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STUDY OF QUOTIENT NEAR-RINGS WITH ADDITIVE MAPS

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Abstract. We consider $\mathcal{N}$ to be a 3-prime field and $\mathcal{P}$ to be a prime ideal of $\mathcal{N}$. In this paper, we study the commutativity of the quotient near-ring $\mathcal{N}/\mathcal{P}$ with left multipliers and derivations satisfying certain identities on P, generalizing some well-known results in the literature. Furthermore, an example is given to illustrate the necessity of our hypotheses.

1. Introduction

Throughout this paper, a left near-ring $\mathcal{N}$ is a triple $(\mathcal{N},+,.)$ with two binary operations " + " and "." such that (i) $(\mathcal{N},+)$ is a group (not necessarily abelian), (ii) $(\mathcal{N},.)$ is a semigroup, (iii) $c.(a+b)=c.a+c.b$ for all $a,b,c\in\mathcal{N}$. The multiplicative center of $\mathcal{N}$ named as $Z(\mathcal{N})$, $\mathcal{N}/\mathcal{P}$ is a quotient near-ring with the multiplicative center $Z(\mathcal{N}/\mathcal{P})$, where $\mathcal{P}$ is a 3-prime ideal of $\mathcal{N}$. Usually, $\mathcal{N}$ will be 3-prime, that is, will have the property that $x\mathcal{N}y=\{0\}$ for $x,y\in\mathcal{N}$ implies $x=0$ or $y=0$; and $\mathcal{N}$ is called 2-torsion free if $\mathcal{N}$ has no element of order 2. For any pair $x,y\in\mathcal{N}$, we write $[x,y]=xy-yx$ and $(x\circ y)=xy+yx$ to denote the commutator and anticommutator, respectively. A derivation on $\mathcal{N}$ is an additive endomorphism d of $\mathcal{N}$ such that $d(xy)=xd(y)+d(x)y$ for all $x,y\in\mathcal{N}$. An additive mapping $H:\mathcal{N}\to\mathcal{N}$ is said to be a left multiplier (resp. right multiplier) if $H(xy)=H(x)y$ (resp. $H(xy)=xH(y)$) for all $x,y\in\mathcal{N}$. Thereby, if H is both a left multiplier and a right multiplier, then H is called a multiplier of $\mathcal{N}$. In [15], S. Mouhssine and A. Boua defined a special derivation $\tilde{d}$ on $\mathcal{N}/\mathcal{P}$ by $\tilde{d}(\bar{x})=\overline{d(x)}$ for all $x\in\mathcal{N}$. Motivated by this new map, here we define a left multiplier $\tilde{H}$ on $\mathcal{N}/\mathcal{P}$ as follows: $\tilde{H}(\bar{x})=\overline{H(x)}$ for all $x\in\mathcal{N}$. A normal subgroup $\mathcal{P}$ of $(\mathcal{N},+)$ is called a left ideal (resp. a right ideal) if $\mathcal{P}\mathcal{N}\subseteq\mathcal{P}$ (resp. $(x+r)y-xy\in\mathcal{P}$ for all $x,y\in\mathcal{N}, r\in\mathcal{P}$), and if $\mathcal{P}$ is both a left ideal and a right ideal, then $\mathcal{P}$ is said to be an ideal of $\mathcal{N}$. According to Groenewald [14], an ideal $\mathcal{P}$ is 3-prime if for $a,b\in\mathcal{N}$, $a\mathcal{N}b\subseteq\mathcal{P}$ implies $a\in\mathcal{P}$ or $b\in\mathcal{P}$. Here we present an example for a near-ring $\mathcal{N}$ which is not a ring and admits a 3-prime ideal $\mathcal{P}$.

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Example 1.1. Let $\mathcal{N} = \{0, a, b, c, d, e, f, g\}$ and define the two laws " + " and "." by:

+	0	a	b	c	d	e	f	g
0	0	a	b	c	d	e	f	g
a	a	b	c	0	e	f	g	d
b	b	c	0	a	f	g	d	e
c	c	0	a	b	g	d	e	f
d	d	g	f	e	0	c	b	a
e	e	d	g	f	a	0	c	b
f	f	e	d	g	b	a	0	c
g	g	f	e	d	c	b	a	0

and

.	0	a	b	c	d	e	f	g
0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	a	0
b	0	b	0	b	0	b	b	0
c	0	c	0	c	0	c	c	0
d	d	d	d	d	d	d	d	d
e	d	e	d	e	d	e	e	d
f	d	f	d	f	d	f	f	d
g	d	g	d	g	d	g	g	d

Then, $\mathcal{P} = \{0, a, b, c\}$ is a 3-prime ideal of the near-ring $\mathcal{N}$.

During the last decades, many authors have studied the commutativity in prime rings and 3-prime near-rings admitting several types of additive mappings defined on these sets, namely automorphisms, derivations, generalized derivations, and semiderivations satisfying appropriate algebraic conditions on appropriate subsets of rings or near-rings (see for example, [1, 3, 6, 15, 16], etc). Recently, Ashraf et al. [6] proved that if a 3-prime near-ring $\mathcal{N}$ admits a nonzero derivation d satisfying $d([x, y]) - [d(x), y] = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring. Also, A. En-guady and A. Boua [13] studied the commutativity of near-rings admitting a left derivation d and a multiplier H satisfying $d([x, u]) - H([x, u]) = 0$ for all $u \in U$, $x \in \mathcal{N}$, where U is a Lie ideal of $\mathcal{N}$.

In this work we will extend and generalize several results existing in the literature (see, [2–6, 8–10]) in different directions by working in quotient near-rings instead of simple near-rings, and also by including other special type of maps.

2. Main results

This section is devoted to the study of the commutativity of a near-ring $\mathcal{N}/\mathcal{P}$ such that $\mathcal{N}$ is a near-ring admitting a derivation d and a left multiplier H satisfying the properties $d([x, y]) - H([x, y]) \in \mathcal{P}$, $d([x, y]) - [d(x), y]) \in \mathcal{P}$, $d(x \circ y) - H(x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, where $\mathcal{P}$ is a 3-prime ideal of $\mathcal{N}$. We begin with some well-known lemmas that are essential for developing the proofs of our main results.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(a) [7, Lemmas 1.3(i)] *If x is an element of $\mathcal{N}$ such that $\mathcal{N}x = \{0\}$ (resp. $x\mathcal{N} = \{0\}$), then $x = 0$.*
(b) [7, Lemmas 1.5)] *If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*
(c) [7, Theorem 2.1] *If $\mathcal{N}$ admits a nonzero derivation d for which $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*
(d) [12, Lemma 2] *Let d be a derivation on $\mathcal{N}$. If $x \in Z(\mathcal{N})$, then $d(x) \in Z(\mathcal{N})$.*

The following theorem generalizes the results [6, Theorem 1(i)] and [10, Theorem 2.2].

Theorem 2.2. *Let $\mathcal{P}$ be a prime ideal of a near-ring $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d and a left multiplier H for which $d(\mathcal{N}) \nsubseteq \mathcal{P}$ or $H(\mathcal{N}) \nsubseteq \mathcal{P}$, then the following assertions are equivalent*:

(i) $d([x,y]) - H([x,y]) \in \mathcal{P}$ *for all* $x, y \in \mathcal{N}$,
(ii) $d([x,y]) - [d(x), y] \in \mathcal{P}$ *for all* $x, y \in \mathcal{N}$,
(iii) $\mathcal{N}/\mathcal{P}$ *is a commutative ring.*

Proof. It is obvious that (iii)⇒(i) and (iii)⇒(ii). So, we need to prove that (i)⇒(iii) and (ii)⇒(iii).

(i)⇒(iii) By hypotheses given, we have $d([x,y]) - H([x,y]) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, which implies that

$$\tilde{d}([\bar{x},\bar{y}]) = \tilde{H}([\bar{x},\bar{y}]) \text{ for all } x, y \in \mathcal{N}. \tag{1}$$

We divide the proof into two cases:
Case 1: Suppose that $H(\mathcal{N}) \subseteq \mathcal{P}$, thereby obtaining $\tilde{d} \neq \bar{0}$ and (1) yields $\tilde{d}([\bar{x},\bar{y}]) = \bar{0}$ for all $x, y \in \mathcal{N}$. Substituting xy for y in the last equation and noting that $[\bar{x},\bar{x}\bar{y}] = \bar{x}[\bar{x},\bar{y}]$, we arrive at $\tilde{d}(\bar{x}) = \bar{0}$ or $\bar{x} \in Z(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{N}$. According to Lemma 2.1(d) and Lemma 2.1(c), we conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring.
Case 2: Assume that $H(\mathcal{N}) \nsubseteq \mathcal{P}$, in this case replacing $\bar{x}$ by $\bar{y}\bar{x}$ in (1), we find that $\bar{y}\tilde{d}([\bar{x},\bar{y}]) + \tilde{d}(\bar{y})[\bar{x},\bar{y}] - \tilde{H}(\bar{y})[\bar{x},\bar{y}] = \bar{0}$ for all $\bar{x},\bar{y} \in \mathcal{N}/\mathcal{P}$. Now, taking $[\bar{u},\bar{v}]$ instead of $\bar{y}$ in the last equation and invoking (1), we obtain

$$[\bar{u},\bar{v}]\tilde{H}([\bar{x},[\bar{u},\bar{v}]]) = \bar{0} \text{ for all } x, u, v \in \mathcal{N} \tag{2}$$

which yields

$$[\bar{u},\bar{v}]\tilde{H}(\bar{x})[\bar{u},\bar{v}] - [\bar{u},\bar{v}]\tilde{H}([\bar{u},\bar{v}])\bar{x} = \bar{0} \text{ for all } x, u, v \in \mathcal{N}. \tag{3}$$

Substituting $\bar{y}\bar{t}$ for $\bar{x}$ in (3), we obtain $[\bar{u},\bar{v}]\tilde{H}(\bar{y})\bar{t}[\bar{u},\bar{v}] = [\bar{u},\bar{v}]\tilde{H}([\bar{u},\bar{v}])\bar{y}\bar{t}$ for all $t, u, v, y \in \mathcal{N}$. Again, in view of (3), $[\bar{u},\bar{v}]\tilde{H}([\bar{u},\bar{v}])\bar{y} = [\bar{u},\bar{v}]\tilde{H}(\bar{y})[\bar{u},\bar{v}]$. Hence,

$$[\bar{u},\bar{v}]\tilde{H}(\bar{y})\bar{t}[\bar{u},\bar{v}] = [\bar{u},\bar{v}]\tilde{H}([\bar{u},\bar{v}])\bar{y}\bar{t} = [\bar{u},\bar{v}]\tilde{H}(\bar{y})[\bar{u},\bar{v}]\bar{t}$$

so that $[\bar{u},\bar{v}]\tilde{H}(\bar{y})\bar{t}[\bar{u},\bar{v}] - [\bar{u},\bar{v}]\tilde{H}(\bar{y})[\bar{u},\bar{v}]\bar{t} = \bar{0}$ for all $t, u, v, y \in \mathcal{N}$ which gives $[\bar{u},\bar{v}]\tilde{H}(\bar{y})[\bar{t},[\bar{u},\bar{v}]] = \bar{0}$ for all $t, u, v, y \in \mathcal{N}$. Putting $y = yz$, we infer that $[\bar{u},\bar{v}]\tilde{H}(\bar{y})\bar{z}[\bar{t},[\bar{u},\bar{v}]] = \bar{0}$ for all $t, u, v, y, z \in \mathcal{N}$. Consequently,

$$[\bar{u},\bar{v}]\tilde{H}(\bar{y})\mathcal{N}/\mathcal{P}[\bar{t},[\bar{u},\bar{v}]] = \{\bar{0}\} \text{ for all } t, u, v, y \in \mathcal{N}.$$

Since $\mathcal{N}/\mathcal{P}$ is 3-prime, the above relation yields that

$$[\bar{u},\bar{v}]\tilde{H}(\bar{y}) = \bar{0} \text{ or } [\bar{t},[\bar{u},\bar{v}]] = \bar{0} \text{ for all } t, u, v, y \in \mathcal{N}. \tag{4}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $[\bar{u}_0,\bar{v}_0]\tilde{H}(\bar{y}) = \bar{0}$ for all $y \in \mathcal{N}$. In particular, putting $y = [r,t]$ and invoking (1), we get

$$[\bar{u}_0,\bar{v}_0]\tilde{d}([\bar{r},\bar{t}]) = \bar{0} \text{ for all } y, r, t \in \mathcal{N}. \tag{5}$$

According to (2) and (1), we have $[\bar{u},\bar{v}]\tilde{d}([\bar{x},[\bar{u},\bar{v}]]) = \bar{0}$ for all $u,v,x \in \mathcal{N}$, so that

$$[\bar{u},\bar{v}]\tilde{d}(\bar{x}[\bar{u},\bar{v}]) = [\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}]\bar{x}) \text{ for all } u,v,x \in \mathcal{N}. \tag{6}$$

Hence, for all $u,v,x \in \mathcal{N}$ we have $[\bar{u},\bar{v}]\tilde{d}(\bar{x})[\bar{u},\bar{v}] + [\bar{u},\bar{v}]\bar{x}\tilde{d}([\bar{u},\bar{v}]) = [\bar{u},\bar{v}]^2\tilde{d}(\bar{x}) + [\bar{u},\bar{v}]\tilde{d}([\bar{u},\bar{v}])\bar{x}$. Replacing x by $[r,t]$ and $[u,v]$ by $[u_0,v_0]$ in the previous relation and using (5), we get

$$\Big([\bar{u}_0,\bar{v}_0]\bar{r}\bar{t} - [\bar{u}_0,\bar{v}_0]\bar{t}\bar{r}\Big)\tilde{d}([\bar{u}_0,\bar{v}_0]) = \bar{0} \text{ for all } r,t \in \mathcal{N}.$$

For $t = H(y)k$, we obtain

$$[\bar{u}_0,\bar{v}_0]\bar{r}\tilde{H}(\bar{y})\bar{k}\tilde{d}([\bar{u}_0,\bar{v}_0]) = \bar{0} \text{ for all } k,r,y \in \mathcal{N},$$

which reduces to

$$[\bar{u}_0,\bar{v}_0]\,\mathcal{N}/\mathcal{P}\,\tilde{H}(\bar{y})\,\mathcal{N}/\mathcal{P}\,\tilde{d}([\bar{u}_0,\bar{v}_0]) = \{\bar{0}\} \text{ for all } y \in \mathcal{N}.$$

By 3-primeness of $\mathcal{P}$, the latter relation shows that

$$[\bar{u}_0,\bar{v}_0] = \bar{0} \text{ or } \tilde{H}(\bar{y}) = \bar{0} \text{ or } \tilde{d}([\bar{u}_0,\bar{v}_0]) = \bar{0} \text{ for all } y \in \mathcal{N}.$$

As $H(\mathcal{N}) \nsubseteq \mathcal{P}$, then $\tilde{H} \neq \bar{0}$ and hence (4) assures that

$$[\bar{u},\bar{v}] = \bar{0} \text{ or } \tilde{d}([\bar{u},\bar{v}]) = \bar{0} \text{ or } [\bar{u},\bar{v}] \in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u,v \in \mathcal{N}.$$

So that,

$$\tilde{d}([\bar{u},\bar{v}]) = \bar{0} \text{ or } [\bar{u},\bar{v}] \in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u,v \in \mathcal{N}. \tag{7}$$

Letting $\bar{a} = [\bar{u}_0,\bar{v}_0] \in Z(\mathcal{N}/\mathcal{P})$ and taking $x = ax$ in (1), we find that $\tilde{d}([\bar{a}\bar{x},\bar{y}]) = \tilde{H}([\bar{a}\bar{x},\bar{y}])$ for all $x,y \in \mathcal{N}$. By defining d and according to (1), we arrive at $\tilde{d}(\bar{a})[\bar{x},\bar{y}] = \bar{0}$ for all $x,y \in \mathcal{N}$. Left multiplying by $\bar{r}$, where $r \in \mathcal{N}$, we get $\tilde{d}(\bar{a})\bar{r}[\bar{x},\bar{y}] = \bar{0}$ for all $r,x,y \in \mathcal{N}$ which, in virtue of the 3-primeness of $\mathcal{N}/\mathcal{P}$, implies that

$$\tilde{d}(\bar{a}) = \bar{0} \text{ or } [\bar{x},\bar{y}] = \bar{0} \text{ for all } x,y \in \mathcal{N}. \tag{8}$$

If $\tilde{d}(\bar{a}) \neq \bar{0}$, then (8) shows that $\mathcal{N}/\mathcal{P}$ is a commutative ring. Otherwise, according to (7), we find that $\tilde{d}([\bar{u},\bar{v}]) = \bar{0}$ for all $u,v \in \mathcal{N}$, which gives that $\mathcal{N}/\mathcal{P}$ is a commutative ring by [11, Theorem 3.1] (it suffices to see that each derivation is a generalized derivation). Consequently, $\mathcal{N}/\mathcal{P}$ is a commutative ring in both cases.

(ii)⇒(iii) Suppose that $d([x,y]) - [d(x),y] \in \mathcal{P}$ for all $x,y \in \mathcal{N}$. This implies that

$$\tilde{d}([\bar{x},\bar{y}]) = [\tilde{d}(\bar{x}),\bar{y}] \text{ for all } x,y \in \mathcal{N}. \tag{9}$$

Replacing y by xy in (9), we get $\tilde{d}([\bar{x},\bar{x}\bar{y}]) = [\tilde{d}(\bar{x}),\bar{x}\bar{y}]$ for all $x,y \in \mathcal{N}$ which implies that $\bar{x}\tilde{d}([\bar{x},\bar{y}]) + \tilde{d}(\bar{x})[\bar{x},\bar{y}] = [\tilde{d}(\bar{x}),\bar{x}\bar{y}]$ for all $x,y \in \mathcal{N}$. Combining this result with (9), we obtain

$$\bar{x}[\tilde{d}(\bar{x}),\bar{y}] + \tilde{d}(\bar{x})[\bar{x},\bar{y}] = [\tilde{d}(\bar{x}),\bar{x}\bar{y}] \text{ for all } x,y \in \mathcal{N}.$$

This implies that, for all $x, y \in \mathcal{N}$, we have

$$\bar{x}\tilde{d}(\bar{x})\bar{y} - \bar{x}\bar{y}\tilde{d}(\bar{x}) + \tilde{d}(\bar{x})[\bar{x}, \bar{y}] = \tilde{d}(\bar{x})\bar{x}\bar{y} - \bar{x}\bar{y}\tilde{d}(\bar{x}). \tag{10}$$

On the other hand, replacing y by x in (9), we arrive at

$$\tilde{d}(\bar{x})\bar{x} = \bar{x}\tilde{d}(\bar{x}) \text{ for all } x \in \mathcal{N}. \tag{11}$$

Using (11) and (10), and after simplification we infer that

$$\tilde{d}(\bar{x})\bar{x}\bar{y} = \tilde{d}(\bar{x})\bar{y}\bar{x} \text{ for all } x, y \in \mathcal{N}.$$

Putting yz instead of y in the latter relation and using it again, we get

$$\tilde{d}(\bar{x})\, \mathcal{N}/\mathcal{P}\, [\bar{x}, \bar{z}] = \{\bar{0}\} \text{ for all } x, z \in \mathcal{N}. \tag{12}$$

In the light of the 3-primeness of $\mathcal{P}$, (12) gives

$$\tilde{d}(\bar{x}) = \bar{0} \text{ or } [\bar{x}, \bar{z}] = \bar{0} \text{ for all } x, z \in \mathcal{N},$$

which may be rewritten as

$$\tilde{d}(\bar{x}) = \bar{0} \text{ or } \bar{x} \in Z(\mathcal{N}/\mathcal{P}) \text{ for all } x \in \mathcal{N},$$

which forces that $\tilde{d}(\bar{x}) \in Z(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{N}$ by Lemma 2.1(d) and thus $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma 2.1(c). □

The next theorem generalizes the result [10, Theorem 2.4].

Theorem 2.3. *Let $\mathcal{N}$ be a 2-torsion near-ring and $\mathcal{P}$ be a prime ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d and a left multiplier H, for which $d(\mathcal{N}) \nsubseteq \mathcal{P}$ or $H(\mathcal{N}) \nsubseteq \mathcal{P}$, satisfying $d(x \circ y) - H(x \circ y) \in \mathcal{P}$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}/\mathcal{P}$ is a commutative ring with characteristic 2.*

Proof. By hypotheses given, we have

$$\tilde{d}(\bar{x} \circ \bar{y}) - \tilde{H}(\bar{x} \circ \bar{y}) = \bar{0} \text{ for all } x, y \in \mathcal{N}. \tag{13}$$

• Firstly, we discuss the case when $H(\mathcal{N}) \subseteq \mathcal{P}$. In this case, (13) becomes in the form $\tilde{d}(\bar{x} \circ \bar{y}) = \bar{0}$ for all $x, y \in \mathcal{N}$. Using the same arguments as those used in the proof of [17, Theorem 3.5], and taking into account the fact that d is a derivation, we arrive to the conclusion $\tilde{d}(\bar{x}) \in Z(\mathcal{N}/\mathcal{P})$ for all $x \in \mathcal{N}$ which implies that $\mathcal{N}/\mathcal{P}$ is a commutative ring by Lemma 2.1(c). Accordingly, for all $x, y, t \in \mathcal{N}$, we have

$$\begin{aligned} \tilde{d}(\bar{x} \circ \bar{y}\bar{t}) &= \bar{0} \\ &= \tilde{d}(\bar{x}(\bar{y}\bar{t} + \bar{y}\bar{t})) \\ &= \tilde{d}(\bar{x})(\bar{y}\bar{t} + \bar{y}\bar{t}) + \bar{x}\tilde{d}(\bar{y} \circ \bar{t}) = \tilde{d}(\bar{x})\bar{y}(\bar{t} + \bar{t}). \end{aligned}$$

In view of $\tilde{d} \neq \bar{0}$ and $\mathcal{N}/\mathcal{P}$ is 3-prime, the last result shows that $2\bar{t} = \bar{0}$ for all $t \in \mathcal{N}$, and hence N/P is a commutative ring of characteristic equal 2.

• Secondly, suppose that $H(\mathcal{N}) \nsubseteq \mathcal{P}$. Replacing y by xy in (13), we obtain $\tilde{d}(\bar{x}(\bar{x} \circ \bar{y})) - \tilde{H}(\bar{x}(\bar{x} \circ \bar{y})) = \bar{0}$ for all $x, y \in \mathcal{N}$. Again, substituting $u \circ v$ for x

in the last equation and applying (13), we arrive at $(\bar{u}\circ\bar{v})\tilde{H}((\bar{u}\circ\bar{v})\circ\bar{y})=\bar{0}$ for all $u,v,y\in\mathcal{N}$. So that

$$(\bar{u}\circ\bar{v})\tilde{H}(\bar{u}\circ\bar{v})\bar{y}=-(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})(\bar{u}\circ\bar{v})\text{ for all }u,v,y\in\mathcal{N}.$$

Now, taking yt instead of y, where $t\in\mathcal{N}$, in the latter expression and using it again, we infer that

$$\Big(-(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})(\bar{u}\circ\bar{v})\Big)\bar{t}=-(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})\bar{t}(\bar{u}\circ\bar{v})\text{ for all }u,v,y,t\in\mathcal{N}.$$

It follows that, $(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})\big(-(\bar{u}\circ\bar{v})\big)\bar{t}=(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})\bar{t}\big(-(\bar{u}\circ\bar{v})\big)$ for all $u,v,y,t\in\mathcal{N}$ in such a way that $(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})[-(\bar{u}\circ\bar{v}),\bar{t}]=\bar{0}$ for all $u,v,y,t\in\mathcal{N}$. Now, taking $y=yr$ the previous expression shows that $(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})\mathcal{N}/\mathcal{P}[-(\bar{u}\circ\bar{v}),\bar{t}]=\{\bar{0}\}$ for all $u,v,y,t\in\mathcal{N}$. In view of the 3-primeness of $\mathcal{N}/\mathcal{P}$, we find that

$$(\bar{u}\circ\bar{v})\tilde{H}(\bar{y})=\bar{0}\ \text{ or }\ -(\bar{u}\circ\bar{v})\in Z(\mathcal{N}/\mathcal{P})\ \text{ for all }\ u,v\in\mathcal{N}. \tag{14}$$

If there are two elements $u_0,v_0\in\mathcal{N}$ such that

$$(\bar{u}_0\circ\bar{v}_0)\tilde{H}(\bar{y})=\bar{0}\text{ for all }\ y\in\mathcal{N}. \tag{15}$$

Replacing y by $(u_0\circ v_0)\circ y$ in (15) and using (13), we get $(\bar{u}_0\circ\bar{v}_0)\tilde{d}\big((\bar{u}_0\circ\bar{v}_0)\circ\bar{y}\big)=\bar{0}$ for all $y\in\mathcal{N}$, means that $(\bar{u}_0\circ\bar{v}_0)\tilde{d}((\bar{u}_0\circ\bar{v}_0)\bar{y})=-(\bar{u}_0\circ\bar{v}_0)\tilde{d}(\bar{y}(\bar{u}_0\circ\bar{v}_0))$ for all $\bar{y}\in\mathcal{N}$. By property defining of d, we obtain $(\bar{u}_0\circ\bar{v}_0)\tilde{d}(\bar{u}_0\circ\bar{v}_0)\bar{y}+(\bar{u}_0\circ\bar{v}_0)^2\tilde{d}(\bar{y})=-(\bar{u}_0\circ\bar{v}_0)\tilde{d}(\bar{y})(\bar{u}_0\circ\bar{v}_0)-(\bar{u}_0\circ\bar{v}_0)\bar{y}\tilde{d}(\bar{u}_0\circ\bar{v}_0)$ for all $y\in\mathcal{N}$. Taking $r\circ s$ in the place of y, where $r,s\in\mathcal{N}$, we get

$$\begin{aligned}&(\bar{u}_0\circ\bar{v}_0)\tilde{d}(\bar{u}_0\circ\bar{v}_0)(\bar{r}\circ\bar{s})+(\bar{u}_0\circ\bar{v}_0)^2\tilde{d}(\bar{r}\circ\bar{s})\\ =\ &-(\bar{u}_0\circ\bar{v}_0)\tilde{d}(\bar{r}\circ\bar{s})(\bar{u}_0\circ\bar{v}_0)-(\bar{u}_0\circ\bar{v}_0)(\bar{r}\circ\bar{s})\tilde{d}(\bar{u}_0\circ\bar{v}_0).\end{aligned}$$

According to (13) and (15), the preceding relation gives

$$(\bar{u}_0\circ\bar{v}_0)(\bar{r}\circ\bar{s})\tilde{d}(\bar{u}_0\circ\bar{v}_0)=\bar{0}\ \text{ for all }\ r,s\in\mathcal{N}$$

in other words,

$$\Big((\bar{u}_0\circ\bar{v}_0)\bar{r}\bar{s}+(\bar{u}_0\circ\bar{v}_0)\bar{s}\bar{r}\Big)\tilde{d}(\bar{u}_0\circ\bar{v}_0)=\bar{0}\ \text{ for all }\ r,s\in\mathcal{N}.$$

Now, taking $r=H(y)$ and invoking (15), we obtain

$$(\bar{u}_0\circ\bar{v}_0)\bar{s}\tilde{H}(\bar{y})\tilde{d}(\bar{u}_0\circ\bar{v}_0)=\bar{0}\ \text{ for all }\ y,s\in\mathcal{N}$$

which reduces to $(\bar{u}_0\circ\bar{v}_0)\mathcal{N}/\mathcal{P}\,\tilde{H}(\bar{y})\tilde{d}(\bar{u}_0\circ\bar{v}_0)=\{\bar{0}\}$ for all $y\in\mathcal{N}$. Since $\mathcal{N}/\mathcal{P}$ is 3-prime, we conclude that

$$\bar{u}_0\circ\bar{v}_0=\bar{0}\ \text{ or }\ \tilde{H}(\bar{y})\tilde{d}(\bar{u}_0\circ\bar{v}_0)=\bar{0}\ \text{ for all }\ y\in\mathcal{N}.$$

Putting yt instead of y, where $t\in\mathcal{N}$, in the last equation and using the 3-primeness of $\mathcal{N}/\mathcal{P}$, we find that

$$\bar{u}_0\circ\bar{v}_0=\bar{0}\ \text{ or }\ \tilde{H}(\bar{y})=\bar{0}\ \text{ or }\ \tilde{d}(\bar{u}_0\circ\bar{v}_0)=\bar{0}\ \text{ for all }\ y\in\mathcal{N}.$$

Since $H\neq 0$, (14) reduces to

$$\tilde{d}(\bar{u}\circ\bar{v})=\bar{0}\ \text{ or }\ -(\bar{u}\circ\bar{v})\in Z(\mathcal{N}/\mathcal{P})\text{ for all }\ u,v\in\mathcal{N}. \tag{16}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $-(\bar{u}_0 \circ \bar{v}_0) \in Z(\mathcal{N}/\mathcal{P})$, in view of Lemma 2.1(d) we have $\tilde{d}(-(\bar{u}_0 \circ \bar{v}_0)) \in Z(\mathcal{N}/\mathcal{P})$. To simplify the notation, let's set $\bar{k} = -(\bar{u}_0 \circ \bar{v}_0)$; returning to the equation (13) and replacing x by kx, we obtain

$$\tilde{d}\big(\bar{k}(\bar{x} \circ \bar{y})\big) = \tilde{H}\big(\bar{k}(\bar{x} \circ \bar{y})\big) \text{ for all } x, y \in \mathcal{N}.$$

Using the definition of d and the property $\bar{k} \in Z(\mathcal{N}/\mathcal{P})$, we get $\tilde{d}\big(\bar{x} \circ \bar{y}\big)\bar{k} + \tilde{d}(\bar{k})(\bar{x} \circ \bar{y}) = \tilde{H}\big(\bar{x} \circ \bar{y}\big)\bar{k}$ for all $x, y \in \mathcal{N}$ which implies that $\tilde{d}(\bar{k})(\bar{x} \circ \bar{y}) = \bar{0}$ for all $x, y \in \mathcal{N}$. Left multiplying the latter relation by $\bar{r}$, where $r \in \mathcal{N}$, and in view of $\tilde{d}(\bar{k}) \in Z(\mathcal{N}/\mathcal{P})$, we conclude that

$$\tilde{d}(\bar{k})\, \mathcal{N}/\mathcal{P}\, (\bar{x} \circ \bar{y}) = \{\bar{0}\} \text{ for all } x, y \in \mathcal{N} \tag{17}$$

and hence by 3-primeness of $\mathcal{N}/\mathcal{P}$ we obtain $\tilde{d}(\bar{k}) = \bar{0}$ or $\bar{x} \circ \bar{y} = \bar{0}$ for all $x, y \in \mathcal{N}$. If the first condition is not verified, clearly the second condition implies that $\bar{x}\bar{y} = -\bar{y}\bar{x}$ for all $x, y \in \mathcal{N}$. Replacing y by yt, where $t \in \mathcal{N}$, we obtain $\bar{x}\bar{y}\bar{t} = \bar{y}(-\bar{x})\bar{t} = \bar{y}\bar{t}(-\bar{x})$ which means that $\bar{y}[\bar{t}, -\bar{x}] = \bar{0}$ for all $x, y, t \in \mathcal{N}$. It follows that $[\bar{t}, -\bar{x}]\bar{y}[\bar{t}, -\bar{x}] = 0$ and hence $[\bar{t}, -\bar{x}]\, \mathcal{N}/\mathcal{P}\, [\bar{t}, -\bar{x}] = \{\bar{0}\}$ for all $t, x \in \mathcal{N}$. In view of the 3-primeness of $\mathcal{N}/\mathcal{P}$, the last result shows that $\mathcal{N}/\mathcal{P}$ is a commutative ring. So, our condition that $\bar{x} \circ \bar{y} = \bar{0}$ yields $\bar{x}(\bar{y} + \bar{y}) = \bar{0}$ for all $x, y \in \mathcal{N}$. Substituting xr for x in the last result and in view of the 3-primeness of $\mathcal{N}/\mathcal{P}$ we conclude that $\mathcal{N}/\mathcal{P}$ is of characteristic 2. Now, suppose that $\tilde{d}(\bar{k}) = \bar{0}$ for all $k = -(\bar{u} \circ \bar{v}) \in Z(\mathcal{N}/\mathcal{P})$, then (16) yields $\tilde{d}(\bar{u} \circ \bar{v}) = \bar{0}$ for all $u, v \in \mathcal{N}$, and therefore in virtue of (13) we find that

$$\tilde{H}(\bar{u} \circ \bar{v}) = \bar{0} \text{ for all } u, v \in \mathcal{N}. \tag{18}$$

Replacing v by uv in (18), we get $\tilde{H}(\bar{u})(\bar{u} \circ \bar{v}) = \bar{0}$ which means that $\tilde{H}(\bar{u})\bar{u}\bar{v} = -\tilde{H}(\bar{u})\bar{v}\bar{u}$ for all $u, v \in \mathcal{N}$. Once again, taking vt instead of v, where $t \in \mathcal{N}$, in the last equation and using it, we arrive at

$$\tilde{H}(\bar{u}) = \bar{0} \text{ or } \bar{u} \in Z(\mathcal{N}/\mathcal{P}) \text{ for all } u \in \mathcal{N}. \tag{19}$$

Let u_0 be an arbitrary element of $\mathcal{N}$ such that $\tilde{H}(\bar{u}_0) = \bar{0}$, according to (18) and additivity of H we have

$$\begin{aligned} \bar{0} &= \tilde{H}(\bar{u}_0 \circ \bar{v}\bar{k}) \\ &= \tilde{H}(\bar{u}_0)\bar{v}\bar{k} + \tilde{H}(\bar{v}\bar{k})\bar{u}_0 \\ &= \tilde{H}(\bar{v})\bar{k}\bar{u}_0 \text{ for all } v, k \in \mathcal{N}. \end{aligned}$$

Using the 3-primeness of $\mathcal{N}/\mathcal{P}$ together $\tilde{H} \neq \bar{0}$, we can conclude that $\bar{u}_0 = \bar{0}$ therefore, from (19) and Lemma 2.1(b), we conclude that $\mathcal{N}/\mathcal{P}$ is a commutative ring. Now, returning to (18), we can see that $\tilde{H}(\bar{u}\bar{t} \circ \bar{v}) = \tilde{H}(\bar{u})\bar{t}(\bar{v} + \bar{v}) = \bar{0}$ for all $u, v, t \in \mathcal{N}$. Consequently, $\mathcal{N}/\mathcal{P}$ is of characteristic 2 which completes the proof. □

The following example shows that the 3-primeness of $\mathcal{P}$ that we used in our results cannot be omitted.

Example 2.4. Consider $\mathcal{M}$ be an any left near-ring and let us define $\mathcal{N}, \mathcal{P}, d, H$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} \mid 0, r, s, t \in \mathcal{M} \right\}, \ \mathcal{P} = \left\{ \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, r \in \mathcal{M} \right\},$$

$$d \begin{pmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H \begin{pmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix}.$$

We can see that $\mathcal{P}$ is an ideal of the near-ring $\mathcal{N}$ which is not 3-prime, d is a derivation of $\mathcal{N}$ and H is a left multiplier of $\mathcal{N}$ which satisfies all identities of our theorems. Furthermore, $\mathcal{N}/\mathcal{P}$ is also a noncommutative ring.

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