Commun. Korean Math. Soc. **39** (2024), No. 2, pp. 421–436
https://doi.org/10.4134/CKMS.c230154
pISSN: 1225-1763 / eISSN: 2234-3024

PARALLEL SHRINKING PROJECTION METHOD FOR FIXED POINT AND GENERALIZED EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLD

Hammed Anuoluwapo Abass and Olawale Kazeem Oyewole

Abstract. In this article, we propose a shrinking projection algorithm for solving a finite family of generalized equilibrium problem which is also a fixed point of a nonexpansive mapping in the setting of Hadamard manifolds. Under some mild conditions, we prove that the sequence generated by the proposed algorithm converges to a common solution of a finite family of generalized equilibrium problem and fixed point problem of a nonexpansive mapping. Lastly, we present some numerical examples to illustrate the performance of our iterative method. Our results extends and improve many related results on generalized equilibrium problem from linear spaces to Hadamard manifolds. The result discuss in this article extends and complements many related results in the literature.

1. Introduction

The theory of Equilibrium Problem (in short, EP) finds its applications in many fields of mathematics such as optimization problems, Nash equilibrium problems, complementarity problems, fixed point problems and variational inequality problems. The EP have been studied extensively in finite and infinite dimensional linear spaces (see, for example [7–10, 16, 20] and the references therein). Let C be a nonempty, closed and convex subset of a topological space X and $F : C \times C \to \mathbb{R}$ be a bifunction, a point $x \in C$ is said to be an EP if

$$F(x, y) \geq 0, \ \forall \ y \in C. \tag{1.1}$$

Several iterative methods have been employed for solving EP (1.1) (see [2, 3, 16, 17, 21, 37, 40] and the references therein). In 2008, Moudafi [24] introduced and studied the Generalized Equilibrium Problem (in short, GEP) which is to find $x \in C$ such that

$$F(x, y) + \langle \psi x, y - x\rangle \geq 0, \ \forall \ y \in C, \tag{1.2}$$

Received June 26, 2023; Accepted October 19, 2023.
2020 *Mathematics Subject Classification.* Primary 47H09, 49J25, 65K10, 90C25.
Key words and phrases. Generalized equilibrium problem, shrinking method, Hadamard manifold, monotone operator, Riemannian manifold.

where $F : C \times C \to \mathbb{R}$ is a bifunction and $\psi : C \to X^*$ is a nonlinear mapping and X^* is the dual space of X. The GEP (1.2) is a combination of EP (1.1) and Variational Inequality Problem (in short, VIP) which is to find $x \in C$ such that

$$\langle \psi x, y - x\rangle \geq 0,\ \forall\ y \in C. \tag{1.3}$$

Remark 1.1. If $\psi \equiv 0$, then GEP (1.2) reduces to EP (1.1) and if $F \equiv 0$, then GEP (1.2) reduces to the so-called VIP (1.3).

The GEP is known to finds its applications in sensor networks, robustness to marginal changes, data compression and equilibrium stability, to mention a few (see [4, 6, 19, 36] and the references therein). Recently, many authors (see [6,16,27,32]) extended the concepts and techniques of the theory of equilibrium problems from Euclidean spaces to nonlinear spaces like Hadamard manifolds. An important motivation for studying equilibrium problems in Hadamard manifolds is that some equilibrium problems on Euclidean spaces can not be solve by the classical technique but can be solve on Hadamard manifolds see [15]. Therefore, the extension of the concepts and techniques of the theory of equilibrium to Riemannian manifolds is very important.

In 2012, Colao *et al.* [15] introduced the concept of equilibrium problem where the associated bifunction is monotone and proved the existence of its solution on Hadamard manifolds. Wang *et al.* [39] studied the notion of monotone and accretive vector fields on Riemannian manifolds. Németh [25] generalized some basic concepts of the existence and uniqueness theorems of the classical theory of variational inequalities from Euclidean spaces to Hadamard manifolds. Zhou and Huang [41] studied the notion of the KKM mapping and proved a generalized KKM theorem on the Hadamard manifold. Noor *et al.* [28] introduced an implicit method for solving equilibrium problem on Hadamard manifolds and Noor *et al.* [27] proposed an explicit method for solving equilibrium problem on Hadamard manifolds.

Let C be a nonempty, closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$. Let $F : C \times C \to \mathbb{R}$ be a bifunction and $\psi : C \to T\mathbb{M}$ be a single-valued vector field, the GEP is to find $x \in C$ such that

$$F(x, y) + \langle \psi x, \exp_x^{-1} y\rangle \geq 0,\ \forall\ y \in C, \tag{1.4}$$

where $\exp^{-1}$ is the inverse of the exponential function $\exp : T\mathbb{M} \to \mathbb{M}$ with $T\mathbb{M}$ the tangent bundle of $\mathbb{M}$. Using the idea in GEP (1.2), it is obvious that variational inequality problem defined in Hadamard manifold is to find $x \in C$ such that

$$\langle \psi x, \exp_x^{-1} y\rangle \geq 0,\ \forall\ y \in C. \tag{1.5}$$

We denote by $GEP(F, \psi)$, the solution set of (1.4).

Very recently, Oyewole *et al.* [31] studied the existence of solution of the generalized equilibrium problem on Hadamard manifold using the KKM lemma. They established a convergence result for approximating a solution to the GEP which is also a fixed point of a nonexpansive mapping using the following viscosity iterative method:

$$\begin{cases} y_n = \exp_{x_n}(1-\beta_n)\exp_{x_n}^{-1} Sx_n; \\ x_{n+1} = \exp_{f(x_n)}(1-\alpha_n)\exp_{f(x_n)}^{-1} T_{r_n}^{F,\psi}, \end{cases} \tag{1.6}$$

where S is a nonexpansive mapping, $f : \mathbb{M} \to \mathbb{M}$ is an α-contraction and $T_{r_n}^{F,\psi}$ (defined in Section 2) is the resolvent of $GEP(F, \psi)$.
It is well-known that in infinite-dimensional Hilbert space, the normal Mann's iterative method has only weak convergence, in general, even for nonexpansive mappings. Of course, weak and strong convergence are only distinguisgable in the infinite-dimensional setting. On the other hand, even when we have to solve infinite-dimensional problems, numerical implementation of algorithms are certainly applied to finite-dimensional approximation of the problems. Nevertheless, it is vital to have convergence theory for the infinite-dimensional case, because it guarantees robustness and stability with respect to discretization schemes employed for obtaining finite-dimensional approximations of infinite-dimensional problems (see [35]). In 2000, Solodov and Svaiter [35] proposed a shrinking iterative method which consists of the proximal point iteration with certain (computational simple) projection steps. They employed their iterative method for finding zeroes of maximal monotone operators in an infinite-dimensional Hilbert space. Since then, several authors working in this direction have employed the shrinking iterative methods for solving different optimization problems in linear spaces (see [1, 13, 29, 30, 33, 35] and the references therein). Very recently, Chang *et al.* [12] proposed a shrinking projection method for solving a finite family of quasi-variational inclusion problems in Hadamard manifolds. It was established under some mild conditions that the sequence generated by their iterative method converges strongly to a common solution of a finite family of quasi-variational inclusion problems.

Spurred by the results of Oyewole *et al.* [31], Colao *et al.* [15], Chang *et al.* [12] and some other related results in literature, we propose a shrinking projection iterative method for solving a finite family of generalized equilibrium problem which is also a fixed point of a nonexpansive mapping in Hadamard manifolds. Using our iterative method, we prove that the sequence generated by our iterative algorithm converges to a solution of the aforementioned problems. We display some numerical examples to show the performance of our iterative method.

In summary, the problem discussed in this article is to find $x \in C$ such that

$$Fix(S) \cap \bigcap_{j=1}^{N} GEP(F_j, \psi_j),$$

where $Fix(S) = \{x \in C : x = Sx\}$ is the fixed point set of a nonlinear mapping S.

We highlight our contributions in this article as follows:

(i) Our result generalizes many related results on EP and GEP from linear spaces to Hadamard manifolds (see [2, 21, 26]).
(ii) The problem considered in [32, 33] of the problem proposed in this article when $j = 1$.
(iii) The result discuss in this paper is new and the proof is concise.

2. Preliminaries

Let $\mathbb{M}$ be an m-dimensional manifold, let $x \in \mathbb{M}$ and let $T_x\mathbb{M}$ be the tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$. We denote by $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ the tangent bundle of $\mathbb{M}$. An inner product $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian metric on $\mathbb{M}$ if $\langle\cdot,\cdot\rangle_x : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{M}$. The corresponding norm induced by the inner product $\mathcal{R}_x\langle\cdot,\cdot\rangle$ on $T_x\mathbb{M}$ is denoted by $\|\cdot\|_x$. We will drop the subscript x and adopt $\|\cdot\|$ for the corresponding norm induced by the inner product. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ is called a Riemannian manifold. In what follows, we denote the Riemannian metric $\mathcal{R}\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle$ when no confusion arises. Given a piecewise smooth curve $\gamma : [a,b] \to \mathbb{M}$ joining x to y (that is, $\gamma(a) = x$ and $\gamma(b) = y$), we define the length $l(\gamma)$ of γ by $l(\gamma) := \int_a^b \|\gamma'(t)\|dt$. The Riemannian distance $d(x,y)$ is the minimal length over the set of all such curves joining x to y. The metric topology induced by d coincides with the original topology on $\mathbb{M}$. We denote by ∇ the Levi-Civita connection associated with the Riemannian metric [34].

Let γ be a smooth curve in $\mathbb{M}$. A vector field X along γ is said to be parallel if $\nabla_{\gamma'}X = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' itself is parallel along γ, then we say that γ is a geodesic and $\|\gamma'\|$ is a constant. If $\|\gamma'\| = 1$, then the geodesic γ is said to be normalized. A geodesic joining x to y in $\mathbb{M}$ is called a minimizing geodesic if its length equals $d(x,y)$. A Riemannian manifold $\mathbb{M}$ equipped with a Riemannian distance d is a metric space $(\mathbb{M}, d)$. A Riemannian manifold $\mathbb{M}$ is said to be complete if for all $x \in \mathbb{M}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. The Hopf-Rinow theorem [34] posits that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimizing geodesic. Moreover, if $(\mathbb{M}, d)$ is a complete metric space, then every bounded and closed subset of $\mathbb{M}$ is compact. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x v := \gamma_v(1,x), \ \forall\ v \in T_x\mathbb{M},$$

where $\gamma_v(\cdot,x)$ is the geodesic starting from x with velocity v (that is, $\gamma_v(0,x) = x$ and $\gamma_v'(0,x) = v$). Then, for any t, we have $\exp_x tv = \gamma_v(t,x)$ and $\exp_x \mathbf{0} = \gamma_v(0,x) = x$. Note that the mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for every $x \in \mathbb{M}$. The exponential map $\exp_x$ has an inverse $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. For any $x, y \in \mathbb{M}$, we have $d(x,y) = \|\exp_y^{-1} x\| = \|\exp_x^{-1} y\|$ (see [34] for more details). The parallel transport $P_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ on the tangent bundle

$T\mathbb{M}$ along $\gamma : [a,b] \to \mathbb{R}$ with respect to ∇ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}v = F(\gamma(b)),\ \forall\ a,b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}\mathbb{M},$$

where F is the unique vector field such that $\nabla_{\gamma'(t)}v = \mathbf{0}$ for all $t \in [a,b]$ and $F(\gamma(a)) = v$. If γ is a minimizing geodesic joining x to y, then we write $P_{y,x}$ instead of $P_{\gamma,y,x}$. Note that for every $a,b,r,s \in \mathbb{R}$, we have

$$P_{\gamma(s),\gamma(r)} \circ P_{\gamma(r),\gamma(a)} = P_{\gamma(s),\gamma(a)} \text{ and } P^{-1}_{\gamma(b),\gamma(a)} = P_{\gamma(a),\gamma(b)}.$$

Also, $P_{\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}\mathbb{M}$ to $T_{\gamma(b)}\mathbb{M}$, that is, the parallel transport preserves the inner product

$$\langle P_{\gamma(b),\gamma(a)}(u), P_{\gamma(b),\gamma(a)}(v)\rangle_{\gamma(b)} = \langle u,v\rangle_{\gamma(a)},\ \forall\ u,v \in T_{\gamma(a)}\mathbb{M}. \tag{2.1}$$

Below is an example of a Hadamard space.

Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{M} = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold equipped with the inner product $\langle x,y\rangle = xy,\ \forall\ x,y \in \mathbb{R}$. Since the sectional curvature of $\mathbb{M}$ is zero [5], $\mathbb{M}$ is an Hadamard manifold. Let $x,y \in \mathbb{M}$ and $v \in T_x\mathbb{M}$ with $\|v\|_2 = 1$. Then $d(x,y) = |\ln x - \ln y|$, $\exp_x tv = xe^{\frac{vx}{t}}$, $t \in (0,+\infty)$, and $\exp_x^{-1} y = x\ln y - x\ln x$.

A subset $C \subset \mathbb{M}$ is said to be convex if for any two points $x,y \in C$, the geodesic γ joining x to y is contained in C. That is, if $\gamma : [a,b] \to \mathbb{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in C$ for all $t \in [0,1]$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called an Hadamard manifold. We denote by $\mathbb{M}$ a finite dimensional Hadamard manifold. Henceforth, unless otherwise stated, we represent by C a nonempty, closed and convex subset of $\mathbb{M}$.

Definition 2.1 ([18])**.** Let C be a nonempty, closed and subset of $\mathbb{M}$ and $\{x_n\}$ be a sequence in $\mathbb{M}$. Then $\{x_n\}$ is said to be Fejèr convergent with respect to C if for all $p \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1},p) \leq d(x_n,p).$$

Definition 2.2 ([34])**.** Let $f : C \to \mathbb{R}$ be a geodesic convex. Let $p \in C$, then a vector $r \in T_p\mathbb{M}$ is said to be a subgradient of f at p if and only if

$$f(q) \geq f(p) + \langle r, \exp_p^{-1} q\rangle,\ \forall\ q \in C. \tag{2.2}$$

Lemma 2.3 ([18])**.** *Let C be a nonempty, closed and closed subset of $\mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be a sequence such that $\{x_n\}$ be a Fejér convergent with respect to C. Then the following hold:*

(i) *For every $p \in C$, $d(x_n,p)$ converges,*
(ii) *$\{x_n\}$ is bounded,*
(iii) *Assume that every cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges to a point in C.*

Definition 2.4. A mapping $S : C \to C$ is said to be

(i) contractive if there exits a constant $k \in (0,1)$ such that

$$d(Sx, Sy) \leq kd(x, y), \ \forall\ x, y \in C. \tag{2.3}$$

If $k = 1$ in (2.3), then S is said to be nonexpansive,

(ii) quasi-nonexpansive if $Fix(S) \neq \emptyset$ and

$$d(Sx, p) \leq d(x, p), \ \forall\ p \in Fix(S) \text{ and } x \in C,$$

(iii) firmly nonexpansive [14] if for all $x, y \in C$, the function $\delta : [0,1] \to [0,\infty]$ defined by

$$\delta(t) := d(\exp_x t \exp_x^{-1} Sx, \exp_y^{-1} Sy), \ \forall\ t \in [0,1]$$

is nonincreasing.

Proposition 2.5 ([14]). *Let $S : C \to C$ be a mapping. Then the following statements are equivalent.*

(i) *S is firmly nonexpansive,*

(ii) *for any $x, y \in C$ and $t \in [0,1]$*

$$d(Sx, Sy) \leq d(\exp_x t \exp_x^{-1} Sx, \exp_y t \exp_y^{-1} Sy),$$

(iii) *for any $x, y \in C$*

$$\langle \exp_{S(x)}^{-1} S(y), \exp_{S(x)}^{-1} x \rangle + \langle \exp_{S(y)}^{-1} S(x), \exp_{S(y)}^{-1} y \rangle \leq 0.$$

Lemma 2.6 ([13]). *Let $S : C \to C$ be a firmly nonexpansive mapping and $Fix(S) \neq \emptyset$. Then for any $x \in C$ and $p \in Fix(S)$, the following conclusion holds*:

$$d^2(Sx, p) \leq d^2(x, p) - d^2(Sx, x).$$

Proposition 2.7 ([34]). *Let $x \in \mathbb{M}$. The exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism. For any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic joining x to y, which is given by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y, \ \forall\ t \in [0,1].$$

For any $x \in \mathbb{M}$ and $C \subset \mathbb{M}$, there exists a unique point $y \in C$ such that $d(x, y) \leq d(x, z)$ for all $z \in C$. This unique point y is called the nearest point projection of x onto the closed and convex set C and is denoted $P_C(x)$.

Lemma 2.8 ([38]). *Let C be a nonempty, closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$.*

(i) *For any $x \in \mathbb{M}$, there exists a unique nearest point projection $y = P_C(x)$. Furthermore, the following inequality holds*:

$$\langle \exp_y^{-1} x, \exp_y^{-1} z \rangle \leq 0, \ \forall\ z \in C.$$

(ii) *$P_C : M \to C$ is a firmly nonexpansive mapping. Therefore from Lemma 2.6, we have*

$$d^2(y, p) \leq d^2(x, p) - d^2(y, x), \ \forall\ x \in \mathbb{M} \text{ and } p \in C.$$

The next lemma presents the relationship between triangles in $\mathbb{R}^2$ and geodesic triangles in Riemannian manifolds (see [11]).

A function $h : \mathbb{M} \to \mathbb{R}$ is said to be geodesic if for any geodesic $\gamma \in \mathbb{M}$, the composition $h \circ \gamma : [u, v] \to \mathbb{R}$ is convex, that is,

$$h \circ \gamma(\lambda u + (1-\lambda)v) \leq \lambda h \circ \gamma(u) + (1-\lambda) h \circ \gamma(v),\ u, v \in \mathbb{R},\ \lambda \in [0,1].$$

Lemma 2.9 ([22])**.** *Let $x_0 \in \mathbb{M}$ and $\{x_n\} \subset \mathbb{M}$ be such that $x_n \to x_0$. Then, for any $y \in \mathbb{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*

The following propositions (see [18]) are very useful in our convergence analysis:

Proposition 2.10. *Let M be a Hadamard manifold and $d : M \times M :\to \mathbb{R}$ be the distance function. Then the function d is convex with respect to the product Riemannian metric. In other words, given any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$, then for all $t \in [0,1]$, we have*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in M$, the function $d(\cdot, y) : M \to \mathbb{R}$ is a convex function.

Proposition 2.11. *Let $\mathbb{M}$ be a Hadamard manifold and $x \in \mathbb{M}$. The map $\Phi_x = d^2(x, y)$ satisfying the following*:

(1) *Φ_x is convex. Indeed, for any geodesic $\gamma : [0,1] \to \mathbb{M}$, the following inequality holds for all $t \in [0,1]$:*

$$d^2(x, \gamma(t)) \leq (1-t) d^2(x, \gamma(0)) + t d^2(x, \gamma(1)) - t(1-t) d^2(\gamma(0), \gamma(1)).$$

(2) *Φ_x is smooth. Moreover, $\partial \Phi_x(y) = -2 \exp_y^{-1} x$.*

Definition 2.12. Let $\mathbb{M}$ be a Hadamard manifold. A mapping $S : \mathbb{M} \to \mathbb{M}$ is said to be demiclosed at 0 if for any sequence $\{x_n\}$ in $\mathbb{M}$ such that $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, then $Sp = p$.

Proposition 2.13 ([23])**.** *Let $S : C \to \mathbb{M}$ be a nonexpansive mapping defined on a closed convex set $C \subseteq M$. Then the fixed point set $Fix(S)$ is closed and convex.*

We need the following results to solve GEP (1.4).

Lemma 2.14 ([31])**.** *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. Let $\psi : C \to T\mathbb{M}$ be a single-valued monotone vector field and $F : C \times C \to \mathbb{R}$ be a bifunction such that $F(x, x) = 0$ satisfying*

(L1) *F is monotone. That is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$.*
(L2) *For all $x \in C$, $F(x, \cdot)$ is convex.*
(L3) *There exists a compact subset $K \subset C$ containing $u_0 \in K$ such that $F(x, u_0) + \langle \psi x, \exp_x^{-1} u_0 \rangle < 0$ whenever $x \in C \setminus K$.*
Then, the GEP (1.4) *is solvable.*

The result stated below describes some properties of the resolvent operator of GEP (1.4) as follows:

Lemma 2.15 ([31]). *Let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying assumptions* (L1)-(L3), *$\psi : C \to T\mathbb{M}$ be a mapping. For $r > 0$, define a set-valued mapping $T_r^{F,\psi} : C \to 2^C$ by*

$$T_r^{F,\psi}(x) = \left\{ z \in C : F(z,y) + \langle \psi z, \exp_z^{-1} y\rangle - \frac{1}{r}\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0 \right\},$$
$$\forall\ y \in C \ \text{and}\ x \in \mathbb{M}.$$

Then, there hold

(i) *$T_r^{F,\psi}$ is single-valued,*
(ii) *$T_r^{F,\psi}$ is firmly nonexpansive,*
(iii) *$Fix(T_r^{F,\psi}) = GEP(F,\psi)$,*
(iv) *$GEP(F,\psi)$ is closed and convex,*
(v) *let $0 < r \leq s$, then for all $x \in C$,*

$$d(x, T_r^{F,\psi}x) \leq 2d(x, T_r^{F,\psi}x),$$

(vi) *for all $x \in C$ and $p \in Fix(T_r^{F,\psi})$,*

$$d^2(p, T_r^{F,\psi}x) + d^2(x, T_r^{F,\psi}x) \leq d^2(x,p).$$

3. Main result

In this section, we introduce a shrinking method for solving a finite family of generalized equilibrium problem and fixed point problem of nonexpansive mapping in Hadamard manifolds. We state and prove our convergence result:

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $j = 1, 2, \dots, N$, let $\psi_j : C \to \mathbb{TM}$ be a monotone vector field and $F_j : C \times C \to \mathbb{R}$ such that $F_j(x,x) = 0$ for all $x \in C$ be a bifunction satisfying conditions* (L1)-(L3). *Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := Fix(S) \bigcap \bigcap_{j=1}^{N} GEP(F_j, \psi_j) \neq \emptyset$. For arbitrary $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be defined iteratively by*

$$\begin{cases} y_n^j = T_{r_n}^{F_j,\psi_j} x_n,\ j = 1, 2, \dots, N; \\ u_n \in \{y_n^j,\ j = 1, 2, \dots, N\} \text{ such that } d(u_n, x_n) = \max\limits_{1\leq j\leq N} d(y_n^j, x_n); \\ w_n = \exp_{u_n}(1-\beta_n)\exp_{u_n}^{-1} Su_n; \\ C_{n+1} = \{v \in C_n : d(w_n, v) \leq d(x_n, v)\}; \\ x_{n+1} = P_{C_{n+1}} x_1,\ \forall\ n \geq 1, \end{cases} \tag{3.1}$$

where the sequences $\{r_n\} \in (0,\infty)$ and $\{\beta_n\} \in (0,1)$ satisfying the following

(i) *$0 < a \leq \beta_n \leq b < 1$ for some $a, b > 0$ for all $n \geq 1$,*
(ii) *$0 < r \leq r_n$,*

then the sequence $\{x_n\}$ *converges to an element* $p \in \Omega$.

Proof. We divide the proof into several steps as follows:
Step 1: We show $\{x_n\}$ is well-defined for every $n \in \mathbb{N}$.
From Theorem 3.1, it can be seen that $C_1 = \mathbb{M}$, thus it is closed and geodesic convex. Suppose that for some $n \geq 2$, C_n is a closed and geodesic convex subset in $\mathbb{M}$. Also, we have from Lemma 2.15(iv) and Proposition 2.13 that $\bigcap_{j=1}^{N} GEP(F_j, \psi_j)$ and $Fix(S)$ are closed and geodesic convex subsets in $\mathbb{M}$. Therefore, the solution set Ω is closed and geodesic convex in $\mathbb{M}$. Now, since $v \longmapsto d(x, v)$ is a geodesic convex function, thus, the set C_{n+1} defined by

$$C_{n+1} := \{v \in C_n : d(w_n, v) \leq d(x_n, v)\}$$

is a geodesic convex subset in $\mathbb{M}$. Let $q \in \Omega$, then we can re-write w_n defined in (3.1) as $w_n = \gamma_n^1(1 - \beta_n)$, where $\gamma_n^1 : [0, 1] \to \mathbb{M}$ is a sequence of geodesic joining u_n to Su_n. By applying the nonexpansive property of S, we have that

$$\begin{aligned} d(w_n, q) &= d(\gamma_n^1(1 - \beta_n), q) \\ &= \beta_n d(\gamma_n^1(0), q) + (1 - \beta_n) d(\gamma_n^1(1), q) \\ &= \beta_n d(u_n, q) + (1 - \beta_n) d(u_n, q) \\ &\leq \beta_n d(u_n, q) + (1 - \beta_n) d(Su_n, q) \qquad (3.2) \\ &= d(u_n, q). \qquad (3.3) \end{aligned}$$

Also from (3.1) and Lemma 2.15, we have

$$d(y_n^j, q) = d(T_{r_n}^{F_j, \psi_j} x_n, q) \leq d(x_n, q). \tag{3.4}$$

This implies that

$$d(u_n, q) = \max_{1 \leq j \leq N} d(y_n^j, q) \leq d(x_n, q). \tag{3.5}$$

Hence, we conclude from (3.3) and (3.5) that

$$d(w_n, q) \leq d(x_n, q). \tag{3.6}$$

This shows that $q \in C_{n+1}, \ \forall \ q \in \Omega$ and since C_n is a nonempty, closed and geodesic convex subset in $\mathbb{M}$, it follows that $\Omega \subset C_{n+1} \subset C_n$ for each $n \geq 1$. Therefore $P_{C_{n+1}} x_1$ is well-defined for every $x_1 \in C$ and the sequence $\{x_n\}$ is well-defined.
Step 2: Next, we prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, x_1)$ exists.
We have established in Step 1 that Ω is a nonempty, closed and geodesic convex subset in $\mathbb{M}$. Then there exists a unique $w \in \Omega$ such that $w = P_\Omega x_1$. Since $x_n = P_{C_n} x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned} &d(x_n, x_1) \leq d(w, x_1), \ \forall \ n \in \mathbb{N} \text{ and} \\ &d(x_n, x_1) \leq d(x_{n+1}, x_1), \ \forall \ n \in \mathbb{N}. \end{aligned}$$

Therefore, $d(x_n, x_1)$ is bounded and it follows that $\{x_n\}$ is bounded. Consequently, $\{y_n^j\}$, $j = 1, 2, \ldots, N$, $\{u_n\}$ and $\{w_n\}$ are bounded. Hence,

$$\lim_{n\to\infty} d(x_n, x_1)$$

exists.

Step 3: We prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\mathbb{M}$ and $\lim_{n\to\infty} x_n = p$. From the construction of C_n, it is clear that $x_m = P_{C_m} x_1 \in C_m \subset C_n$ for $m > n \geq 1$. By Lemma 2.8(ii), we have that

$$d^2(x_n, x_m) = d^2(P_{C_n} x_1, x_m) \leq d^2(x_1, x_m) - d^2(x_n, x_1) \to 0 \text{ as } n, m \to \infty. \tag{3.7}$$

Since $\lim_{n\to\infty} d(x_n, x_1)$ exists, it follows that $\{x_n\}$ is a Cauchy sequence. By the completeness of $\mathbb{M}$ and closedness of C, there exists an element $p \in C$ such that $\lim_{n\to\infty} x_n = p \in \mathbb{M}$.

Step 4: We next show that $\lim_{n\to\infty} d(y_n^j, x_n) = 0 = \lim_{n\to\infty} d(u_n, Su_n)$. From (3.7), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have from Proposition 2.11, (3.5) and Lemma 2.15(vi) that

$$\begin{aligned}
d^2(w_n, q) &= d^2(\gamma_n^1(1-\beta_n), q) \\
&\leq (1-\beta_n)d^2(\gamma_n^1(0), q) + \beta_n d^2(\gamma_n^1(1), q) - \beta_n(1-\beta_n)d^2(\gamma_n^1(0), \gamma_n^1(1)) \\
&= (1-\beta_n)d^2(u_n, q) + \beta_n d^2(Su_n, q) - \beta_n(1-\beta_n)d^2(u_n, Su_n) \\
&\leq (1-\beta_n)d^2(u_n, q) + \beta_n d^2(u_n, q) - \beta_n(1-\beta_n)d^2(u_n, Su_n) \\
&= d^2(u_n, q) - \beta_n(1-\beta_n)d^2(u_n, Su_n) \qquad (3.9) \\
&\leq d^2(y_n^j, q) - \beta_n(1-\beta_n)d^2(u_n, Su_n) \\
&\leq d^2(x_n, q) - d^2(y_n^j, x_n) - \beta_n(1-\beta_n)d^2(u_n, Su_n). \qquad (3.10)
\end{aligned}$$

But, from the definition of C_{n+1}, we obtain that

$$d(w_n, x_{n+1}) \leq d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty, \tag{3.11}$$

which implies that

$$d(w_n, x_n) \leq d(w_n, x_{n+1}) + d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Thus, we obtain from (3.10) that

$$\begin{aligned}
&d^2(y_n^j, x_n) - \beta_n(1-\beta_n)d^2(u_n, Su_n) \\
&\leq d^2(x_n, q) - d^2(w_n, q) \\
&= \big(d(x_n, q) - d(w_n, q)\big)\big(d(x_n, q) + d(w_n, q)\big) \\
&\leq d(x_n, w_n)\big(d(x_n, q) + d(w_n, q)\big) \to 0 \text{ as } n \to \infty. \qquad (3.13)
\end{aligned}$$

Using condition (i) of (3.1), we obtain that

$$\lim_{n\to\infty} d(u_n, Su_n) = 0 = \lim_{n\to\infty} d(y_n^j, x_n). \tag{3.14}$$

It is obvious from (3.14) that

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \tag{3.15}$$

By replacing q with u_n in (3.2) and applying (3.14), we have

$$d(w_n, u_n) \leq \beta_n d(u_n, u_n) + (1-\beta_n) d(Su_n, u_n) \to 0 \text{ as } n \to \infty. \tag{3.16}$$

From (3.15) and (3.16), we get

$$d(w_n, x_n) \leq d(w_n, u_n) + d(u_n, x_n) \to 0 \text{ as } n \to \infty. \tag{3.17}$$

Step 5: We show that $p \in \Omega$.
Since $T_{r_n}^{F_j,\psi_j}$, $j = 1, 2, \dots, N$, is nonexpansive and demiclosed at 0, we have from (3.14) that $p \in Fix(T_{r_n}^{F_j,\psi_j}) = \bigcap_{j=1}^N GEP(F_j, \psi_j)$. Similarly, from (3.14), we obtain that $p \in Fix(S)$. Thus, we conclude that $p \in \Omega$.
Step 6: Finally, we show that $p = P_\Omega x_1$.
We know that $x_n = P_{C_n} x_1$ and $\Omega \subset C_n$, then it follows from Lemma 2.8(i) that

$$\langle \exp_{x_n}^{-1} x_1, \exp_{x_n}^{-1} x^* \rangle \leq 0, \ \forall\, x^* \in \Omega. \tag{3.18}$$

By taking $n \to \infty$ in (3.18), we get

$$\langle \exp_{p}^{-1} x_1, \exp_{p}^{-1} x^* \rangle \leq 0, \ \forall\, x^* \in \Omega. \tag{3.19}$$

This implies that $p = P_\Omega x_1$ as required. □

We now state some of the consequences of our result.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $j = 1, 2, \dots, N$, let $\psi_j : C \to \mathbb{TM}$ be a monotone vector field and $F_j : C \times C \to \mathbb{R}$ such that $F_j(x, x) = 0$ for all $x \in C$ be a bifunction satisfying conditions* (L1)-(L3). *Assume that $\Omega := \bigcap_{j=1}^N GEP(F_j, \psi_j) \neq \emptyset$. For arbitrary $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be defined iteratively by*

$$\begin{cases} y_n^j = T_{r_n}^{F_j,\psi_j} x_n, \ j = 1, 2, \dots, N; \\ u_n \in \{y_n^j, \ j = 1, 2, \dots, N\} \text{ such that } d(u_n, x_n) = \max_{1\leq j\leq N} d(y_n^j, x_n); \\ C_{n+1} = \{v \in C_n : d(u_n, v) \leq d(x_n, v)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \forall\, n \geq 1, \end{cases} \tag{3.20}$$

where the sequences $\{r_n\} \in (0, \infty)$ and $\{\beta_n\} \in (0, 1)$ satisfying the following

(i) $0 < r \leq r_n$,

then the sequence $\{x_n\}$ converges to an element $p \in \Omega$.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a Hadamard manifold $\mathbb{M}$. For $j = 1, 2, \dots, N$, let $\psi : C \to \mathbb{TM}$ be a monotone vector field and $F : C \times C \to \mathbb{R}$ such that $F(x, x) = 0$ for all $x \in C$ be a bifunction satisfying conditions* (L1)-(L3). *Let $S : C \to C$ be a nonexpansive mapping* *such that* $\Omega := Fix(S)\bigcap GEP(F,\psi) \neq \emptyset$*. For arbitrary* $x_1 \in \mathbb{M}$ *and* $C_1 = \mathbb{M}$*, let* $\{x_n\}$ *be defined iteratively by*

$$\begin{cases} y_n = T_{r_n}^{F,\psi} x_n; \\ w_n = \exp_{u_n}(1-\beta_n)\exp_{y_n}^{-1} Sy_n; \\ C_{n+1} = \{v \in C_n : d(w_n, v) \leq d(x_n, v)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \forall\ n \geq 1, \end{cases} \tag{3.21}$$

where the sequences $\{r_n\} \in (0,\infty)$ *and* $\{\beta_n\} \in (0,1)$ *satisfying the following*

(i) $0 < a \leq \beta_n \leq b < 1$ *for some* $a, b > 0$ *for all* $n \geq 1$*,*
(ii) $0 < r \leq r_n$*,*

then the sequence $\{x_n\}$ *converges to an element* $p \in \Omega$*.*

4. Numerical example

We now present a numerical examples on Hadamard manifolds to illustrate the convergence of our Algorithm (3.1).

Let $M = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $(M, \langle \cdot,\cdot \rangle)$ be the Riemannian manifold and $\langle \cdot,\cdot \rangle$ the Riemannian metric defined by $\langle p, q\rangle = \frac{1}{x^2}pq$ for all vectors $p, q \in T_xM$. The tangent space at $x \in M$ denoted $T_xM = \mathbb{R}$. Furthermore, the parallel transport is the identity mapping on the tangent bundle TM. The Riemannian distance $d : M \times M \to \mathbb{R}_+$ is given by

$$d(x,y) = \left|\ln \frac{x}{y}\right|, \ \forall\ x, y \in M.$$

Then, $(M, \langle \cdot,\cdot \rangle)$ is a Hadamard manifold and the unique geodesic $\gamma : \mathbb{R} \to M$ starting from $\gamma(0) = x$ such that $\gamma'(0) = v \in T_xM$ is defined by $\gamma(s) = xe^{\frac{sv}{x}}$. Thus, $\exp_x sv = xe^{\frac{sv}{x}}$ and the inverse exponential map $\exp_x^{-1} y = \gamma'(0) = x \ln \frac{y}{x}$. Let $C = (0, +\infty)$ be a geodesic subset of $\mathbb{R}_{++}$, $\psi_j : C \to TM$ be single valued vector fields defined by $\psi_j(x) = \frac{x}{2j} \ln x$ and $F_j : C \times C \to \mathbb{R}$ be a bifunction defined by $F_j(x, y) = -\frac{1}{2j} \ln \frac{y}{x}$, $\forall j = 1, \ldots, N$, $x \in C$. Then by Lemma 2.15, there exists $z \in C$ such that

$$0 \leq F_j(z, y) + \langle A_j(z), \exp_z^{-1} y\rangle - \frac{1}{r}\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle, \ \forall\ j = 1, 2, \ldots, N.$$

By simple calculation, we obtain that

$$z = \exp\left(\frac{2j \ln x + r}{2j + r}\right).$$

Next, we define the mapping $S : C \to C$ by $S(x) = x$. We find that Ω is nonempty since $e \in \Omega$. Let $\beta_n = \frac{n}{7n+3}$ and $r_n = \frac{n}{3n+2}$ and choose $d(x_{n+1}, x_n) \leq \epsilon$ as the stopping criterion, where $\epsilon = 10^{-4}$. We plot the graph of $\{x_n\}$. We also plot the graph of errors. The result of this experiment is presented in Figure 1.

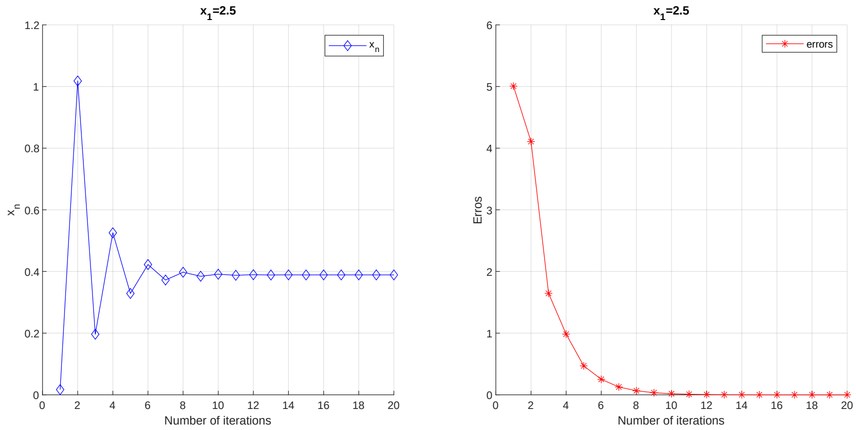

FIGURE 1. Left: x_n against number of iteration; Right: errors against number of iteration.

5. Conclusion

We have studied a common solution of a finite family of generalized equilibrium problem and fixed point problem of a nonexpansive mapping on Hadamard manifolds. Using a shrinking projection algorithm, we prove that the sequences generated by our algorithm converges to a solution of finite family of generalized equilibrium problem and fixed point problem of a nonexpansive mapping. Lastly, we presented a numerical example to show the performance of our algorithm.

References

[1] H. A. Abass, C. Izuchukwu, O. T. Mewomo, and Q. L. Dong, *Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach space*, Fixed Point Theory **21** (2020), no. 2, 397–411.

[2] H. A. Abass, F. U. Ogbuisi, and O. T. Mewomo, *Common solution of split equilibrium problem and fixed point problem with no prior knowledge of operator norm*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **80** (2018), no. 1, 175–190.

[3] H. A. Abass, O. K. Oyewole, L. O. Jolaoso, and K. O. Aremu, *Modified inertial Tseng method for solving variational inclusion and fixed point problems on Hadamard manifolds*, Appl. Anal. (2023).

[4] F. Akutsah, H. A. Abass, A. A. Mebawondu, and O. K. Narain, *On split generalized mixed equilibrium and fixed point problems of an infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces*, Asian-Eur. J. Math. **15** (2022), no. 5, Paper No. 2250082, 20 pp. https://doi.org/10.1142/S1793557122500826

[5] Q. H. Ansari, F. Babu, and X. Li, *Variational inclusion problems in Hadamard manifolds*, J. Nonlinear Convex Anal. **19** (2018), no. 2, 219–237.

[6] F. Babu, A. Ali, and A. H. Alkhaldi, *An extragradient method for non-monotone equilibrium problems on Hadamard manifolds with applications*, Appl. Numer. Math. **180** (2022), 85–103. https://doi.org/10.1016/j.apnum.2022.05.007

[7] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibrium problems*, J. Optim. Theory Appl. **90** (1996), no. 1, 31–43. https://doi.org/10.1007/BF02192244
[8] M. Bianchi and S. Schaible, *Equilibrium problems under generalized convexity and generalized monotonicity*, J. Global Optim. **30** (2004), no. 2-3, 121–134. https://doi.org/10.1007/s10898-004-8269-9
[9] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), no. 1-4, 123–145.
[10] H. Brezis, L. Nirenberg, and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital. (4) **6** (1972), 293–300.
[11] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften, 319, Springer, Berlin, 1999. https://doi.org/10.1007/978-3-662-12494-9
[12] S. Chang, J. C. Yao, M. Liu, L. C. Zhao, and J. H. Zhu, *Shrinking projection algorithm for solving a finite family of quasi-variational inclusion problems in Hadamard manifold*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 4, Paper No. 166, 11 pp. https://doi.org/10.1007/s13398-021-01105-4
[13] S. Chang, J. C. Yao, C. Wen, and L. J. Qin, *Shrinking projection method for solving inclusion problem and fixed point problem in reflexive Banach spaces*, Optimization **70** (2021), no. 9, 1921–1936. https://doi.org/10.1080/02331934.2020.1763988
[14] C. Chong, G. López, and V. Martín-Márquez, *Iterative algorithms for nonexpansive mappings on Hadamard manifolds*, Taiwanese J. Math. **14** (2010), no. 2, 541–559. https://doi.org/10.11650/twjm/1500405806
[15] V. Colao, G. López, G. Marino, and V. Martín-Márquez, *Equilibrium problems in Hadamard manifolds*, J. Math. Anal. Appl. **388** (2012), no. 1, 61–77. https://doi.org/10.1016/j.jmaa.2011.11.001
[16] V. Colao, G. Marino, and H. K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl. **344** (2008), no. 1, 340–352. https://doi.org/10.1016/j.jmaa.2008.02.041
[17] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), no. 1, 117–136.
[18] O. P. Ferreira and P. R. Oliveira, *Proximal point algorithm on Riemannian manifolds*, Optimization **51** (2002), no. 2, 257–270. https://doi.org/10.1080/02331930290019413
[19] E. C. Godwin, H. A. Abass, C. Izuchukwu, and O. T. Mewomo, *On split equality equilibrium, monotone variational inclusion and fixed point problems in Banach spaces*, Asian-Eur. J. Math. **15** (2022), no. 7, Paper No. 2250139, 29 pp. https://doi.org/10.1142/S179355712250139X
[20] A. N. Iusem and W. Sosa, *New existence results for equilibrium problems*, Nonlinear Anal. **52** (2003), no. 2, 621–635. https://doi.org/10.1016/S0362-546X(02)00154-2
[21] U. Kamraksa and R. Wangkeeree, *Generalized equilibrium problems and fixed point problems for nonexpansive semigroups in Hilbert spaces*, J. Global Optim. **51** (2011), no. 4, 689–714. https://doi.org/10.1007/s10898-011-9654-9
[22] C. Li, G. López, and V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc. (2) **79** (2009), no. 3, 663–683. https://doi.org/10.1112/jlms/jdn087
[23] G. López and V. Martín-Márquez, *Approximation methods for nonexpansive type mappings in Hadamard manifolds*, in Fixed-point algorithms for inverse problems in science and engineering, 273–299, Springer Optim. Appl., 49, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-9569-8_14
[24] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal. **9** (2008), no. 1, 37–43.
[25] S. Z. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal. **52** (2003), no. 5, 1491–1498. https://doi.org/10.1016/S0362-546X(02)00266-3

[26] M. O. Nnakwe and C. C. Okeke, *A common solution of generalized equilibrium problems and fixed points of pseudo-contractive-type maps*, J. Appl. Math. Comput. **66** (2021), no. 1-2, 701–716. https://doi.org/10.1007/s12190-020-01457-x

[27] M. A. Noor and K. I. Noor, *Some algorithms for equilibrium problems on Hadamard manifolds*, J. Inequal. Appl. **2012** (2012), 230, 8 pp. https://doi.org/10.1186/1029-242X-2012-230

[28] M. A. Noor, S. Zainab, and Y. Yao, *Implicit methods for equilibrium problems on Hadamard manifolds*, J. Appl. Math. **2012** (2012), Art. ID 437391, 9 pp. https://doi.org/10.1155/2012/437391

[29] C. C. Okeke, L. O. Jolaoso, and R. Nwokoye, *A self-adaptive shrinking projection method with an inertial technique for split common null point problems in Banach spaces*, Axioms **9** (2020), no. 4.

[30] M. A. Olona, T. O. Alakoya, A. O. Owolabi, and O. T. Mewomo, *Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings*, Demonstr. Math. **54** (2021), no. 1, 47–67. https://doi.org/10.1515/dema-2021-0006

[31] O. K. Oyewole, L. O. Jolaoso, K. O. Aremu, and M. Aphane, *On the existence and approximation of solutions of generalized equilibrium problem on Hadamard manifolds*, J. Inequal. Appl. **2022** (2022), Paper No. 142, 22 pp. https://doi.org/10.1186/s13660-022-02883-0

[32] O. K. Oyewole and S. Reich, *An inertial subgradient extragradient method for approximating solution to equilibrium problems in Hadamard manifold*, Axioms **12** (2023), 256.

[33] S. Saewan and P. Kumam, *The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-ϕ-nonexpansive mappings*, Fixed Point Theory Appl. **2011** (2011), 9, 25 pp.

[34] T. Sakai, *Riemannian geometry*, translated from the 1992 Japanese original by the author, Translations of Mathematical Monographs, 149, Amer. Math. Soc., Providence, RI, 1996. https://doi.org/10.1090/mmono/149

[35] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), no. 1, Ser. A, 189–202. https://doi.org/10.1007/s101079900113

[36] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), no. 3, 1025–1033. https://doi.org/10.1016/j.na.2008.02.042

[37] G. C. Ugwunnadi, H. A. Abass, M. Aphane, and O. K. Oyewole, *Inertial Halpern-typemethod for solving split feasibility and fixed point problems via dynamical stepsize in real Banach spaces*, Annali dell'Universita di Ferrara (2023). https://doi.org/10.1007/s11565-023-00473-6

[38] R. Walter, *On the metric projection onto convex sets in riemannian spaces*, Arch. Math. (Basel) **25** (1974), 91–98. https://doi.org/10.1007/BF01238646

[39] J. H. Wang, G. López, V. Martín-Márquez, and C. Li, *Monotone and accretive vector fields on Riemannian manifolds*, J. Optim. Theory Appl. **146** (2010), no. 3, 691–708. https://doi.org/10.1007/s10957-010-9688-z

[40] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, *Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems*, Taiwanese J. Math. **10** (2006), no. 6, 1497–1514. https://doi.org/10.11650/twjm/1500404570

[41] L. W. Zhou and N. J. Huang, *Generalized KKM theorems on Hadamard manifolds*, (2009), Accessed 27 Mar 2015.

Hammed Anuoluwapo Abass
Department of Mathematics and Applied Mathematics
Sefako Makgatho Health Science University
P.O. Box 94, Pretoria 0204, South Africa
Email address: hammedabass548@gmail.com, hammed.abass@smu.ac.za

Olawale Kazeem Oyewole
Department of Mathematics
The Technion – Israel Institute of Technology
32000 Haifa, Israel
Email address: oyewoleolawalekazeem@gmail.com, oyewoleok@campus.technion.ac.il