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A GENERALIZATION OF THE SYMMETRY PROPERTY OF A RING VIA ITS ENDOMORPHISM

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Abstract. Lambek introduced the concept of symmetric rings to expand the commutative ideal theory to noncommutative rings. In this study, we propose an extension of symmetric rings called strongly α-symmetric rings, which serves as both a generalization of strongly symmetric rings and an extension of symmetric rings. We define a ring R as strongly α-symmetric if the skew polynomial ring $R[x;\alpha]$ is symmetric. Consequently, we provide proofs for previously established outcomes regarding symmetric and strongly symmetric rings, directly derived from the results we have obtained. Furthermore, we explore various properties and extensions of strongly α-symmetric rings.

1. Introduction

Let R be an associative ring with identity and α be a non-zero and non-identity endomorphism of R. We denote the polynomial ring with an indeterminate x over R by $R[x]$ and the degree of $f(x) \in R[x]$ by the notation $deg\, f$.

Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements. Reversible rings are defined as a generalization of commutative rings by Cohn [4] as follows: A ring R is called *reversible* if whenever $a, b \in R$ satisfies $ab = 0$, then $ba = 0$. These classes of rings have also found application in one of the most famous conjectures in ring theory known as Köthe's Conjecture. Cohn [4] showed that reversible rings satisfy Köthe's Conjecture.

According to Lambek [23], a ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$. It is obvious that a commutative ring R is symmetric. Also, if R is a symmetric ring, then it is reversible. But, in general, the other aspects need not have to be satisfied (see [24, Example 5] and [1, Example II.5]).

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In [19], Krempa introduced *rigid* endomorphisms. An endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$, where $a \in R$. For a ring R if there exists a rigid endomorphism α, then R is called *α-rigid*. It is well-known that any rigid endomorphism of a ring is a monomorphism and it is showed in [9, Propositon 5] that α-rigid rings are reduced.

Armendariz rings are defined by Rege and Chhawchharia in [29] as a generalization of reduced rings. Let $p(x) = p_0 + p_1x + \cdots + p_mx^m$ and $q(x) = q_0 + q_1x + \cdots + q_nx^n \in R[x]$. If $p(x)q(x) = 0$ implies $p_iq_j = 0$ for each i and j, then R is called an *Armendariz ring*. Also, note that Armendariz rings can be used to establish an association between the annihilators of R and its polynomial extension $R[x]$.

According to [7], a *skew polynomial ring* over a coefficient ring R (also called an *Ore extension of endomorphism type*) is defined as the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$, where $r \in R$ and it is denoted by $R[x; \alpha]$. This property makes the study of Ore extensions of endomorphism type more difficult than that of the polynomial rings. Note that for any skew polynomial ring $R[x; \alpha]$ of R we have $\alpha(1) = 1$, since $1 \cdot x = x \cdot 1 = \alpha(1)x$.

Armendariz property of a ring is extended to skew polynomial rings by considering the polynomials in $R[x; \alpha]$ instead of $R[x]$. R is called *α-Armendariz* (resp., *α-skew Armendariz*) if for $p(x) = \sum_{i=0}^{m} p_ix^i$ and $q(x) = \sum_{j=0}^{n} q_jx^j$ in $R[x; \alpha]$, $p(x)q(x) = 0$ implies $p_iq_j = 0$ (resp., $p_i\alpha^i(q_j) = 0$) for all i, j (see [10] and [13] for more details).

In [18, Example 2.1], Kim and Lee showed that polynomial rings over reversible rings need not be reversible. Following [31], Yang and Liu consider reversible rings over which polynomial rings are reversible and called them *strongly reversible*. According to Bell [3], a one-sided ideal I of a ring R is said to have the *insertion-of factors-principle* (or simply *IFP*), if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. A ring R is called an *IFP ring* if the zero ideal of R has the IFP. Also, note that polynomial rings over IFP rings need not be IFP by [15, Example 2]. Following [21], Kwak et al. called a ring R *strongly IFP* if $R[x]$ has IFP. Huh et al. proved in [14, Example 3.1] that polynomial rings over symmetric rings need not be symmetric. In [6], Eltiyeb and Ayoub investigated strongly symmetric rings over which polynomial rings are symmetric and a ring R is called *strongly symmetric* if whenever polynomials $p(x), q(x), r(x)$ in $R[x]$ satisfy $p(x)q(x)r(x) = 0$, then $p(x)r(x)q(x) = 0$. Recall that a ring R is called *abelian* if every idempotent element of R is central.

Another approach to generalize reversible and IFP properties is obtained by considering the properties on Ore extensions of endomorphism type. Following Jin et al. [16], a ring R is called *strongly α-skew reversible* if the skew polynomial ring $R[x; \alpha]$ is reversible and in [2], Başer et al. called a ring R *strongly α-IFP* if the skew polynomial ring $R[x; \alpha]$ has IFP.

Motivated by the above, in this paper, we introduce a new class of rings which is called *strongly α-symmetric* ring to extend the symmetry property on skew polynomials. The following diagram describes all known implications. Also, note that no other implications in the diagram hold in general.

$$\begin{tikzcd}
 & \boxed{\alpha\text{-rigid}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\textbf{strongly } \boldsymbol{\alpha}\textbf{-symmetric}} \arrow[r] \arrow[d] & \boxed{\text{strongly symmetric}} \arrow[r] \arrow[d] & \boxed{\text{symmetric}} \arrow[d] \\
\boxed{\text{strongly } \alpha\text{-skew reversible}} \arrow[r] \arrow[d] & \boxed{\text{strongly reversible}} \arrow[r] \arrow[d] & \boxed{\text{reversible}} \arrow[d] \\
\boxed{\text{strongly } \alpha\text{-IFP}} \arrow[r] \arrow[dr] & \boxed{\text{strongly IFP}} \arrow[r] \arrow[d] & \boxed{\text{IFP}} \arrow[dl] \\
 & \boxed{\text{abelian}} &
\end{tikzcd}$$

In Section 2, we examine the relationships between several classes of rings and strongly α-symmetric rings and prove some statements about the links given in the above diagram. We also provide some examples of strongly α-symmetric rings and counterexamples to several naturally raised situations.

In Section 3, as suggested by the literature, there is a considerable interest whether strongly α-symmetric property is preserved under extensions. For this aim, we examine whether some ring extensions over strongly α-symmetric ring R again possess this property, where α is an endomorphism of the ring R. With this generalization, several known results relating to symmetric rings can be obtained as corollaries of our results.

2. Properties of strongly α-symmetric rings

In this section, we introduce the concept of a strongly α-symmetric ring for an endomorphism α and investigate its ring-theoretical properties. Firstly, we begin with the following example which illustrates the need to introduce the symmetric property of skew polynomial rings.

Example 2.1. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Since R is reduced, R is symmetric. Let $\alpha : R \to R$ be an endomorphism of R defined by $\alpha((a, b)) = (b, a)$. If we consider $p(x) = (1, 1)$, $q(x) = (1, 0)$ and $r(x) = (0, 1)x$ in $R[x; \alpha]$, then we have $p(x)q(x)r(x) = 0$. On the other hand, $p(x)r(x)q(x) = (1, 1)(0, 1)x(1, 0) = (1, 1)(0, 1)\alpha((1, 0))x = (0, 1)x \neq (0, 0)$. Hence, $R[x; \alpha]$ is not symmetric.

Inspired by this example, we give the following definition.

Definition. Let R be a ring and α be an endomorphism of R. Then R is called *strongly α-symmetric* if $R[x; \alpha]$ is symmetric.

Every strongly α-symmetric ring is symmetric, but the converse is not true by Example 2.1. Any α-rigid ring R (i.e., $R[x; \alpha]$ is reduced) is clearly strongly

α-symmetric. However, there exists a strongly α-symmetric ring which is not α-rigid by [25, Example 3.4]. It is clear that any domain R with a monomorphism α is strongly α-symmetric since R is α-rigid. Note that every subring S of a strongly α-symmetric ring with $\alpha(S) \subseteq S$ is also strongly α-symmetric. Any strongly α-symmetric ring is clearly strongly α-IFP, but the converse may not be true.

Example 2.2. Let $R = \mathbb{Z}[x]$. Consider the endomorphism

$$\alpha : R \to R$$

defined by $\alpha(p(x)) = p(0)$, where $p(x) \in \mathbb{Z}[x]$. Then by [2, Example 2.3], we know that R is strongly α-IFP. On the other hand, for the polynomials $p(y) = 1 + x$, $q(y) = xy$ and $r(y) = x \in \mathbb{Z}[x][y; \alpha]$ we have $p(y)q(y)r(y) = 0$, but $p(y)r(y)q(y) \neq 0$. Therefore, R is not strongly α-symmetric.

Following Kwak [20, Definition 2.1], an endomorphism α of a ring R is called *right* (*resp., left*) *symmetric* if whenever $abc = 0$ for $a, b, c \in R$, then $ac\alpha(b) = 0$ (resp., $\alpha(b)ac = 0$). A ring R is called *right* (*resp., left*) *α-symmetric* if there exists a right (resp., left) symmetric endomorphism α of R. R is called *α-symmetric* if it is both right and left α-symmetric.

Proposition 2.3. *Let R be a ring with an endomorphism α. If R is strongly α-symmetric, then R is both symmetric and α-symmetric.*

Proof. Let R be a strongly α-symmetric ring. Then it is clear that R is symmetric. Assume now that $abc = 0$, where $a, b, c \in R$ and consider the polynomials $p(x) = a$, $q(x) = b$ and $r(x) = cx$ in $R[x; \alpha]$. Then $p(x)q(x)r(x) = 0$. Since R is strongly α-symmetric, we have $p(x)r(x)q(x) = 0$ and so, $ac\alpha(b) = 0$. Hence, we obtain that R is right α-symmetric. Moreover, we have $\alpha(b)ac = 0$ since R is reversible. Therefore, R is left α-symmetric. ☐

Let R be a ring with an endomorphism α. Then by [11, Definition 2.1], R is called *α-skew quasi Armendariz* if whenever $p(x)R[x; \alpha]q(x) = 0$ for $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j \in R[x; \alpha]$, then $p_i R \sigma^i(q_j) = 0$ for all i, j. Also, note that any α-skew Armendariz ring is α-skew quasi Armendariz, when α is an epimorphism by [11]. But, α-skew quasi Armendariz rings need not be α-skew Armendariz even if α is an automorphism by [11, Example 2.2(1)]. In the following theorem, our aim is to show that over strongly α-symmetric rings these concepts are equivalent.

Theorem 2.4. *Let R be a ring with an endomorphism α. If R is strongly α-symmetric, then R is α-skew Armendariz if and only if R is α-skew quasi Armendariz.*

Proof. It is enough to show that R is α-skew Armendariz when R is α-skew quasi Armendariz. Let $p(x)q(x) = 0$, where $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j$ in $R[x; \alpha]$. Then $p(x)q(x)r(x) = 0$ for each $r(x) \in R[x; \alpha]$. By the assumption, we have $p(x)r(x)q(x) = 0$ for each $r(x) \in R[x; \alpha]$. Hence,

$p(x)R[x;\alpha]q(x) = 0$ and since R is α-skew quasi Armendariz, $p_iR\alpha^i(q_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore, $p_i\alpha^i(q_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$ as required. □

Note that for a ring R being "α-skew Armendariz" and "strongly α-symmetric" are independent of each other by the following examples.

Example 2.5. (1) Consider $R = \mathbb{Z}_2[x]$ with an the endomorphism α, where α is defined by $\alpha(p(x)) = p(0)$ for $p(x) \in \mathbb{Z}_2[x]$. Then R is α-skew Armendariz by [10, Example 5], but is not strongly α-symmetric. Indeed, for $p(y) = 1, q(y) = (\bar{1} + x)y, r(y) = x \in \mathbb{Z}_2[x][y;\alpha]$, we have $p(y)q(y)r(y) = 0$ and $p(y)r(y)q(y) = x(\bar{1} + x)y \neq 0$.

(2) Let $S = \left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{0} & \bar{a} \end{pmatrix} \mid \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}$. By [10, Example 14], S is not I_S-skew Armendariz, where I_S denotes the identity map of S. On the other hand, it can be seen that S is strongly I_S-symmetric.

Theorem 2.6. *Let R be an α-Armendariz ring. Then the followings are equivalent*:

(1) *R is right α-symmetric.*
(2) *R is symmetric.*
(3) *R is strongly α-symmetric.*

Proof. (1) $\Rightarrow$ (2) Assume that R is right α-symmetric. Let $a, b, c \in R$ such that $abc = 0$. Since R is right α-symmetric, we have $ac\alpha(b) = 0$. Then $acb = 0$ by [13, Proposition 1.3(ii)] since R is α-Armendariz. Hence, R is symmetric.

(2) $\Rightarrow$ (3) It is clear by [13, Theorem 3.6(i)] since R is α-Armendariz.

(3) $\Rightarrow$ (1) Assume that R is strongly α-symmetric and $abc = 0$, where $a, b, c \in R$. Let $p(x) = a, q(x) = b$ and $r(x) = cx$ in $R[x;\alpha]$. Then $p(x)q(x)r(x) = 0$ and by the assumption, we get $p(x)r(x)q(x) = 0$. Therefore, $ac\alpha(b) = 0$ as required. □

Note that the condition of R being "α-Armendariz" in Theorem 2.6 is not superfluous by Example 2.5(1). In [13, Example 1.9], it is proved that $R = \mathbb{Z}_2[x]$ is not α-Armendariz. Since R is a commutative domain, then R is symmetric and right α-symmetric for any endomorphism.

Corollary 2.7 ([14, Proposition 3.4])**.** *Let R be an Armendariz ring. Then R is symmetric if and only if $R[x]$ is symmetric.*

The following lemma can be seen by using [16, Lemma 2.3], since any strongly α-symmetric ring is strongly α-skew reversible. For the sake of completeness, we include the statements.

Lemma 2.8. *Let R be a strongly α-symmetric ring. Then we have the following results*:

(1) *R is symmetric.*
(2) *α is a monomorphism.*

(3) *For any $a, b \in R$ and nonnegative integer m and n, we have* $a\alpha^m(b) = 0 \Leftrightarrow ab = 0 \Leftrightarrow ba = 0 \Leftrightarrow \alpha^m(b)\alpha^n(a) = 0 \Leftrightarrow \alpha^n(a)\alpha^m(b) = 0$.

(4) *R is abelian and $\alpha(e) = e$ for any $e^2 = e \in R$.*

Following [8], a ring R with an endomorphism α is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. It is a well-known fact that if R is a α-compatible ring, then α is a monomorphism.

Lemma 2.9. *Let R be a ring with an endomorphism α. If R is strongly α-symmetric, then R is α-compatible.*

Proof. If R is strongly α-symmetric, then R is strongly α-skew reversible and by using [16, Corollary 2.4(1)], we obtain that R is α-compatible. $\square$

On the other hand, α-compatible rings need not be strongly α-symmetric by [16, Example 2.11]. In the following theorem, we show the relation between α-compatible rings and strongly α-symmetric rings.

Theorem 2.10. *Let R be a ring with an endomorphism α. Assume that R is α-skew Armendariz. Then R is strongly α-symmetric if and only if R is symmetric and α-compatible.*

Proof. By using Lemma 2.9 and the fact that strongly α-symmetric property is inherited by its subrings, it suffices to show the necessity. Suppose that R is symmetric and α-compatible. Let $p(x)q(x)r(x) = 0$, where $p(x) = \sum_{i=0}^{m} p_i x^i$, $q(x) = \sum_{j=0}^{n} q_j x^j$ and $r(x) = \sum_{k=0}^{l} r_k x^k$ in $R[x; \alpha]$. Then we have $p_i\alpha^i(q_j)\alpha^{i+j}(r_k) = 0$ for all $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le l$. Since R is α-compatible, we obtain $p_iq_jr_k = 0$. By symmetric property, we get $p_ir_kq_j = 0$. Hence, $p_i\alpha^i(r_k)\alpha^{i+j}(q_j) = 0$ for all $0 \le i \le m$, $0 \le j \le n$, $0 \le k \le l$ and this implies that $p(x)r(x)q(x) = 0$. $\square$

Note that the statements of R being "an α-compatible ring" and "an α-skew Armendariz ring" in Theorem 2.10 can not be dropped by the following examples.

Example 2.11. Consider the polynomial ring $R = \mathbb{Z}_2[x]$ and the endomorphism $\alpha : R \to R$ defined by $\alpha(p(x)) = p(0)$, where $p(x) \in \mathbb{Z}_2[x]$. Since R is a domain, it is symmetric. Also by [10, Example 5], R is α-skew Armendariz. On the other hand, R is not strongly α-symmetric by Example 2.5(1) and since α is not a monomorphism, we get that R is not α-compatible.

Example 2.12. Inspired by [14, Example 3.1], let us construct a ring R as follows:

Let $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the ring of integers modulo 2. Let α be an automorphism of A defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively. Let B be the set that consists of all polynomials in A with zero constant terms and I be the ideal of A generated by

$$a_0b_0,\ b_0a_0,\ a_2b_2,\ b_2a_2,\ a_0a_0,\ a_2a_2,\ b_0b_0,\ b_2b_2$$
$$a_0rb_0,\ b_0ra_0,\ a_2rb_2,\ b_2ra_2,\ a_0ra_0,\ a_2ra_2,\ b_0rb_0,\ b_2rb_2,\ r_1r_2r_3r_4$$
$$a_0b_1+a_1b_0,\ b_0a_1+b_1a_0,\ a_1b_2+a_2b_1,\ b_1a_2+b_2a_1,\ a_0a_1+a_1a_0,\ b_0b_1+b_1b_0$$
$$a_1a_2+a_2a_1,\ b_1b_2+b_2b_1,\ a_0b_2+a_1b_1+a_2b_0,\ b_0a_2+b_1a_1+b_2a_0,\ a_0a_2+a_1a_1+a_2a_0$$
$$b_0b_2+b_1b_1+b_2b,\ (a_0+a_1+a_2)r(b_0+b_1+b_2),\ (b_0+b_1+b_2)r(a_0+a_1+a_2)$$
$$(a_0+a_1+a_2)r(a_0+a_1+a_2),\ (b_0+b_1+b_2)r(b_0+b_1+b_2)$$

for $r, r_1, r_2, r_3, r_4 \in B$. It can be seen that $B^4 \subseteq I$. Now set $R = A/I$. Since $\alpha(I) \subseteq I$, an automorphism $\bar{\alpha}$ of R can be induced by defining $\bar{\alpha}(a+I) = \alpha(a)+I$, where $a \in A$. Also, note that $\alpha^2 = I_R$. By [16, Example 2.5], R is $\bar{\alpha}$-compatible, but not strongly $\bar{\alpha}$-skew reversible. Therefore, R is not strongly $\bar{\alpha}$-symmetric. Moreover, by [14, Example 2.10(1)], R is not $\bar{\alpha}$-skew Armendariz.

Now we show that R is symmetric. Recall that $p \in A$ a *monomial* of degree n if it is a product of exactly n number of indeterminates. The set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$ is denoted by H_n. Then H_n is finite for any n. In addition, the ideal I is homogeneous (i.e., if $\sum_{i=1}^{t} r_i \in I$ with $r_i \in H_i$ then every r_i is in I) by the construction.

Claim. If $p_1q_1s_1 \in I$ where $p_1, q_1, s_1 \in H_1$, then $p_1q_1s_1 \in I$.

Proof of Claim. Based on the construction of I we have the following cases;
Case 1:

$$(p_1=a_0, q_1=b_0, s_1=r),\ (p_1=a_0, q_1=r, s_1=b_0),\ (p_1=b_0, q_1=a_0, s_1=r),$$
$$(p_1=b_0, q_1=r, s_1=a_0),\ (p_1=a_0, q_1=a_0, s_1=r),\ (p_1=a_0, q_1=r, s_1=a_0),$$
$$(p_1=b_0, q_1=b_0, s_1=r),\ (p_1=b_0, q_1=r, s_1=b_0),\ (p_1=r, q_1=a_0, s_1=b_0),$$
$$(p_1=r, q_1=b_0, s_1=a_0),\ (p_1=a_2, q_1=b_2, s_1=r),\ (p_1=a_2, q_1=r, s_1=b_2),$$
$$(p_1=b_2, q_1=a_2, s_1=r),\ (p_1=b_2, q_1=r, s_1=a_2),\ (p_1=a_2, q_1=a_2, s_1=r),$$
$$(p_1=a_2, q_1=r, s_1=a_2),\ (p_1=b_2, q_1=b_2, s_1=r),\ (p_1=b_2, q_1=r, s_1=b_2),$$
$$(p_1=r, q_1=a_2, s_1=b_2),\ (p_1=r, q_1=b_2, s_1=a_2),\ (p_1=r, q_1=a_0, s_1=a_0),$$
$$(p_1=r, q_1=a_2, s_1=a_2),\ (p_1=r, q_1=b_0, s_1=b_0),\ (p_1=r, q_1=b_2, s_1=b_2),$$
$$(p_1=a_0+a_1+a_2, q_1=r, s_1=b_0+b_1+b_2),\ (p_1=a_0+a_1+a_2, q_1=b_0+b_1+b_2, s_1=r),$$
$$(p_1=r, q_1=a_0+a_1+a_2, s_1=b_0+b_1+b_2),\ (p_1=b_0+b_1+b_2, q_1=r, s_1=a_0+a_1+a_2),$$
$$(p_1=b_0+b_1+b_2, q_1=a_0+a_1+a_2, s_1=r),\ (p_1=r, q_1=b_0+b_1+b_2, s_1=a_0+a_1+a_2),$$
$$(p_1=a_0+a_1+a_2, q_1=a_0+a_1+a_2, s_1=r),\ (p_1=a_0+a_1+a_2, q_1=r, s_1=a_0+a_1+a_2),$$
$$(p_1=r, q_1=a_0+a_1+a_2, s_1=a_0+a_1+a_2),\ (p_1=b_0+b_1+b_2, q_1=b_0+b_1+b_2, s_1=r),$$
$$(p_1=b_0+b_1+b_2, q_1=r, s_1=b_0+b_1+b_2),\ (p_1=r, q_1=b_0+b_1+b_2, s_1=b_0+b_1+b_2),$$

where $r \in H_1$ and these cases are clear by the construction of I.
Case 2:

- If $(p_1=a_0, q_1=b_1, s_1=a_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_0b_1a_0 = a_0b_1a_0 + a_1b_0a_0 = (a_0b_1+a_1b_0)a_0 \in I$ and $a_0a_0b_1 \in I$.
- If $(p_1=b_0, q_1=a_1, s_1=b_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_0a_1b_0 = b_0a_1b_0 + b_1a_0b_0 = (b_0a_1+b_1a_0)b_0 \in I$ and $b_0b_0a_1 \in I$.

- If $(p_1 = b_2, q_1 = a_1, s_1 = b_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_2a_1b_2 = b_2a_1b_2 + b_2a_2b_1 = b_2(a_1b_2 + a_2b_1) \in I$ and $b_2b_2a_1 \in I$.
- If $(p_1 = a_2, q_1 = b_1, s_1 = a_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_2b_1a_2 = a_2b_1a_2 + a_2b_2a_1 = a_2(b_1a_2 + b_2a_1) \in I$ and $a_2a_2b_1 \in I$.
- If $(p_1 = b_0, q_1 = a_1, s_1 = a_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_0a_1a_0 = b_0a_1a_0 + b_0a_0a_1 = b_0(a_0a_1 + a_1a_0) \in I$ and $b_0a_0a_1 \in I$.
- If $(p_1 = a_0, q_1 = a_1, s_1 = b_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_0a_1b_0 = a_0a_1b_0 + a_1a_0b_0 = (a_0a_1 + a_1a_0)b_0 \in I$ and $a_0b_0a_1 \in I$.
- If $(p_1 = a_0, q_1 = b_1, s_1 = b_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_0b_1b_0 = a_0b_1b_0 + a_0b_0b_1 = a_0(b_0b_1 + b_1b_0) \in I$ and $a_0b_0b_1 \in I$.
- If $(p_1 = b_0, q_1 = b_1, s_1 = a_0)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_0b_1a_0 = b_0b_1a_0 + b_1b_0a_0 = (b_0b_1 + b_1b_0)a_0 \in I$ and $b_0a_0b_1 \in I$.
- If $(p_1 = a_2, q_1 = a_1, s_1 = b_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_2a_1b_2 = a_2a_1b_2 + a_1a_2b_2 = (a_1a_2 + a_2a_1)b_2 \in I$ and $a_2b_2a_1 \in I$.
- If $(p_1 = b_2, q_1 = a_1, s_1 = a_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_2a_1a_2 = b_2a_1a_2 + b_2a_2a_1 = b_2(a_1a_2 + a_2a_1) \in I$ and $b_2a_2a_1 \in I$.
- If $(p_1 = a_2, q_1 = b_1, s_1 = b_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $a_2b_1b_2 = a_2b_1b_2 + a_2b_2b_1 = a_2(b_1b_2 + b_2b_1) \in I$ and $a_2b_2b_1 \in I$.
- If $(p_1 = b_2, q_1 = b_1, s_1 = a_2)$, then $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$, since $b_2b_1a_2 = b_2b_1a_2 + b_1b_2a_2 = (b_1b_2 + b_2b_1)a_2 \in I$ and $b_2a_2b_1 \in I$.

Thus, we obtain that $p_1q_1s_1 \in I$ implies $p_1s_1q_1 \in I$ for each case and it proves our claim. Now let $p = p_1+p_2+p_3+p_4,\ q = q_1+q_2+q_3+q_4,\ s = s_1+s_2+s_3+s_4$ with $p_i, q_i, s_i \in H_i$ for $i = 1, 2, 3$ and $p_4, q_4, s_4 \in I$. Let $pqs \in I$, where $p, q, s \in A$. We want to see that R is symmetric. Since each monomial of degree ≥ 4 is contained in I, then we have $pqs = p_1q_1s_1 + s' \in I$, where $s' \in I$. Hence, $p_1q_1s_1 \in I$ and $p_1s_1q_1 \in I$. Therefore, we have $psq \in I$ as required.

Recall that an ideal I of R is called an *α-ideal* if $\alpha(I) \subseteq I$, where α is an endomorphism of R.

Definition ([12, Definition 1.1])**.** Let R be a ring with an automorphism α. For an α-ideal I of R, I is called *strongly α-semiprime ideal* of R if $aR\alpha(a) \subseteq I$ implies $a \in I$ for any $a \in R$. R is called a *strongly α-semiprime* ring if the zero ideal is strongly α-semiprime.

For an automorphism α of R, every α-rigid ring is strongly α-semiprime. Also, recall that any α-rigid ring is strongly α-symmetric. The following proposition shows when strongly α-symmetric rings are α-rigid.

Proposition 2.13. *Let R be a ring with an automorphism α. Then R is α-rigid if and only if R is strongly α-semiprime and R is strongly α-symmetric.*

Proof. Suppose that R is strongly α-semiprime and strongly α-symmetric ring and let $a\alpha(a) = 0$, where $a \in R$. Now, consider the polynomials $p(x) = ax$ and $q(x) = a$ in $R[x;\alpha]$. Then we have $p(x)q(x)r(x) = 0$ for any $r(x) \in R[x;\alpha]$. Since R is strongly α-symmetric, we obtain $p(x)r(x)q(x) = 0$ for any $r(x) \in$

$R[x;\alpha]$. This implies that $aR\alpha(a)=0$ since α is onto. By using the fact that R is strongly α-semiprime, we get $a=0$. $\square$

The following examples show that the concepts of strongly α-symmetric ring and strongly α-semiprime ring are independent of each other.

Example 2.14. (1) Let F be a field with $\mathrm{char}(F)\neq 2$. Consider the ring $R=M_2(F)$ and let $\alpha: R\to R$ be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then it is proved in [12, Example 2.1] that R is strongly α-semiprime. On the other hand, for the polynomials $p(x)=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $q(x)=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}x$ and $r(x)=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in $R[x;\alpha]$, we have $p(x)q(x)r(x)=0$, but

$$p(x)r(x)q(x)=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}x=\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}x\neq O.$$

Hence, R is not strongly α-symmetric. Moreover, R is not α-rigid since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\alpha\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)=0$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\neq 0$. This example shows that in Theorem 2.13, being strongly α-symmetric is not superfluous.

(2) Consider the ring $R=\left\{\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a\in\mathbb{Z},\ t\in\mathbb{Q}\right\}$ and let α be the identity endomorphism of R. R is not strongly α-semiprime since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)=0$$

for any $a\in\mathbb{Z}$ and $t\in\mathbb{Q}$. On the other hand, suppose that $p(x)q(x)r(x)=0$, where $p(x)=\sum_{i=0}^{m}\begin{pmatrix} a_i & t_i \\ 0 & a_i \end{pmatrix}x^i$, $q(x)=\sum_{j=0}^{n}\begin{pmatrix} b_j & t'_j \\ 0 & b_j \end{pmatrix}x^j$ and $r(x)=\sum_{k=0}^{l}\begin{pmatrix} c_k & t''_k \\ 0 & c_k \end{pmatrix}x^k$ in $R[x;\alpha]$. Then we obtain

$$\begin{aligned}
0 &= p(x)q(x)r(x)\\
&= \sum_{t=0}^{m+n+l}\left(\sum_{t=s+k}\left(\sum_{s=i+j}\begin{pmatrix} a_i & t_i \\ 0 & a_i \end{pmatrix}\bar{\alpha}^i\begin{pmatrix} b_j & t'_j \\ 0 & b_j \end{pmatrix}\bar{\alpha}^s\begin{pmatrix} c_k & t''_k \\ 0 & c_k \end{pmatrix}\right)\right)x^t\\
&= \sum_{t=0}^{m+n+l}\sum_{t=s+k}\sum_{s=i+j}\begin{pmatrix} a_ib_jc_k & a_ib_jt''_k+a_it'_jc_k+t_ib_jc_k \\ 0 & a_ib_jc_k \end{pmatrix}x^t\\
&= p(x)r(x)q(x).
\end{aligned}$$

Therefore, R is strongly α-symmetric.

Corollary 2.15. *Let R be a ring with an automorphism α. Then the following statements are equivalent*:

(1) *R is strongly α-semiprime and strongly α-symmetric.*
(2) *R is α-rigid.*
(3) *$R[x, x^{-1}; \alpha]$ is reduced.*

Proof. (1) $\Leftrightarrow$ (2) is obtained by Theorem 2.13. It can be seen that (2) $\Leftrightarrow$ (3) follows from [28, Theorem 3]. □

By using Proposition 2.13, we obtain a generalization of the following result.

Corollary 2.16 ([14, Proposition 2.7(1)]). *R is reduced if and only if R is semiprime and R is symmetric.*

Proposition 2.17. *Let R be a ring with an endomorphism α and I_λ be an ideal of R with $\alpha(I_\lambda) \subseteq I_\lambda$ for all $\lambda \in \Lambda$. Let $\alpha_\lambda : R/I_\lambda \to R/I_\lambda$ be the induced endomorphism of R/I_λ. If R is a subdirect sum of strongly α_λ-symmetric rings for all $\lambda \in \Lambda$, then R is a strongly α-symmetric ring.*

Proof. Since R is a subdirect sum of strongly α_λ-symmetric rings, by [27, Theorem 3], we have R/I_λ is a strongly α_λ-symmetric ring for all $\lambda \in \Lambda$ and $\cap_{\lambda\in\Lambda} I_\lambda = 0$. Suppose that $p(x)q(x)r(x) = 0$, where $p(x) = \sum_{i=0}^{m} p_i x^i$, $q(x) = \sum_{j=0}^{n} q_j x^j$ and $r(x) = \sum_{k=0}^{l} r_k x^k$ in $R[x; \alpha]$. Then $\bar{p}(x)\bar{q}(x)\bar{r}(x) = \bar{0}$ in $\big(R/I_\lambda\big)[x; \alpha_\lambda]$ for all $\lambda \in \Lambda$. Since R/I_λ is strongly α_λ-symmetric for all $\lambda \in \Lambda$, we can deduce that $\bar{p}(x)\bar{r}(x)\bar{q}(x) = \bar{0}$. Then $\sum_{t=s+j} \sum_{s=i+k} p_i \alpha^i(r_k)\alpha^s(q_j) \in I_\lambda$ for all $\lambda \in \Lambda$. Since $\cap_{\lambda\in\Lambda} I_\lambda = 0$, we obtain that

$$\sum_{t=s+j} \sum_{s=i+k} p_i \alpha^i(r_k)\alpha^s(q_j) = 0.$$

Therefore, $p(x)r(x)q(x) = 0$ as required. □

Let α_i be an endomorphism of a ring R_i for each $i \in I$. Then the map $\alpha : \prod_{i\in I} R_i \to \prod_{i\in I} R_i$ defined by $\alpha((a_i)) = (\alpha_i(a_i))$, where $(a_i) \in \prod_{i\in I} R_i$ is an endomorphism. The proof of the following lemma can be obtained by routine computations.

Lemma 2.18. *Let R_i be a ring with an endomorphism α_i for each $i \in I$. Then the followings are equivalent*:

(i) *R_i is strongly α_i-symmetric for each $i \in I$.*
(ii) *The direct product $\prod_{i\in I} R_i$ is strongly α-symmetric.*
(iii) *The direct sum $\bigoplus_{i\in I} R_i$ is strongly α-symmetric.*

Recall that a ring R is called *local* if $R/J(R)$ is a division ring, where $J(R)$ denotes the Jacobson radical of R. R is called *semilocal* if $R/J(R)$ is semisimple Artinian and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$ (see [23] for more details). Also, note that local rings are abelian and semilocal.

Proposition 2.19. *Let R be a ring with an endomorphism α. Then we have the followings.*

(i) *R is a strongly α-symmetric and semiperfect ring if and only if $R = \bigoplus_{i=1}^{n} R_i$ such that R_i is local and strongly α_i-symmetric ring, where α_i is an endomorphism of R_i for all $i = 0, 1, \ldots, n$.*

(ii) *Let R be a ring and e be a central idempotent of R. Then eR and $(1-e)R$ are strongly α-symmetric if and only if R is strongly α-symmetric.*

Proof. (i) Suppose that R is strongly α-symmetric and semiperfect. Since R is semiperfect, by using [22, Corollary 3.7.2], R has a finite orthogonal set $\{e_1, \ldots, e_n\}$ of local idempotents whose sum is 1. Then $R = \sum_{i=1}^{n} e_i R$ such that $e_i R e_i$ is a local ring for all $i = 1, \ldots, n$. Since R is strongly α-symmetric, then R is abelian and $e_i R e_i = e_i R$. Also, by Lemma 2.8(iv), $\alpha(e_i R) \subseteq e_i R$ for all $i = 1, \ldots, n$. Let α_i be an endomorphism of $e_i R$ induced by α. Then $e_i R$ is strongly α_i-symmetric and local subring of R.

Conversely, let R be a finite direct sum of strongly α_i-symmetric local rings R_i for all $i = 0, 1, \ldots, n$. Since local rings are semiperfect and R is strongly α-symmetric by Lemma 2.18, then we get R is semiperfect.

(ii) It can be seen by using the fact $R \cong eR \oplus (1-e)R$ and Lemma 2.18. $\square$

3. Extensions of strongly α-symmetric rings

In this section, we investigate properties of strongly α-symmetric rings and their extensions. First, we deal with the polynomial extensions of strongly α-symmetric rings. Note that an endomorphism α of a ring R can be extended to an endomorphism $\bar{\alpha}$ of the polynomial ring $R[x]$, where

$$\bar{\alpha}\left(\sum_{i=0}^{m} a_i x^i\right) = \sum_{i=0}^{m} \alpha(a_i) x^i.$$

It is obvious that the polynomial rings over commutative (resp., reduced) rings are commutative (resp., reduced). But, this is not true for IFP, reversible and symmetric rings by [15, Example 2], [18, Example 2.1] and [14, Example 3.1], respectively.

Based on these results, one may ask whether a polynomial ring over a strongly α-symmetric ring is strongly $\bar{\alpha}$-symmetric. We remark that the idea of the following proof is similar to [10, Theorem 6].

Theorem 3.1. *Let R be a ring and let α be an endomorphism of R with $\alpha^t = I_R$ for some positive integer t, where I_R denotes the identity endomorphism of R. Then R is strongly α-symmetric if and only if $R[x]$ is strongly $\bar{\alpha}$-symmetric.*

Proof. It is enough to prove that $R[x]$ is strongly $\bar{\alpha}$-symmetric when R is strongly α-symmetric. Assume that

$$\begin{aligned} p(y) &= p_0 + p_1 y + \cdots + p_m y^m, \\ q(y) &= q_0 + q_1 y + \cdots + q_n y^n, \\ r(y) &= r_0 + r_1 y + \cdots + r_l y^l \end{aligned}$$

in $R[x][y;\bar{\alpha}]$ such that $p(y)q(y)r(y)=0$. We also let

$$\begin{aligned} p_i &= p_{i_0}+p_{i_1}x+\cdots+p_{i_{u_i}}x^{u_i},\\ q_j &= q_{j_0}+q_{j_1}x+\cdots+q_{j_{v_j}}x^{v_j},\\ r_k &= r_{k_0}+r_{k_1}x+\cdots+r_{k_{w_k}}x^{w_k} \end{aligned}$$

in $R[x]$, where $u_i, v_j, w_k \geq 0$ for each $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. We want to show that $p(y)r(y)q(y)=0$. Take a positive integer s such that

$$s > \max\{\deg p_i, \deg q_j, \deg r_k\}$$

for any i, j and k, where the degree is considered as polynomials in $R[x]$. Also, note that we assume the degree of zero polynomial is zero. Let

$$\begin{aligned} p(x^{ts+1}) &= p_0+p_1x^{ts+1}+\cdots+p_mx^{mts+m},\\ q(x^{ts+1}) &= q_0+q_1x^{ts+1}+\cdots+q_nx^{nts+n},\\ r(x^{ts+1}) &= r_0+r_1x^{ts+1}+\cdots+r_lx^{lts+l}. \end{aligned}$$

Then the set of coefficients of the p_i(resp., q_j and r_k) equals the set of coefficients of $p(x^{ts+1})$ (resp., $q(x^{ts+1})$ and $r(x^{ts+1})$). Since $p(y)q(y)r(y)=0$, x commutes with elements of R in the polynomial ring $R[x]$ and $\alpha^{ts}=I_R$, we get $p(x^{ts+1})q(x^{ts+1})r(x^{ts+1})=0 \in R[x,\alpha]$. Then $p(x^{ts+1})r(x^{ts+1})q(x^{ts+1})= 0 \in R[x,\alpha]$ since R is strongly α-symmetric. Thus, we obtain $p(y)r(y)q(y)=0$ as required. □

In the following example, we show that there exists a non-identity endomorphism α of a strongly α-symmetric ring R such that $\alpha^t=I_R$ for some positive integer t.

Example 3.2. Consider the ring

$$R=\left\{\begin{pmatrix} a & \bar{b}\\ 0 & a\end{pmatrix} \mid a\in\mathbb{Z},\ \bar{b}\in\mathbb{Z}_4\right\}$$

and let α be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & \bar{b}\\ 0 & a\end{pmatrix}\right)=\begin{pmatrix} a & -\bar{b}\\ 0 & a\end{pmatrix}$. By [13, Example 1.10], R is α-Armendariz and we also have R is symmetric since it is commutative. Then by using Theorem 2.6, we obtain R is strongly α-symmetric and $\alpha^2=I_R$.

Let R be a ring and u be an element of R. Recall that if $ur=0$ implies $r=0$ for $r\in R$, then the element u is called $r$ight regular and left regular elements is defined, similarly. An element is called *regular* if it is both left and right regular. Following [7], a ring R is called *left Ore* for given $a,b\in R$ with b regular, there exist $a_1, b_1\in R$ with b_1 regular such that $b_1a=a_1b$. In [26, Theorem 2.1.12], it is showed that the classical left quotient ring $S^{-1}R$ exists if and only if S is a left Ore set and the set $\bar{S}=\{s+ass(S)\in R/ass(S)\mid s\in S\}$ consists of regular elements, where $ass(S):=\{r\in R\mid sr=0 \text{ for some } s\in S\}$. In [14, Theorem 4.1]), it is proved that R is symmetric if and only if $Q(R)$ is symmetric. In the following theorem, we consider the classical left quotient rings of strongly α-symmetric rings. Let R be a ring with the classical left quotient ring $Q(R)$. Then each automorphism α of R can be extended to an endomorphism $\bar{\alpha}$ of $Q(R)$ with $\bar{\alpha}(b^{-1}a) = \alpha(b)^{-1}\alpha(a)$, where $a, b \in R$ with b regular.

Theorem 3.3. *Let R be a ring with an automorphism α. If the classical left quotient ring $Q(R)$ of R exists, then R is strongly α-symmetric if and only if $Q(R)$ is strongly $\bar{\alpha}$-symmetric.*

Proof. It is enough to prove that $Q(R)$ is strongly $\bar{\alpha}$-symmetric whenever R is strongly α-symmetric. Let $p(x) = \sum_{i=0}^{m} u_1^{-1} p_i x^i$, $q(x) = \sum_{j=0}^{n} v_1^{-1} q_j x^j$ and $r(x) = \sum_{k=0}^{l} w_1^{-1} r_k x^k \in Q(R)[x;\bar{\alpha}]$ such that $p(x)q(x)r(x) = 0$, where $p_i, q_j, r_k \in R$ and u, v, w are regular elements in R for $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le l$. Then we obtain

$$\begin{aligned} 0 &= p(x)q(x)r(x) \\ &= u_1^{-1}\Big(\sum_{i=0}^{m} p_i x^i v_1^{-1}\Big)\Big(\sum_{j=0}^{n} q_j x^j w_1^{-1}\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\ &= \Big(\sum_{i=0}^{m} p_i \alpha^i(v_1)^{-1} x^i\Big)\Big(\sum_{j=0}^{n} q_j \alpha^j(w_1)^{-1} x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big). \end{aligned}$$

There exist $p_i', q_j' \in R$ and regular elements $v_2, w_2 \in R$ such that

$$p_i \alpha^i(v_1)^{-1} = v_2^{-1} p_i', \tag{1}$$

$$q_j \alpha^j(w_1)^{-1} = w_2^{-1} q_j' \tag{2}$$

for $0 \le i \le m$ and $0 \le j \le n$. Thus, we have

$$\begin{aligned} 0 &= \Big(\sum_{i=0}^{m} v_2^{-1} p_i' x^i\Big)\Big(\sum_{j=0}^{n} w_2^{-1} q_j' x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\ &= v_2^{-1}\Big(\sum_{i=0}^{m} p_i' \alpha^i(w_2)^{-1} x^i\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big). \end{aligned}$$

There exist $p_i'' \in R$ and regular element $w_3 \in R$ such that

$$p_i' \alpha^i(w_2)^{-1} = w_3^{-1} p_i'' \tag{3}$$

for $0 \le i \le m$. Then we have $\Big(\sum_{i=0}^{m} p_i'' x^i\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) = 0.$ Using strongly α-symmetric property of R, we deduce that

$$\Big(\sum_{i=0}^{m} p_i'' x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big) = 0. \tag{4}$$

Since R is strongly α-IFP, we obtain that

$$
\begin{aligned}
0 &= \Big(\sum_{i=0}^{m} p_i'' x^i\Big) w_2 v_1 \Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big) \\
&= \Big(\sum_{i=0}^{m} p_i'' \alpha^i(w_2 v_1) x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big). \qquad (5)
\end{aligned}
$$

By multiplying (5) on the left hand side with $(w_3 v_2)^{-1}$ and using (1) and (3), we obtain

$$
\begin{aligned}
0 &= \Big(\sum_{i=0}^{m} (w_3 v_2)^{-1} p_i'' \alpha^i(w_2 v_1) x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big) \\
&= \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j' x^j\Big).
\end{aligned}
$$

Thus, we get

$$
\Big(\sum_{j=0}^{n} q_j' x^j\Big) w_1 \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) = 0 \qquad (6)
$$

since R is strongly α-skew reversible and strongly α-IFP. If we multiply (6) by w_2^{-1} on the left hand side and use (2), then we have

$$
\begin{aligned}
0 &= w_2^{-1}\Big(\sum_{j=0}^{n} q_j' \alpha^j(w_1) x^j\Big)\Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\
&= \Big(\sum_{j=0}^{n} q_j x^j\Big)\Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\
&= \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j x^j\Big). \qquad (7)
\end{aligned}
$$

By using (7) and similar arguments as above, we get $p(x)r(x)q(x) = 0$. Hence, $Q(R)$ is strongly $\bar{\alpha}$-symmetric. □

Let R be a ring and α be an endomorphism of R. The set of all central regular elements of R is denoted by Δ. It can be seen that Δ is a multiplicatively closed subset of R. Then the map

$$\bar{\alpha} : \Delta^{-1}R \to \Delta^{-1}R$$

defined by $\bar{\alpha}(u^{-1}r) = \alpha(u)^{-1}\alpha(r)$ is also an automorphism, where $r \in R$ and $u \in \Delta$.

In the following proposition, we are able to remove the condition in [14, Lemma 3.2] which states that "$\alpha(u) = u$ for any central regular element u".

And hence, we obtain a generalization of [14, Lemma 3.2] without any condition on the automorphism α.

Proposition 3.4. *Let R be a ring and α be an endomorphism of R. Then R is strongly α-symmetric if and only if $\Delta^{-1}R$ is strongly $\bar{\alpha}$-symmetric.*

Proof. Let $p(x)q(x)r(x) = 0$, where $p(x)=\sum_{i=0}^{m} u^{-1}p_i x^i$, $q(x)=\sum_{j=0}^{n} v^{-1}q_j x^j$ and $r(x) = \sum_{k=0}^{l} w^{-1}r_k x^k \in \Delta^{-1}R[x;\bar{\alpha}]$ with $p_i, q_j, r_k \in R$ and u, v, w are central regular elements in R for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. Then we have

$$\begin{aligned}
0 &= p(x)q(x)r(x)\\
&= \Big(\sum_{i=0}^{m} u^{-1}p_i x^i\Big)\Big(\sum_{j=0}^{n} v^{-1}q_j x^j\Big)\Big(\sum_{k=0}^{l} w^{-1}r_k x^k\Big)\\
&= u^{-1}\Big(\sum_{i=0}^{m} p_i\alpha^i(v)^{-1}x^i\Big)\Big(\sum_{j=0}^{n} q_j\alpha^j(w)^{-1}x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\\
&= u^{-1}\Big(\sum_{i=0}^{m} \alpha^i(v)^{-1}p_i x^i\Big)\Big(\sum_{j=0}^{n} \alpha^j(w)^{-1}q_j x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big).
\end{aligned}$$

There exist $p'_i, p'_j \in R$ and central regular elements $v_1, w_1 \in R$ such that

$$\alpha^i(v)^{-1}p_i = v_1^{-1}p'_i, \tag{8}$$

$$\alpha^j(w)^{-1}q_j = w_1^{-1}q'_j \tag{9}$$

for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Then we have

$$\begin{aligned}
0 &= u^{-1}v_1^{-1}\Big(\sum_{i=0}^{m} p'_i\alpha^i(w_1)^{-1}x^i\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\\
&= u^{-1}v_1^{-1}\Big(\sum_{i=0}^{m} \alpha^i(w_1)^{-1}p'_i x^i\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big).
\end{aligned}$$

There exist $p''_i \in R$ and central regular element $w_2 \in R$ such that

$$\alpha^i(w_1)^{-1}p'_i = w_2^{-1}p''_i \tag{10}$$

for all $0 \leq i \leq m$. Then we obtain

$$0 = u^{-1}v_1^{-1}w_2^{-1}\Big(\sum_{i=0}^{m} p''_i x^i\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big).$$

Hence, we get $\Big(\sum_{i=0}^{m} p''_i x^i\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) = 0$. Since R is strongly α-symmetric, we deduce that

$$0 = \Big(\sum_{i=0}^{m} p''_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big).$$

By using the fact that R is strongly α-IFP, we have

$$0 = \Big(\sum_{i=0}^{m} p''_i x^i\Big) v w_1 \Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big). \tag{11}$$

If we multiply (11) by $(w_2 v_1)^{-1}$ on the left hand side and if we use (8) and (10), we get

$$\begin{aligned} 0 &= (w_2 v_1)^{-1}\Big(\sum_{i=0}^{m} p''_i \alpha^i(v w_1) x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big) \\ &= \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q'_j x^j\Big). \end{aligned}$$

Since R is strongly α-skew reversible and strongly α-IFP, we obtain that

$$0 = \Big(\sum_{j=0}^{n} q'_j x^j\Big) w \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big). \tag{12}$$

By multiplying (12) with w_1^{-1} on the left hand side and using (9), we get

$$\begin{aligned} 0 &= w_1^{-1}\Big(\sum_{j=0}^{n} q'_j x^j\Big) w \Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\ &= \Big(\sum_{j=0}^{n} w_1^{-1} q'_j \alpha^j(w) x^j\Big)\Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big) \\ &= \Big(\sum_{j=0}^{n} q_j x^j\Big)\Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big). \end{aligned}$$

Then we have $\Big(\sum_{i=0}^{m} p_i x^i\Big)\Big(\sum_{k=0}^{l} r_k x^k\Big)\Big(\sum_{j=0}^{n} q_j x^j\Big) = 0$. Therefore, $\Delta^{-1}R$ is strongly $\bar{\alpha}$-symmetric. □

Recall that the ring of Laurent polynomials over a ring R consists of all formal sums $\sum_{i=k}^{n} p_i x^i$ with usual addition and multiplication, where $p_i \in R$ and k, n are (possibly negative) integers and denoted by $R[x, x^{-1}]$. Note that an endomorphism α of R can be extend an endomorphism $\bar{\alpha}$ of $R[x, x^{-1}]$ with $\bar{\alpha}(\sum_{i=k}^{n} p_i x^i) = \sum_{i=k}^{n} \alpha(p_i) x^i$.

Corollary 3.5. *Let R be a ring with an endomorphism α. Then $R[x]$ is strongly $\bar{\alpha}$-symmetric if and only if so is $R[x, x^{-1}]$.*

Proof. Let $\Delta = \{1, x, x^2, \ldots\}$. It can be easily seen that Δ is a multiplicatively closed subset of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$. By Proposition 3.4, it follows that $R[x, x^{-1}]$ is strongly $\bar{\alpha}$-symmetric. □

Therefore, we obtain a generalization of the following results.

Corollary 3.6 ([14, Lemma 3.2])**.** (1) *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then R is symmetric if and only if so is $\Delta^{-1}R$.*

(2) *For a ring R, $R[x]$ is symmetric if and only if so is $R[x; x^{-1}]$.*

Let R be a ring with a monomorphism α. Recall that the Jordan's construction of R by α is the minimal extension of R to which α extends as an automorphism. Let

$$A(R, \alpha) = \{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$$

be a subset of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. The multiplication is defined by $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Also, note that $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$ for each $j \geq 0$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following operations:

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)},$$

$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$$

for $r, s \in R$ and $i, j \geq 0$. Furthermore, $A(R, \alpha)$ is an extension of R by α and the map α can be extended to an automorphism $\bar{\alpha}$ of $A(R, \alpha)$ with $\bar{\alpha}(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$. In [17], Jordan proved that for any pair of (R, α) such an extension always exists and $A(R, \alpha)$ is called *Jordan extension of R by α*.

Proposition 3.7. *Let R be a ring with a monomorphism α. Then R is strongly α-symmetric if and only if $A(R, \alpha)$ is strongly $\bar{\alpha}$-symmetric.*

Proof. Suppose that R is strongly α-symmetric and let $p(y)q(y)r(y) = 0$, where $p(y) = \sum_{i=0}^m p_i y^i$, $q(y) = \sum_{j=0}^n q_j y^j$ and $r(y) = \sum_{k=0}^l r_k y^k$ in $A(R, \alpha)[y; \bar{\alpha}]$ such that $p_i = x^{-u_i}p'_i x^{u_i}$, $q_j = x^{-v_j}q'_j x^{v_j}$, $r_k = x^{-w_k}r'_k x^{w_k}$ for $p'_i, q'_j, r'_k \in R$ and $u_i, v_j, w_k \geq 0$ for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. Then we have

$$p_i = x^{-\mu}p''_i x^{\mu}, \ q_j = x^{-\mu}q''_j x^{\mu} \text{ and } r_k = x^{-\mu}r''_k x^{\mu}$$

for some $\mu \geq 0$, where $p''_i, q''_j, r''_k \in R$ for all $0 \leq i \leq m$, $0 \leq j \leq n$ and $0 \leq k \leq l$. Since $p(y)q(y)r(y) = 0$, we have

$$\sum_{t=0}^{m+n+l} \left(\sum_{t=s+k} \left(\sum_{s=i+j} p_i \bar{\alpha}^i(q_j)\bar{\alpha}^s(r_k) \right) \right) y^t = 0.$$

Thus

$$\sum_{t=s+k}\sum_{s=i+j} p_i\bar{\alpha}^i(q_j)\bar{\alpha}^s(r_k) = \sum_{t=s+k}\sum_{s=i+j} x^{-\mu}p_i''\alpha^i(q_j'')\alpha^s(r_k'')x^{\mu} = 0. \tag{13}$$

Let $p'(y) = \sum_{i=0}^{m} p_i'' y^i$, $q'(y) = \sum_{j=0}^{n} q_j'' y^j$ and $r'(y) = \sum_{k=0}^{l} r_k'' y^k \in R[x;\alpha]$. Then by (13), we get $p'(y)q'(y)r'(y) = 0$ and $p'(y)r'(y)q'(y) = 0$ since R is strongly α-symmetric. Therefore, $p(y)r(y)q(y) = 0$ and hence, $A(R,\alpha)$ is strongly $\bar{\alpha}$-symmetric. □

Let R be a ring with an endomorphism α and I be an ideal of R. The map $\bar{\alpha}$ defined by $\bar{\alpha}(r+I) = \alpha(r)+I$ is an endomorphism of R/I, where $r \in R$. Note that although R is a strongly α-symmetric ring, R/I need not be a strongly $\bar{\alpha}$-symmetric ring. Indeed, when we consider the ring R is the ring of quaternions with integer coefficients and α is a monomorphism of R, then R is a domain and so strongly α-symmetric; while for any odd prime integer q, we have $R/qR \cong Mat_2(\mathbb{Z}_q)$ by [7, Exercise 3A]. Notice that $Mat_2(\mathbb{Z}_q)$ is not strongly $\bar{\alpha}$-symmetric since it is not abelian and thus, the factor ring R/qR is not strongly $\bar{\alpha}$-symmetric.

Let S be a subset of the ring R and then the set $r_R(S) = \{c \in R \mid Sc = 0\}$ is called the *right annihilator* of S in R. The *left annihilator* of S in R, $l_R(S)$, is defined similarly.

Proposition 3.8. *Let R be a ring with an endomorphism α. Suppose that R is a strongly α-symmetric ring. If I is a one-sided annihilator in R and $\alpha(I) \subseteq I$, then R/I is strongly $\bar{\alpha}$-symmetric.*

Proof. Let $I = r_R(S)$ for some $S \subseteq R$. We write $\bar{R} = R/I$ and $\bar{r} = r+I$, where $r \in R$. We have R is symmetric and so, has IFP since R is strongly α-symmetric. Thus, by [30, Lemma 1.2], I is an ideal of R. Let $\bar{p}(x)\bar{q}(x)\bar{r}(x) = \bar{0}$, where $\bar{p}(x) = \sum_{i=0}^{m} \bar{p}_i x^i$, $\bar{q}(x) = \sum_{j=0}^{n} \bar{q}_j x^j$ and $\bar{r}(x) = \sum_{k=0}^{l} \bar{r}_k x^k$ in $\bar{R}[x;\bar{\alpha}]$. Then $p(x)q(x)r(x) \in I[x;\alpha]$ and so $Sp(x)q(x)r(x) = 0$. Since R is strongly α-symmetric, we obtain $Sp(x)r(x)q(x) = 0$ and this implies that $\bar{p}(x)\bar{r}(x)\bar{q}(x) = \bar{0}$. Therefore, R/I is strongly $\bar{\alpha}$-symmetric. The left annihilator case can be shown similarly. □

As a converse of Proposition 3.8, we give the following result.

Proposition 3.9. *Let R be a ring with an endomorphism α and I be an α-ideal of R. If R/I is a strongly $\bar{\alpha}$-symmetric ring and I is an α-rigid ring without identity, then R is strongly α-symmetric.*

Proof. Let's take $p(x), q(x), r(x) \in R[x;\alpha]$ such that $p(x)q(x)r(x) = 0$. Then $\bar{p}(x)\bar{q}(x)\bar{r}(x) = \bar{0}$ and we get $\bar{p}(x)\bar{r}(x)\bar{q}(x) = \bar{0}$ since R/I is strongly $\bar{\alpha}$-symmetric. Thus, $p(x)r(x)q(x) \in I[x,\alpha]$. By [10, Proposition 3], we have that $I[x;\alpha]$ is reduced and hence, symmetric. Also, by [23, Proposition 1], we get that all the possible products of $p(x), q(x)$ and $r(x)$ is in $I[x;\alpha]$. Then we obtain

$$(q(x)r(x)p(x))^2 = q(x)r(x)[p(x)q(x)r(x)]p(x) = 0$$

and so $q(x)r(x)p(x) = 0$ since $I[x;\alpha]$ is reduced. Thus, we have

$$\begin{aligned} 0 &= p(x)r(x)[q(x)r(x)p(x)]r(x)q(x) \\ &= [p(x)r(x)q(x)][r(x)p(x)r(x)q(x)] \\ &= [r(x)p(x)r(x)q(x)][p(x)r(x)q(x)] \end{aligned} \tag{14}$$

by using the fact that $I[x;\alpha]$ is symmetric and hence, reversible. If we multiply (14) on the right hand side by $p(x)$, then we obtain

$$\begin{aligned} 0 &= r(x)[p(x)r(x)q(x)p(x)r(x)q(x)p(x)] \\ &= [p(x)r(x)q(x)p(x)r(x)q(x)p(x)]r(x). \end{aligned} \tag{15}$$

If we multiply (15) on the right hand side by $q(x)$, then we get $(p(x)r(x)q(x))^3 = 0$ and hence, $p(x)r(x)q(x) = 0$. Therefore, R is strongly α-symmetric. □

As a consequence of Proposition 3.8 and Proposition 3.9, we give the following corollary.

Corollary 3.10. (1) [14, Proposition 3.5] *Let R be a symmetric ring and I be an ideal of R. If I is an annihilator in R, then R/I is symmetric.*

(2) [14, Proposition 3.6(1)] *Let R be a ring and I be a proper ideal of R. If R/I is symmetric and I is reduced* (*as a ring without identity*), *then R is symmetric.*

The following example shows that the condition on I in Proposition 3.9 is necessary.

Example 3.11. Let R be the ring $U_2(F)$, where F is a division ring and α be an automorphism of R is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. Then R is not strongly α-symmetric because R is not abelian. Also $A\alpha(A) = 0$, but for $0 \neq A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in I$. Thus, the α-ideal $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ of R is not α-rigid. In addition, the factor ring $R/I \cong F$ is reduced and $\bar{\alpha}$ is the identity map on R/I and hence, R/I is $\bar{\alpha}$-rigid. Therefore, R/I is strongly $\bar{\alpha}$-symmetric.

Let S be a commutative ring and R be an algebra over S. Following [5], the Dorroh extension of R by S is defined by $D = R \oplus S$ with multiplication $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. For an S-algebra homomorphism α of R, α can be extended to an S-algebra homomorphism $\bar{\alpha} : D \to D$ with $\bar{\alpha}((r, s)) = (\alpha(r), s)$.

Theorem 3.12. *Let S be a commutative domain, R be an algebra over S and α be an S-algebra homomorphism of R. Then R is strongly α-symmetric if and only if the Dorroh extension D of R by S is strongly $\bar{\alpha}$-symmetric.*

Proof. We assume that R is strongly α-symmetric. Suppose that $p(x)q(x)r(x) = 0$, where $p(x)=\sum_{i=0}^{m}(a_i, b_i)x^i$, $q(x)=\sum_{j=0}^{n}(c_j, d_j)x^j$ and $r(x)=\sum_{k=0}^{l}(e_k, f_k)x^k \in D[x;\bar{\alpha}]$. Then we obtain that

$$\begin{aligned}
0 &= \Big(\sum_{s=0}^{m+n}\Big(\sum_{s=i+j}(a_i, b_i)\bar{\alpha}^i(c_j, d_j)x^s\Big)\Big)\Big(\sum_{k=0}^{l}(e_k, f_k)x^k\Big)\\
&= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i, b_i)\bar{\alpha}^i(c_j, d_j)\bar{\alpha}^s(e_k, f_k)\Big)\Big)x^t\\
&= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i, b_i)(\alpha^i(c_j), d_j)(\alpha^s(e_k), f_k)\Big)\Big)x^t. \qquad (16)
\end{aligned}$$

Let $p'(x) = \sum_{i=0}^{m} b_i x^i$, $q'(x) = \sum_{j=0}^{n} d_j x^j$ and $r'(x) = \sum_{k=0}^{l} f_k x^k \in S[x]$. Then by (16), we get $p'(x)q'(x)r'(x) = 0$. Since $S[x]$ is a domain, either $p'(x) = 0$ or $q'(x) = 0$ or $r'(x) = 0$. Let $p'(x) = 0$. If we use the facts that R is an S-algebra and $\alpha^i(s) = s$ for each $s \in S$ and $i \in \mathbb{N}$, then the equation (16) becomes

$$\begin{aligned}
0 &= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i, 0)(\alpha^i(c_j), d_j)(\alpha^s(e_k), f_k)\Big)\Big)x^t\\
&= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i\alpha^i(c_j) + d_j a_i, 0)(\alpha^s(e_k), f_k)\Big)\Big)x^t\\
&= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i\alpha^i(c_j)\alpha^s(e_k) + d_j a_i\alpha^s(e_k) + f_k a_i\alpha^i(c_j)\\
&\qquad + f_k d_j a_i, 0)\Big)\Big)x^t\\
&= \sum_{t=0}^{m+n+l}\Big(\sum_{t=s+k}\Big(\sum_{s=i+j}(a_i\alpha^i(c_j)\alpha^s(e_k) + a_i\alpha^i(d_j)\alpha^s(e_k) + a_i\alpha^i(c_j)\alpha^s(f_k)\\
&\qquad + a_i\alpha^i(d_j)\alpha^s(f_k), 0)\Big)\Big)x^t.
\end{aligned}$$

Let $p''(x) = \sum_{i=0}^{m} a_i x^i$, $q''(x) = \sum_{j=0}^{n}(c_j + d_j)x^j$ and $r''(x) = \sum_{k=0}^{l}(e_k + f_k)x^k \in R[x;\alpha]$. Then $p''(x)q''(x)r''(x) = 0$. Since R is strongly α-symmetric, we have $p''(x)r''(x)q''(x) = 0$ and this implies that $p(x)r(x)q(x) = 0$. For the cases $q'(x) = 0$ and $r'(x) = 0$, the proof can be seen by using similar arguments. $\square$

Corollary 3.13 ([14, Proposition 4.2(2)])**.** *Let S be a commutative ring and R be an algebra over S. If S is domain and R is symmetric, then the Dorroh extension D is symmetric.*

In the following result, we give a criteria to transfer strongly α-symmetric property from one ring to another.

Proposition 3.14. *Let $\phi : R \to R'$ be a ring isomorphism. Then R is a strongly α-symmetric ring if and only if R' is a strongly $\phi\alpha\phi^{-1}$-symmetric ring.*

Proof. The proof is obvious since an isomorphism from $R[x;\alpha]$ to $S[x;\phi\alpha\phi^{-1}]$ can be defined via the isomorphism ϕ. □

We denote the ring of full matrices (resp., upper triangular matrices) of $n\times n$ type on R by $Mat_n(R)$ (resp., $U_n(R)$) for $n \geq 2$. The following rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R, i = 1, \ldots, n, j = 2, \ldots, n \right\}$$

and

$$V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{(i+1)(j+1)} \text{ for } i = 1, \ldots, n-2,\ j = 2, \ldots, n-1\}$$

are subrings of $Mat_n(R)$. It is easy to see that $V_n(R) \cong R[x]/(x^n)$, where (x^n) denotes the ideal of $R[x]$ generated by x^n. An endomorphism α of a ring R can be extended to an endomorphism $\bar{\alpha}$ of $D_n(R)$ (resp., $V_n(R)$) with $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. We use the same notation $\bar{\alpha}$ for the extension endomorphism of $D_n(R)$ and $V_n(R)$. It is known that $Mat_n(R)$ and $U_n(R)$ do not have IFP for $n \geq 2$ since $U_2(R)$ is not abelian. Also, note that by [18, Example 1.3], $D_n(R)$ does not have IFP for $n \geq 4$. Thus, $Mat_n(R)$ and $U_n(R)$ are not strongly $\bar{\alpha}$-IFP and so, are not strongly $\bar{\alpha}$-symmetric for $n \geq 2$. Similarly, $D_n(R)$ is not strongly $\bar{\alpha}$-symmetric for $n \geq 4$. In [2, Proposition 3.7], it is proved that $D_2(R)$ and $D_3(R)$ are strongly $\bar{\alpha}$-IFP whenever R is an α-rigid ring. Naturally, one might ask whether $D_2(R)$ and $D_3(R)$ are strongly $\bar{\alpha}$-symmetric when R is an α-rigid ring. In the following example, we eliminate the case for $D_3(R)$.

Example 3.15. Recall that $D_3(\mathbb{R})[x;\bar{\alpha}] \cong D_3(\mathbb{R}[x;\alpha])$. Consider the matrices

$$p = \begin{pmatrix} 1 & x^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ q = \begin{pmatrix} 0 & 3x^4 & x^5 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0 & 2x^2 & 3x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $D_3(\mathbb{R})[x;\bar{\alpha}]$. Then we have $pqr = 0$, but $prq \neq 0$. Thus, $D_3(\mathbb{R})$ is not strongly $\bar{\alpha}$-symmetric.

In the following theorem, it is proved that $n\times n$ upper triangular matrix ring over a strongly α-symmetric ring has a subring which is strongly $\bar{\alpha}$-symmetric.

Theorem 3.16. *Let R be a ring with an endomorphism α. If R is α-rigid, then $V_n(R)$ is strongly $\bar{\alpha}$-symmetric.*

Proof. Firstly, note that $V_n(R)[x;\bar{\alpha}] \cong V_n(R[x;\alpha])$ and hence, $p \in V_n(R)[x;\bar{\alpha}]$ can be written as $p = (p_1, p_2, \ldots, p_n)$ for some $p_i \in R[x;\alpha]$. We assume that $p(x)q(x)r(x) = 0$, where $p(x) = \sum_{i=0}^{m} A_i x^i = (p_1, p_2, \ldots, p_n)$, $q(x) = \sum_{j=0}^{n} B_j x^j = (q_1, q_2, \ldots, q_n)$ and $r(x) = \sum_{k=0}^{l} C_k x^k = (r_1, r_2, \ldots, r_n) \in V_n(R)[x;\bar{\alpha}]$, for $A_i = (a_{st}^{(i)})$, $B_j = (b_{uv}^{(j)})$, $C_k = (c_{zw}^{(k)}) \in V_n(R)$ for $1 \leq u, s, t, u, v, z, w \leq n$. In this case we obtain that the following system of equations in $R[x;\alpha]$:

$$p_1q_1r_1 = 0, \tag{17}$$

$$p_1q_1r_2 + p_1q_2r_1 + p_2q_1r_1 = 0, \tag{18}$$

$$p_1q_1r_3 + p_1q_2r_2 + p_1q_3r_1 + p_2q_1r_2 + p_2q_2r_1 + p_3q_1r_1 = 0, \tag{19}$$

$$\vdots$$

$$p_1q_1r_{n-1} + p_1q_2r_{n-2} + \cdots + p_1q_{n-1}r_1 + \cdots + p_{n-1}q_1r_1 = 0, \tag{20}$$

$$p_1q_1r_n + p_1q_2r_{n-1} + \cdots + p_1q_nr_1 + \cdots + p_nq_1r_1 = 0. \tag{21}$$

By the assumption, $R[x;\alpha]$ is reduced and therefore, $p(x)q(x) = 0$ requires $p(x)R[x;\sigma]\ q(x) = 0$ and $q(x)p(x) = 0$. Also, $p(x)q(x)^2 = 0$ implies $p(x)q(x) = 0$ for any $p(x), q(x) \in R[x;\alpha]$. Considering these facts and (17), we have $p_1r_1q_1 = 0$ and $r_1p_1q_1 = 0$. If the equality (18) is multiplied by q_1r_1 from the right hand side, then we have $p_2q_1r_1 = 0$ and $p_2r_1q_1 = 0$. Now, (18) becomes

$$p_1q_1r_2 + p_1q_2r_1 = 0. \tag{22}$$

If we multiply (22) on the right hand side by q_2r_1, we have $p_1q_2r_1 = 0$ and thus, $p_1q_1r_2 = 0$. Hence, we obtain $p_1r_1q_2 = 0$ and $p_1r_2q_1 = 0$. Similarly, If equality (19) is multiplied by q_1r_1 from the right hand side, then we get $p_3q_1r_1 = 0$ and $p_3r_1q_1 = 0$. So, (19) becomes

$$p_1q_1r_3 + p_1q_2r_2 + p_1q_3r_1 + p_2q_1r_2 + p_2q_2r_1 = 0. \tag{23}$$

If equality (23) is multiplied by q_2r_1 from the right hand side, then we have $p_2q_2r_1 = 0$ and $p_2r_1q_2 = 0$. Hence, (23) becomes

$$p_1q_1r_3 + p_1q_2r_2 + p_1q_3r_1 + p_2q_1r_2 = 0. \tag{24}$$

If we multiply (24) on the right hand side by q_1r_2, q_3r_1, q_2r_2, respectively, we obtain $p_2q_1r_2 = 0$, $p_1q_3r_1 = 0$, $p_1q_2r_2 = 0$ and $p_1q_1r_3 = 0$. Inductively, assume that $p_iq_jr_k = 0$, where $1 \leq i, j, k \leq n-1$ and $i+j+k = n+1$ for $n \geq 2$. If we multiply (21) on the right hand side by q_1r_1, we get $p_nq_1r_1 = 0$ and $p_nr_1q_1 = 0$. Then (21) becomes

$$p_1q_1r_n + p_1q_2r_{n-1} + \cdots + p_1q_nr_1 + \cdots + p_{n-1}q_2r_1 = 0. \tag{25}$$

If we multiply (25) on the right hand side by q_2r_1, we have $p_{n-1}q_2r_1 = 0$ and $p_{n-1}r_1q_2 = 0$. Continuing this procedure, by multiplying the equation by the appropriate q_jr_k on the right side, we get $p_iq_jr_k = 0$ and hence, $p_ir_kq_j = 0$, where $1 \leq i, j, k \leq n$ such that $i + j + k = n + 2$. Consequently, we get $p(x)r(x)q(x) = 0$ and therefore, $V_n(R)$ is strongly $\bar{\alpha}$-symmetric. $\square$

By Theorem 3.16, it may be asked whether $V_n(R)$ is strongly $\bar{\alpha}$-symmetric for a strongly α-symmetric ring R. But, the following example shows that this statement is not correct.

Example 3.17. Consider

$$R = \left\{ \begin{pmatrix} a & \bar{b} \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z},\ \bar{b} \in \mathbb{Z}_4 \right\}$$

and let α be an endomorphism of R defined by $\alpha\left(\begin{pmatrix} a & \bar{b} \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -\bar{b} \\ 0 & a \end{pmatrix}$. By Example 3.2, we know that R is strongly α-symmetric. For

$$A = \begin{pmatrix} \begin{pmatrix} 1 & \bar{0} \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & \bar{0} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & \bar{0} \\ 0 & 1 \end{pmatrix} \end{pmatrix}, B = \begin{pmatrix} \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & \bar{1} \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$C = \begin{pmatrix} \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & \bar{1} \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \bar{1} \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

in $V_2(R)[x; \bar{\alpha}]$ we have $ABC = 0$, but

$$ACB = \begin{pmatrix} \begin{pmatrix} 0 & \bar{0} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \bar{2} \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \bar{0} \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Therefore, $V_2(R)$ is not strongly $\bar{\alpha}$-symmetric.

Corollary 3.18. *Let R be a ring and α be an endomorphism of R. If R is α-rigid, then $R[x]/(x^n)$ is strongly $\bar{\alpha}$-symmetric, where (x^n) denotes the ideal of $R[x]$ generated by x^n.*

Proof. It is clear since $R[x]/(x^n) \cong V_n(R)$. $\square$

Let R be a ring and let M be an (R, R)-bimodule. The ring $T(R, M) = R \oplus M$ is called the *trivial extension* of R by M with the componentwise addition and the multiplication defined as following:

$$(r, m)(r', m') = (rr', rm' + mr').$$

It is easy to see that this ring is isomorphic to the ring $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ with the usual matrix operations. An endomorphism α of the ring R can be extended to an endomorphism $\bar{\alpha}$ of $T(R,R)$ by $\bar{\alpha}\left(\begin{pmatrix} r & r' \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \alpha(r) & \alpha(r') \\ 0 & \alpha(r) \end{pmatrix}$, where $r, r' \in R$. Also, note that $T(R,0) \cong R$.

Corollary 3.19. *Let R be a ring with an endomorphism α. If R is an α-rigid ring, then $T(R,R)$ is strongly $\bar{\alpha}$-symmetric.*

Proof. It is clear by Theorem 3.16, since $T(R,R) \cong V_2(R)$. $\square$

If we consider Example 2.5(2), it may be seen that the converse of the Corollary 3.19 is not true. On the other hand, we obtain a generalization of the following results by using Theorem 3.16.

Corollary 3.20 ([14, Theorem 2.3])**.** *Let R be a reduced ring. Then $R[x]/(x^n)$ is symmetric, where (x^n) denotes the ideal of $R[x]$ generated by x^n.*

Corollary 3.21 ([14, Corollary 2.4])**.** *If R is a reduced ring, then $T(R,R)$ is symmetric.*

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References

[1] D. D. Anderson and V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), no. 6, 2847–2852. https://doi.org/10.1080/00927879908826596

[2] M. Başer, B. Hıçyilmaz, F. Kaynarca, T. K. Kwak, and Y. Lee, *Insertion-of-factors-property on skew polynomial rings*, J. Korean Math. Soc. **52** (2015), no. 6, 1161–1178. https://doi.org/10.4134/JKMS.2015.52.6.1161

[3] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368. https://doi.org/10.1017/S0004972700042052

[4] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648. https://doi.org/10.1112/S0024609399006116

[5] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), no. 2, 85–88. https://doi.org/10.1090/S0002-9904-1932-05333-2

[6] A. Eltiyeb and E. Ayoub, *On strongly symmetric rings*, Inter. J. Math. Appl. **5** (2017), 61–65.

[7] K. R. Goodearl and R. B. Warfield Jr., *An introduction to noncommutative Noetherian rings*, second edition, London Mathematical Society Student Texts, 61, Cambridge Univ. Press, Cambridge, 2004. https://doi.org/10.1017/CBO9780511841699

[8] E. Hashemi and A. Moussavi, *Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar. **107** (2005), no. 3, 207–224. https://doi.org/10.1007/s10474-005-0191-1

[9] C. Y. Hong, N. K. Kim, and T. K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure Appl. Algebra **151** (2000), no. 3, 215–226. https://doi.org/10.1016/S0022-4049(99)00020-1

[10] C. Y. Hong, N. K. Kim, and T. K. Kwak, *On skew Armendariz rings*, Comm. Algebra **31** (2003), no. 1, 103–122. https://doi.org/10.1081/AGB-120016752

[11] C. Y. Hong, N. K. Kim, and Y. Lee, *Skew polynomial rings over semiprime rings*, J. Korean Math. Soc. **47** (2010), no. 5, 879–897. https://doi.org/10.4134/JKMS.2010.47.5.879

[12] C. Y. Hong and T. K. Kwak, *On minimal strongly prime ideals*, Comm. Algebra **28** (2000), no. 10, 4867–4878. https://doi.org/10.1080/00927870008827127
[13] C. Y. Hong, T. K. Kwak, and S. T. Rizvi, *Extensions of generalized Armendariz rings*, Algebra Colloq. **13** (2006), no. 2, 253–266. https://doi.org/10.1142/S100538670600023X
[14] C. Huh, H. K. Kim, N. K. Kim, and Y. Lee, *Basic examples and extensions of symmetric rings*, J. Pure Appl. Algebra **202** (2005), no. 1-3, 154–167. https://doi.org/10.1016/j.jpaa.2005.01.009
[15] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761. https://doi.org/10.1081/AGB-120013179
[16] H. L. Jin, F. Kaynarca, T. K. Kwak, and Y. Lee, *On commutativity of skew polynomials at zero*, Bull. Korean Math. Soc. **54** (2017), no. 1, 51–69. https://doi.org/10.4134/BKMS.b150623
[17] D. A. Jordan, *Bijective extensions of injective ring endomorphisms*, J. London Math. Soc. (2) **25** (1982), no. 3, 435–448. https://doi.org/10.1112/jlms/s2-25.3.435
[18] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223. https://doi.org/10.1016/S0022-4049(03)00109-9
[19] J. Krempa, *Some examples of reduced rings*, Algebra Colloq. **3** (1996), no. 4, 289–300.
[20] T. K. Kwak, *Extensions of extended symmetric rings*, Bull. Korean Math. Soc. **44** (2007), no. 4, 777–788. https://doi.org/10.4134/BKMS.2007.44.4.777
[21] T. K. Kwak, Y. Lee, and S. J. Yun, *The Armendariz property on ideals*, J. Algebra **354** (2012), 121–135. https://doi.org/10.1016/j.jalgebra.2011.12.019
[22] J. Lambek, *Lectures on rings and modules*, Blaisdell Publishing Co., Waltham, MA, 1966.
[23] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368. https://doi.org/10.4153/CMB-1971-065-1
[24] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002), no. 3, 311–318. https://doi.org/10.1016/S0022-4049(02)00070-1
[25] G. Marks, *A taxonomy of* 2*-primal rings*, J. Algebra **266** (2003), no. 2, 494–520. https://doi.org/10.1016/S0021-8693(03)00301-6
[26] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics (New York), John Wiley & Sons, Ltd., Chichester, 1987.
[27] N. H. McCoy, *Subdirect sums of rings*, Bull. Amer. Math. Soc. **53** (1947), 856–877. https://doi.org/10.1090/S0002-9904-1947-08867-4
[28] A. R. Nasr-Isfahani and A. Moussavi, *Skew Laurent polynomial extensions of Baer and p.p.-rings*, Bull. Korean Math. Soc. **46** (2009), no. 6, 1041–1050. https://doi.org/10.4134/BKMS.2009.46.6.1041
[29] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17. http://projecteuclid.org/euclid.pja/1195510144
[30] G. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60. https://doi.org/10.2307/1996398
[31] G. Yang and Z. K. Liu, *On strongly reversible rings*, Taiwanese J. Math. **12** (2008), no. 1, 129–136. https://doi.org/10.11650/twjm/1500602492

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