID.*
(ii) *$\{\gamma_n\}_{n=0}^{\infty}$ is ID.*
(iii) *$\{\gamma_{n+j}\}_{n=0}^{\infty}$ is ID for every $j = 0, 1, \ldots$.*

Proof. Assume that W_α is MID, i.e., for any $p > 0$, W_{α^p} is subnormal. Note that the moment sequence of W_{α^p} is $\{\gamma_n^p\}_{n=0}^{\infty}$. It thus follows from Theorem 3.3 that $\gamma^p = \{\gamma_n^p\}_{n=0}^{\infty}$ is semipositive definite for any $p > 0$. □

Corollary 3.5. *Let W_α and W_β be MID weighted shifts. Then*

(i) *λW_α is MID for any $\lambda \geq 0$.*
(ii) *The p-Schur power $W_\alpha^{\circ p}$ is MID for any $p \geq 0$.*
(iii) *The Schur product $W_\alpha W_\beta = W_{\alpha\beta}$ is MID.*

4. $(k, 0)$-CPD weighted shifts

We consider weighted shift operators having the property of conditionally positive definite moment matrices. Let $a = \{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers, $k \geq 1$, and $i \geq 0$. We denote that $M_{\nabla^{2m}(a)}(i, k-m)$ is the Hankel matrix of size $k+1$ by $k+1$ given by

$$M_a(i,k) = \begin{pmatrix} a_i & a_{i+1} & \dots & a_{i+k} \\ a_{i+1} & a_{i+2} & \dots & a_{i+k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i+k} & a_{i+k+1} & \dots & a_{i+2k} \end{pmatrix}.$$

Definition (see [4])**.** Let W_α be a weighted shift with nonzero weight sequence $\{\alpha_n\}_{n=0}^\infty$. Then for each k, $0 \le m \le 2k$, W_α is said to be $(k, 2m)$-*CPD* if the moment sequence $\{\gamma_k\}_{k=0}^\infty$ is $(k,2m)$-CPD, i.e., for every $i \ge 0$, the matrix $M_{\nabla^{2m}(\gamma)}(i, k-m)$ is CPD.

For fixed $k \ge 1$, we consider $(k,0)$-CPD weighted shift W_α with $\{\gamma_n\}_{n=0}^\infty$ is the moment sequence, i.e., for every $n \ge 0$, the following Hankel matrix $M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^k$ is CPD.

Example 4.1. For $x > 0$, let T_x be the weighted shift whose weight sequence is given by

$$\sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \dots.$$

Then

(i) T_x is subnormal $\Leftrightarrow 0 < x \le \frac{1}{2}$.
(ii) T_x is k-hyponormal $\Leftrightarrow 0 < x \le \frac{(k+1)^2}{2k(k+2)}$.
In particular, T_x is 2-hyponormal $\Leftrightarrow 0 < x \le \frac{9}{16}$.
(iii) T_x is quadratically hyponormal (weakly 2-hyponormal) $\Leftrightarrow 0 < x \le \frac{2}{3}$.
(iv) T_x is $(2,0)$-CPD if and only if $0 < x \le \frac{4}{7}$.

Proof. The proof of (i), (ii) and (iii) is easily seen in [14, Theorem 4.3.5]. Also, we need only prove (iv). It is sufficient to show that for each $n = 0, 1, \dots$

$$M_\gamma(n,2) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} \text{ is CPD.}$$

Note that

$$\gamma_0 = 1, \gamma_1 = x, \gamma_2 = \frac{2}{3}x, \gamma_3 = \frac{1}{2}x, \gamma_4 = \frac{2}{5}x, \dots, \gamma_n = \frac{2}{n+1}x, \dots.$$

For $n \ge 1$,

$$M_\gamma(n,2) = x \begin{pmatrix} \frac{2}{n+1} & \frac{2}{n+2} & \frac{2}{n+3} \\ \frac{2}{n+2} & \frac{2}{n+3} & \frac{2}{n+4} \\ \frac{2}{n+3} & \frac{2}{n+4} & \frac{2}{n+5} \end{pmatrix} \ge 0,$$

and so $M_\gamma(n,2)$ is CPD. We now need only to check for $n=0$. Indeed,

$$M_\gamma(0,2) = x\begin{pmatrix} \frac{1}{x} & 1 & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \end{pmatrix},$$

and it thus follows from Lemma 2.15 that $M_\gamma(0,2)$ is CPD if and only if the following matrix B is positive semidefinite

$$B = \begin{pmatrix} \frac{1}{x} - \frac{4}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{15} \end{pmatrix}.$$

Note that $B \geq 0 \Leftrightarrow x \leq \frac{4}{7}$. Thus T_x $(k,0)$-CPD if and only if $0 < x \leq \frac{4}{7}$. $\square$

Proposition 4.2. *Let W_α be a unilateral weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Then*

(i) *If W_α is hyponormal then W_α is $(1,0)$-CPD. Moreover, if W_α is paranormal then W_α is $(1,0)$-CPD.*

(ii) *If W_α is $(k+1,0)$-CPD then W_α is $(k,0)$-CPD.*

Proof. (i) Assume that $\{\gamma_n\}_{n=0}^{\infty}$ is the moment sequence of W_α. Then W_α is hyponormal, i.e.,

$$\begin{pmatrix} \gamma_n & \gamma_{n+1} \\ \gamma_{n+1} & \gamma_{n+2} \end{pmatrix} \geq 0 \Leftrightarrow \sqrt{\gamma_n\gamma_{n+2}} \geq \gamma_{n+1}. \tag{4}$$

Since $\frac{\gamma_n+\gamma_{n+2}}{2} \geq \sqrt{\gamma_n\gamma_{n+2}}$, it thus follows from (4) that

$$\gamma_n - 2\gamma_{n+1} + \gamma_{n+2} \geq 0.$$

This implies

$$\begin{pmatrix} \gamma_n & \gamma_{n+1} \\ \gamma_{n+1} & \gamma_{n+2} \end{pmatrix} \text{ is CPD.}$$

By Definition 4, W_α is $(1,0)$-CPD, and this is the desired conclusion.
Next, we assume that W_α is paranormal. Recall that W_α is paranormal if and only if for every $n \geq 0$,

$$\alpha_n^2\alpha_{n+1}^2 - 2\lambda\alpha_n^2 + \lambda^2 \geq 0, \ \ \forall \lambda > 0. \tag{5}$$

Note that W_α is $(1,0)$-CPD if and only if for every $n \geq 0$,

$$\begin{aligned} &\begin{pmatrix} \gamma_n & \gamma_{n+1} \\ \gamma_{n+1} & \gamma_{n+2} \end{pmatrix} \text{ is CPD} \\ &\Leftrightarrow \gamma_n - 2\gamma_{n+1} + \gamma_{n+2} \geq 0 \\ &\Leftrightarrow 1 - 2\alpha_n^2 + \alpha_n^2\alpha_{n+1}^2 \geq 0. \end{aligned}$$

By choosing $\lambda = 1$ in (5), we get W_α is conditionally hyponormal.

(ii) Assume that W_α is $(k+1,0)$-CPD. By Definition 4,

$$M_\gamma(n,k+1) = \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} & \gamma_{n+k+1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} & \gamma_{n+k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} & \gamma_{n+2k+1} \\ \gamma_{n+k+1} & \gamma_{n+k+2} & \cdots & \gamma_{n+2k+1} & \gamma_{n+2k+2} \end{pmatrix} \text{ is CPD.}$$

This implies

$$M_\gamma(n,k) = \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \text{ is CPD.}$$

Hence W_α is $(k,0)$-CPD, and the proof is complete. □

Remark 4.3. The converse of the first statement in Proposition 4.2 need not necessarily hold. For example, W_α is the weighted shift whose weight sequence is given by

$$\frac{1}{2}, \frac{1}{4}, x, x, \dots,$$

where $x \geq \frac{1}{4}$. We see that the above W_α is not hyponormal. But W_α is $(1,0)$-CPD. Indeed, Then $\gamma_0 = 1$, $\gamma_1 = \frac{1}{4}$, $\gamma_2 = \frac{1}{64}$, $\gamma_n = \frac{1}{64}x^{2(n-2)}$ for $n \geq 2$.
For $n \geq 2$,

$$\begin{pmatrix} \gamma_n & \gamma_{n+1} \\ \gamma_{n+1} & \gamma_{n+2} \end{pmatrix} = \frac{1}{64} \begin{pmatrix} x^{2(n-2)} & x^{2(n-1)} \\ x^{2(n-1)} & x^{2n} \end{pmatrix} \geq 0$$

So we only need to check for $n = 0$ and $n = 1$. For $n = 1$, since $x \geq \frac{1}{4}$,

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64}x^2 \end{pmatrix} \geq 0.$$

For $n = 0$, we see that

$$\begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{64} \end{pmatrix} \text{ is CPD.}$$

Hence W_α is $(1,0)$-CPD.

Moreover, G.R. Exner, I.B. Jung, and S.S. Park proved for general operators that k-hyponormality implies $2k$-contractivity (see [9, Theorem 1.2]). Actually, we need only the weakly positivity of $E_k(T) := ([T^{*i+j}, T^{i+j}])_{i,j=0}^k)$ to get $2k$-contractivity. Applying the technique in the proof of [9, Theorem 1.2], we show that if a weighted shift W_α is $(k,0)$-CPD, i.e., the moment matrix $M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^k$ is CPD for any $n \geq 0$ then W_α is $2k$-contractive. Indeed, if $M_\gamma(n,k)$ is CPD, we put λ is the column vector of length $k+1$ with

j-th coordinate $\lambda_j = (-1)^j \binom{k}{j}$, $j = 0, 1, \ldots, k$. Observe that $\sum_{j=0}^{k} \lambda_j = 0$. Since $M_\gamma(n,k)$ is CPD, we get $\langle M_\gamma(n,k)\lambda, \lambda\rangle \geq 0$. By Vandermonde's identity, for each $n \geq 0$ we get

$$\langle M_\gamma(n,k)\lambda, \lambda\rangle = \sum_{l=0}^{2k}(-1)^l \sum_{j=l-k}^{l} \binom{k}{j}\binom{k}{l-j}\gamma_{n+l} = \sum_{l=0}^{2k}(-1)^l \binom{2k}{l}\gamma_{n+l}.$$

Lemma 4.4. *Let W_α be a unilateral weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. If W_α is $(k,0)$-CPD then W_α is $2k$-contractive.*

We observe that if the shift W_α is $2k$-contractive then it also is $2j$-contractive for any $j \leq k$. And this result is a specific case of [4, Proposition 3.7].

Corollary 4.5. *Let W_α be a contractive unilateral weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Then the following conditions are equivalent.*

(i) *W_α is $(k,0)$-CPD for every $k \geq 1$.*
(ii) *W_α is subnormal.*

Proof. Assume that W_α is $(k,0)$-CPD for every $k \geq 1$. By Lemma 4.4, W_α is $2k$-contractive for every k. Since W_α is contractive, W_α is $2k$-hypercontractive for every k, i.e. W_α is j-contractive for every $j \geq 1$. This implies W_α is subnormal. □

Concluding Remarks

Observe that the properties of a weighted shift can be explicitly expressed by its moment sequence. As mentioned in Section 4, we consider $(k,0)$-CPD weighted shifts having the property of conditionally positive definite moment matrices. However, we use the term "k-CPD" instead of "$(k,0)$-CPD" for convenience. Also, we give a definition of "k-CPD sequences". A sequence of real numbers $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ is said to be k-CPD if the Hankel moment matrix $M_\gamma(n,k) := (\gamma_{n+i+j})_{i,j=0}^{k}$ is CPD for any $n \geq 0$. Observe also that a weighted shift is k-CPD if and only if its moment sequence is k-CPD. We can investigate more properties of k-CPD weighted shifts by making some relations among properties of moment sequences, such as k-monotone, k-CPD sequences,. . . .

We also try to extend the notion of the moment infinitely divisibility for weighted shifts to arbitrary contractions on Hilbert space as we mentioned in Question 2.29. Moreover, we can consider MID weighted shifts on quaternionic Hilbert spaces. Some results that are too similar to the case of complex Hilbert spaces may be omitted. We plan to pursue these matters in future research.

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