Commun. Korean Math. Soc. **39** (2024), No. 2, pp. 331–343
https://doi.org/10.4134/CKMS.c230213
pISSN: 1225-1763 / eISSN: 2234-3024

NONLINEAR MIXED $*$-JORDAN TYPE n-DERIVATIONS ON $*$-ALGEBRAS

Raof Ahmad Bhat, Abbas Hussain Shikeh, and Mohammad Aslam Siddeeque

Abstract. Let $\mathfrak{R}$ be a $*$-algebra with unity I and a nontrivial projection P_1. In this paper, we show that under certain restrictions if a map $\Psi : \mathfrak{R} \to \mathfrak{R}$ satisfies

$$\begin{aligned}&\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n)\\ &= \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n\end{aligned}$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ and $S_i = I$ for all $i \in \{1, 2, \ldots, n-3\}$, where $n \geq 3$, then Ψ is an additive $*$-derivation.

1. Introduction

Throughout the text, we tacitly assume that $\mathfrak{R}$ is an associative $*$-algebra with unity over the field $\mathbb{C}$ of complex numbers. For $S, K \in \mathfrak{R}$, let $S \circ K = SK + KS$, $S \bullet K = SK + KS^*$ and $S \diamond K = S^*K + K^*S$ represent Jordan product, Jordan $*$-product and bi-skew Jordan product of S and K respectively. In a number of research disciplines, Jordan and Jordan $*$-product are becoming increasingly relevant and hence their study attracted the attention of numerous researchers (for reference see [5–11, 13, 16–20]). An additive map $\Psi : \mathfrak{R} \to \mathfrak{R}$ is termed as an additive derivation if $\Psi(SK) = \Psi(S)K + S\Psi(K)$ for all $S, K \in \mathfrak{R}$. Moreover, if $\Psi(S^*) = \Psi(S)^*$ holds for all $S \in \mathfrak{R}$, then Ψ is coined as an additive $*$-derivation. Let $\Psi : \mathfrak{R} \to \mathfrak{R}$ be a map (not necessarily additive). Then Ψ is called a nonlinear Jordan $*$-derivation if

$$\Psi(S \bullet K) = \Psi(S) \bullet K + S \bullet \Psi(K)$$

holds for all $S, K \in \mathfrak{R}$ and a nonlinear $*$-Jordan n-derivation if

$$\Psi(S_1 \bullet S_2 \bullet \cdots \bullet S_n) = \sum_{k=1}^{n} S_1 \bullet S_2 \bullet \cdots \bullet S_{k-1} \bullet \Psi(S_k) \bullet S_{k+1} \bullet \cdots \bullet S_{n-1} \bullet S_n$$

Received August 17, 2023; Revised October 9, 2023; Accepted November 3, 2023.
2020 *Mathematics Subject Classification.* 16W25, 47B47, 46K15.
Key words and phrases. $*$-algebra, $*$-derivation, mixed Jordan n-derivation.

holds for all $S_1, S_2, \dots, S_n \in \mathfrak{R}$, where $S_1 \bullet S_2 \bullet \cdots \bullet S_n = (\cdots((S_1 \bullet S_2) \bullet S_3) \cdots \bullet S_n)$ and $n \geq 2$ is any fixed integer. Moreover, for any fixed integer $n \geq 3$, Ψ is called a nonlinear mixed $*$-Jordan n-derivation if

$$\Psi(S_1 \circ S_2 \circ \cdots \circ S_{n-1} \diamond S_n) = \sum_{k=1}^{n} S_1 \circ S_2 \circ \cdots \circ S_{k-1} \circ \Psi(S_k) \circ S_{k+1} \circ \cdots \circ S_{n-1} \diamond S_n$$

holds for all $S_1, S_2, \dots, S_n \in \mathfrak{R}$, where $S_1 \circ S_2 \circ \cdots \diamond S_n = (\cdots((S_1 \circ S_2) \circ S_3) \cdots \diamond S_n)$ (for details see [4]). Throughout the text, a map (not necessary additive) $\Psi : \mathfrak{R} \to \mathfrak{R}$ is called a nonlinear mixed $(\diamond, \bullet)$-Jordan n-derivation if

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ and $S_i = I$ for all $i \in \{1, 2, \dots, n-3\}$.

From past few years, the study of Jordan product, Jordan $*$-product, bi-skew Jordan product, mixed Jordan product and mixed Jordan $*$-product have attracted the attention of many algebraists. For instance, Liang and Zhang [12] proved that on factor von Neumann algebras, every nonlinear mixed Lie triple derivation is an additive $*$-derivation. Zhou et al. [21] proved that on prime $*$-algebra, every nonlinear mixed Lie triple derivation is an additive $*$-derivation. Taghavi et al. [14] proved that on a prime $*$-algebra $\mathfrak{A}$, a map Ψ satisfying

$$\Psi([A, B]_*) = [\Psi(A), B]_* + [A, \Psi(B)]_*,$$

where $[A, B]_* = A^*B - B^*A$ for all $A, B \in \mathfrak{A}$ is additive. Moreover, they showed that if $\Psi(\alpha I)$ is self-adjoint for $\alpha \in \{1, i\}$, then Ψ is a $*$-derivation. Also in [15] Taghavi et al. proved that if a map Ψ preserves $*$-Lie n-derivation on a prime $*$-algebra $\mathfrak{A}$, then Ψ is additive. Moreover, if $\Psi(i\frac{I}{2})$ is self-adjoint, then Ψ is a $*$-derivation. Li et al. [11] proved that every nonlinear $*$-Jordan derivation is an additive $*$-derivation. Ferreira et al. [4] showed that every nonlinear mixed $*$-Jordan n-derivation is an additive $*$-derivation.

Inspired by the aforementioned works, we will investigate the structure of nonlinear mixed $(\diamond, \bullet)$-Jordan n-derivations on $*$-algebras. Precisely, we show that for any fixed integer $n \geq 3$, if a map $\Psi : \mathfrak{R} \to \mathfrak{R}$ satisfies

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ and $S_i = I$ for all $i \in \{1, 2, \dots, n-3\}$, then Ψ is additive. Furthermore, we will prove that if $\Psi(\alpha \mathrm{I})$ is self-adjoint for $\alpha \in \{i, 1\}$, then Ψ is an additive $*$-derivation.

2. Main result

Theorem 2.1. *Let $\mathfrak{R}$ be a unital $*$-algebra containing a nontrivial projection P_1 and satisfying*

$$S\mathfrak{R}P_1 = 0 \textit{ implies } S = 0 \tag{2.1}$$

and

$$S\mathfrak{R}(I - P_1) = 0 \textit{ implies } S = 0, \tag{2.2}$$

where $S \in \mathfrak{R}$. Suppose that a map $\Psi : \mathfrak{R} \to \mathfrak{R}$ satisfies

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ and $S_i = I$ for all $i \in \{1, 2, \ldots, n-3\}$, where $n \geq 3$. Then Ψ is additive. Moreover, if $\Psi(\alpha I)$ is self-adjoint for $\alpha \in \{i, 1\}$, then Ψ is an additive $*$-derivation.

Proof. Let $P_2 = I - P_1$ and $\mathfrak{R}_{ij} = P_i \mathfrak{R} P_j$ for $i, j = 1, 2$. By Peirce decomposition of $\mathfrak{R}$, we have $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$. Note that any $S \in \mathfrak{R}$ can be written as $S = S_{11} + S_{12} + S_{21} + S_{22}$, where $S_{ij} \in \mathfrak{R}_{ij}$ for $i, j = 1, 2$ (see [1–3]). Now to show the additivity of Ψ on $\mathfrak{R}$, we use the above partition on $\mathfrak{R}$ and establish some lemmas that will show that Ψ is additive on each $\mathfrak{R}_{ij}$ for $i, j = 1, 2$. Besides, the following multiplicative relations also hold:
(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}$ $(i, j, l = 1, 2)$.
(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$ $(k = 1, 2)$ if $j \neq k$. □

Thus Theorem 2.1 is a consequence of following lemmas.

Lemma 2.2. $\Psi(0) = 0$.

Proof. Trivially

$$\begin{aligned}\Psi(0) &= \Psi(I \diamond I \diamond \cdots \diamond I \diamond 0 \bullet 0) \\ &= \Psi(I) \diamond I \diamond \cdots \diamond I \diamond 0 \bullet 0 + \cdots + I \diamond I \diamond \cdots \diamond I \diamond 0 \bullet \Psi(0) \\ &= 0.\end{aligned}$$

□

Lemma 2.3. *Let $S_{12} \in \mathfrak{R}_{12}$ and $S_{21} \in \mathfrak{R}_{21}$. We have $\Psi(S_{12}+S_{21}) = \Psi(S_{12}) + \Psi(S_{21})$.*

Proof. Let $M = \Psi(S_{12} + S_{21}) - \Psi(S_{12}) - \Psi(S_{21})$. Since

$$I \diamond I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet S_{12} = I \diamond I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet S_{21} = 0,$$

applying Lemma 2.2, we have

$$\begin{aligned}&\Psi(I) \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet (S_{12} + S_{21}) + \cdots + I \diamond \cdots \diamond I \diamond \Psi(P_2 - P_1) \\ &\bullet (S_{12} + S_{21}) + I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet \Psi(S_{12} + S_{21}) \\ &= \Psi(I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet (S_{12} + S_{21})) \\ &= \Psi(I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet S_{12}) + \Psi(I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet S_{21})\end{aligned}$$

$$= \Psi(I) \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet (S_{12} + S_{21}) + \cdots$$
$$+ I \diamond \cdots \diamond I \diamond \Psi(P_2 - P_1) \bullet (S_{12} + S_{21})$$
$$+ I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet (\Psi(S_{12}) + \Psi(S_{21})).$$

From this, we have

$$I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet M = 0$$

which implies that $P_2M - P_1M + MP_2 - MP_1 = 0$. Multiplying both sides by P_1, we get $P_1MP_1 = 0$. Similarly, multiplying both sides by P_2, we get $P_2MP_2 = 0$.

Now, again $I \diamond \cdots \diamond P_1 \diamond S_{21} \bullet P_2 = 0$. So, we have

$$I \diamond \cdots \diamond P_1 \diamond (S_{12} + S_{21}) \bullet P_2 = I \diamond \cdots \diamond P_1 \diamond S_{12} \bullet P_2.$$

Utilizing Lemma 2.2, we have

$$\Psi(I) \diamond \cdots \diamond P_1 \diamond (S_{12} + S_{21}) \bullet P_2 + \cdots + I \diamond \cdots \diamond P_1 \diamond \Psi(S_{12} + S_{21}) \bullet P_2$$
$$+ I \diamond \cdots \diamond P_1 \diamond (S_{12} + S_{21}) \bullet \Psi(P_2)$$
$$= \Psi(I \diamond \cdots \diamond P_1 \diamond (S_{12} + S_{21}) \bullet P_2)$$
$$= \Psi(I \diamond \cdots \diamond P_1 \diamond S_{12} \bullet P_2) + \Psi(I \diamond \cdots \diamond P_1 \diamond S_{21} \bullet P_2)$$
$$= \Psi(I) \diamond \cdots \diamond P_1 \diamond (S_{12}+S_{21}) \bullet P_2 + \cdots + I \diamond \cdots \diamond P_1 \diamond (\Psi(S_{12})+\Psi(S_{21})) \bullet P_2$$
$$+ I \diamond \cdots \diamond P_1 \diamond (S_{12}+S_{21}) \bullet \Psi(P_2).$$

Hence, we get $I \diamond \cdots \diamond P_1 \diamond M \bullet P_2 = 0$, i.e., $P_1MP_2 + P_2M^*P_1 = 0$. Multiplying by P_1 from left, we get $P_1MP_2 = 0$. Similarly, we can show that $P_2MP_1 = 0$. Hence, $M = 0$, i.e., $\Psi(S_{12} + S_{21}) = \Psi(S_{12}) + \Psi(S_{21})$. □

Lemma 2.4. *For every $S_{11} \in \mathfrak{R}_{11}$, $S_{12} \in \mathfrak{R}_{12}$, $S_{21} \in \mathfrak{R}_{21}$ and $S_{22} \in \mathfrak{R}_{22}$, we have*

(i) $\Psi(S_{11} + S_{12} + S_{21}) = \Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{21})$.
(ii) $\Psi(S_{12} + S_{21} + S_{22}) = \Psi(S_{12}) + \Psi(S_{21}) + \Psi(S_{22})$.

Proof. (i) Let $M = \Psi(S_{11} + S_{12} + S_{21}) - \Psi(S_{11}) - \Psi(S_{12}) - \Psi(S_{21})$. Since $I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11} = 0$, using Lemma 2.3, we have

$$\Psi(I) \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{11} + S_{12} + S_{21}) + \cdots$$
$$+ I \diamond \cdots \diamond I \diamond \Psi(P_2) \bullet (S_{11} + S_{12} + S_{21})$$
$$+ I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet \Psi(S_{11} + S_{12} + S_{21})$$
$$= \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{11} + S_{12} + S_{21}))$$
$$= \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11}) + \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{12} + S_{21}))$$
$$= \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11}) + \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet S_{12})$$
$$+ \Psi(I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet S_{21})$$
$$= \Psi(I) \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{11} + S_{12} + S_{21})$$
$$+ \cdots + I \diamond \cdots \diamond I \diamond \Psi(P_2) \bullet (S_{11} + S_{12} + S_{21})$$

$$+ I \diamond \cdots \diamond P_2 \bullet (\Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{21})).$$

Hence, we get

$$I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet M = 0.$$

The above relation reduces to $P_2M + MP_2 = 0$. This implies that $P_2MP_2 = P_2MP_1 = P_1MP_2 = 0$.

Also, we have

$$I \diamond I \diamond \cdots \diamond I \diamond (P_1 - P_2) \bullet S_{12} = I \diamond I \diamond \cdots \diamond I \diamond (P_1 - P_2) \bullet S_{21} = 0.$$

Applying Lemma 2.2, it follows that

$$\begin{aligned}
&\Psi(I) \diamond \cdots \diamond (P_1 - P_2) \bullet (S_{11} + S_{12} + S_{21}) + \cdots \\
&+ I \diamond \cdots \diamond \Psi(P_1 - P_2) \bullet (S_{11} + S_{12} + S_{21}) \\
&+ I \diamond \cdots \diamond (P_1 - P_2) \bullet \Psi(S_{11} + S_{12} + S_{21}) \\
= {}& \Psi(I \diamond \cdots \diamond (P_1 - P_2) \bullet (S_{11} + S_{12} + S_{21})) \\
= {}& \Psi(I \diamond \cdots \diamond I \diamond (P_1 - P_2) \bullet S_{11}) + \Psi(I \diamond \cdots \diamond I \diamond (P_1 - P_2) \bullet S_{12}) \\
&+ \Psi(I \diamond \cdots \diamond (P_1 - P_2) \bullet S_{21}) \\
= {}& \Psi(I) \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet (S_{11} + S_{12} + S_{21}) + \cdots \\
&+ I \diamond \cdots \diamond \Psi(P_2 - P_1) \bullet (S_{11} + S_{12} + S_{21}) \\
&+ I \diamond \cdots \diamond (P_2 - P_1) \bullet (\Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{21})).
\end{aligned}$$

Solving this, we get

$$I \diamond \cdots \diamond I \diamond (P_2 - P_1) \bullet M = 0$$

which implies that $P_2M - P_1M + MP_2 - MP_1 = 0$. Multiplying both sides by P_1, we get $P_1MP_1 = 0$. Hence $M = 0$. In a similar way, we can prove the other part also. □

Lemma 2.5. *For any $S_{ij} \in \mathfrak{R}_{ij}$, $1 \le i, j \le 2$, we have*

$$\Psi(\sum_{i,j=1}^{2} S_{ij}) = \sum_{i,j=1}^{2} \Psi(S_{ij}).$$

Proof. Let $M = \Psi(S_{11} + S_{12} + S_{21} + S_{22}) - \Psi(S_{11}) - \Psi(S_{12}) - \Psi(S_{21}) - \Psi(S_{22})$. Since $I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11} = 0$, invoking Lemma 2.4, we have

$$\begin{aligned}
&\Psi(I) \diamond \cdots \diamond P_2 \bullet (S_{11} + S_{12} + S_{21} + S_{22}) + \cdots \\
&+ I \diamond \cdots \diamond \Psi(P_2) \bullet (S_{11} + S_{12} + S_{21} + S_{22}) \\
&+ I \diamond \cdots \diamond P_2 \bullet \Psi(S_{11} + S_{12} + S_{21} + S_{22}) \\
= {}& \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{11} + S_{12} + S_{21} + S_{22})) \\
= {}& \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet (S_{12} + S_{21} + S_{22})) + \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11}) \\
= {}& \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet S_{11}) + \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet S_{12}) \\
&+ \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet S_{21}) + \Psi(I \diamond \cdots \diamond I \diamond P_2 \bullet S_{22})
\end{aligned}$$

$$
\begin{aligned}
&= \Psi(I) \diamond \cdots \diamond I \diamond P_2 \bullet (S_{11} + S_{12} + S_{21} + S_{22}) + \cdots \\
&\quad + I \diamond \cdots \diamond I \diamond \Psi(P_2) \bullet (S_{11} + S_{12} + S_{21} + S_{22}) \\
&\quad + I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet (\Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{21}) + \Psi(S_{22})).
\end{aligned}
$$

Hence, $I \diamond I \diamond \cdots \diamond I \diamond P_2 \bullet M = 0$. This implies that $P_1MP_2 = P_2MP_1 = P_2MP_2 = 0$. Similarly, we can show that $P_1MP_1 = 0$. Thus $M = 0$, i.e., $\Psi(S_{11} + S_{12} + S_{21} + S_{22}) = \Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{21}) + \Psi(S_{22})$. □

Lemma 2.6. *For any $S_{ij}, K_{ij} \in \mathfrak{R}_{ij}$ with $i \neq j$, $\Psi(S_{ij} + K_{ij}) = \Psi(S_{ij}) + \Psi(K_{ij})$.*

Proof. Let $N = \Psi(S_{ij} + K_{ij}) - \Psi(S_{ij}) - \Psi(K_{ij})$. Since

$$I \diamond I \diamond \cdots \diamond I \diamond (\frac{P_i + S_{ij} + S_{ij}^*}{2^{n-2}}) \bullet (P_j + K_{ij}) = K_{ij} + S_{ij} + S_{ij}^* K_{ij} + S_{ij}^* + K_{ij} S_{ij}^*,$$

invoking Lemmas 2.2-2.5, we get

$$
\begin{aligned}
&\Psi(S_{ij} + K_{ij}) + \Psi(S_{ij}^* K_{ij}) + \Psi(S_{ij}^*) + \Psi(K_{ij} S_{ij}^*) \\
&= \Psi(I \diamond \cdots \diamond I \diamond (\frac{P_i + S_{ij} + S_{ij}^*}{2^{n-2}}) \bullet (P_j + K_{ij})) \\
&= \Psi(I) \diamond \cdots \diamond I \diamond (\frac{P_i + S_{ij} + S_{ij}^*}{2^{n-2}}) \bullet (P_j + K_{ij}) + \cdots \\
&\quad + I \diamond \cdots \diamond \Psi(\frac{P_i + S_{ij} + S_{ij}^*}{2^{n-2}}) \bullet (P_j + K_{ij}) \\
&\quad + I \diamond \cdots \diamond I \diamond (\frac{P_i + S_{ij} + S_{ij}^*}{2^{n-2}}) \bullet \Psi(P_j + K_{ij}) \\
&= \Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{P_i}{2^{n-2}} \bullet P_j) + \Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{P_i}{2^{n-2}} \bullet K_{ij}) \\
&\quad + \Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{S_{ij} + S_{ij}^*}{2^{n-2}} \bullet P_j) + \Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{S_{ij} + S_{ij}^*}{2^{n-2}} \bullet K_{ij}) \\
&= \Psi(K_{ij}) + \Psi(S_{ij} + S_{ij}^*) + \Psi(S_{ij}^* K_{ij} + K_{ij} S_{ij}^*) \\
&= \Psi(K_{ij}) + \Psi(S_{ij}) + \Psi(S_{ij}^*) + \Psi(S_{ij}^* K_{ij}) + \Psi(K_{ij} S_{ij}^*).
\end{aligned}
$$

Therefore, $\Psi(S_{ij} + K_{ij}) = \Psi(S_{ij}) + \Psi(K_{ij})$.

Lemma 2.7. *For any $S_{ii}, K_{ii} \in \mathfrak{R}_{ii}$, $1 \leq i \leq 2$, we have $\Psi(S_{ii} + K_{ii}) = \Psi(S_{ii}) + \Psi(K_{ii})$*

Proof. Let $Q = \Psi(S_{ii} + K_{ii}) - \Psi(S_{ii}) - \Psi(K_{ii})$. Since $I \diamond \cdots \diamond I \diamond P_j \bullet S_{ii} = I \diamond I \diamond \cdots \diamond I \diamond P_j \bullet K_{ii} = 0$, it follows that

$$
\begin{aligned}
&\Psi(I) \diamond \cdots \diamond I \diamond P_j \bullet (S_{ii} + K_{ii}) + \cdots + I \diamond \cdots \diamond I \diamond \Psi(P_j) \bullet (S_{ii} + K_{ii}) \\
&\quad + I \diamond \cdots \diamond I \diamond P_j \bullet \Psi(S_{ii} + K_{ii}) \\
&= \Psi(I \diamond \cdots \diamond I \diamond P_j \bullet (S_{ii} + K_{ii})) \\
&= \Psi(I \diamond \cdots \diamond I \diamond P_j \bullet S_{ii}) + \Psi(I \diamond \cdots \diamond I \diamond P_j \bullet K_{ii}) \\
&= \Psi(I) \diamond \cdots \diamond I \diamond P_j \bullet (S_{ii} + K_{ii}) + \cdots + I \diamond \cdots \diamond I \diamond \Psi(P_j) \bullet (S_{ii} + K_{ii})
\end{aligned}
$$

$$+ I \diamond \cdots \diamond I \diamond P_j \bullet (\Psi(S_{ii}) + \Psi(K_{ii})).$$

From this, we get

$$I \diamond I \diamond \cdots \diamond I \diamond P_j \bullet Q = 0.$$

Now simplifying this, we get $P_jQ + QP_j = 0$. Hence $P_iQP_j = P_jQP_i = P_jQP_j = 0$. Now, we have to only show that $P_iQP_i = 0$.

For any $C_{ij} \in \mathfrak{R}_{ij}$ with $i \neq j$ and applying Lemma 2.6, we have

$$\begin{aligned}
&\Psi(I) \diamond \cdots \diamond I \diamond C_{ij} \bullet (S_{ii} + K_{ii}) + \cdots + I \diamond \cdots \diamond I \diamond \Psi(C_{ij}) \bullet (S_{ii} + K_{ii}) \\
&\quad + I \diamond \cdots \diamond I \diamond C_{ij} \bullet \Psi(S_{ii} + K_{ii}) \\
&= \Psi(I \diamond \cdots \diamond I \diamond C_{ij} \bullet (S_{ii} + K_{ii})) \\
&= \Psi(I \diamond \cdots \diamond I \diamond C_{ij} \bullet S_{ii}) + \Psi(I \diamond \cdots \diamond I \diamond C_{ij} \bullet K_{ii}) \\
&= \Psi(I) \diamond \cdots \diamond I \diamond C_{ij} \bullet (S_{ii} + K_{ii}) + \cdots + I \diamond \cdots \diamond I \diamond \Psi(C_{ij}) \bullet (S_{ii} + K_{ii}) \\
&\quad + I \diamond \cdots \diamond I \diamond C_{ij} \bullet (\Psi(S_{ii}) + \Psi(K_{ii})).
\end{aligned}$$

Hence $I \diamond \cdots \diamond I \diamond C_{ij} \bullet Q = 0$. Solving this relation, we get $C_{ij}Q + C_{ij}^*Q + QC_{ij}^* + QC_{ij} = 0$, i.e., $C_{ij}Q_{ii} + C_{ij}^*Q_{ii} + Q_{ii}C_{ij}^* + Q_{ii}C_{ij} = 0$. Finally, we have $Q_{ii}C_{ij} = 0$. It follows from conditions (2.1) and (2.2) that $P_iQP_i = 0$. Thus, $Q = 0$. □

Lemma 2.8. Ψ *is additive.*

Proof. For every $S, K \in \mathfrak{R}$, we write $S = S_{11} + S_{12} + S_{21} + S_{22}$ and $K = K_{11} + K_{12} + K_{21} + K_{22}$. By using Lemmas 2.5-2.7, we get

$$\begin{aligned}
\Psi(S + K) &= \Psi(S_{11} + S_{12} + S_{21} + S_{22} + K_{11} + K_{12} + K_{21} + K_{22}) \\
&= \Psi(S_{11} + K_{11}) + \Psi(S_{12} + K_{12}) + \Psi(S_{21} + K_{21}) + \Psi(S_{22} + K_{22}) \\
&= \Psi(S_{11}) + \Psi(K_{11}) + \Psi(S_{12}) + \Psi(K_{12}) + \Psi(S_{21}) + \Psi(K_{21}) \\
&\quad + \Psi(S_{22}) + \Psi(K_{22}) \\
&= \Psi(S_{11} + S_{12} + S_{21} + S_{22}) + \Psi(K_{11} + K_{12} + K_{21} + K_{22}) \\
&= \Psi(S) + \Psi(K).
\end{aligned}$$

□

Lemma 2.9. $\Psi(I) = 0$.

Proof. Invoking Lemma 2.8, we have

$$\begin{aligned}
2^{n-1}\Psi(I) &= \Psi(I \diamond I \diamond \cdots \diamond I \bullet I) \\
&= \Psi(I) \diamond I \diamond \cdots \diamond I \bullet I + \cdots \\
&\quad + I \diamond I \diamond \cdots \diamond \Psi(I) \bullet I + I \diamond I \diamond \cdots \diamond I \bullet \Psi(I) \\
&= (n-1)2^{n-2}\{(\Psi(I) + \Psi(I)^*) \bullet I)\} + 2^{n-1}\Psi(I).
\end{aligned}$$

This implies that $\{\Psi(I) + \Psi(I)^*\} \bullet I = 0$. Hence, $\Psi(I) + \Psi(I)^* = 0$. Since by given hypothesis, $\Psi(I)$ is self adjoint, hence from last relation, we get $\Psi(I) = 0$. □

Lemma 2.10. *Ψ preserves '$*$', i.e., $\Psi(E^*) = \Psi(E)^*$ for all $E \in \mathfrak{R}$.*

Proof. Invoking Lemma 2.8, we have

$$\Psi(I \diamond I \diamond I \diamond \cdots \diamond I \diamond \frac{E}{2^{n-2}} \bullet I) = \Psi(E + E^*) = \Psi(E) + \Psi(E^*).$$

On the other way, using Lemmas 2.8 and 2.9, we have

$$\begin{aligned}\Psi(I \diamond I \diamond I \diamond \cdots \diamond I \diamond \frac{E}{2^{n-2}} \bullet I) &= I \diamond I \diamond \cdots \diamond \Psi(\frac{E}{2^{n-2}}) \bullet I \\ &= (2^{n-3}I) \diamond \Psi(\frac{E}{2^{n-2}}) \bullet I \\ &= \{\Psi(\frac{E}{2}) + \Psi(\frac{E}{2})^*\} \bullet I \\ &= \Psi(E) + \Psi(E)^*.\end{aligned}$$

Comparing the above two relations, we conclude that $\Psi(E^*) = \Psi(E)^*$. □

Lemma 2.11. (i) $\Psi(iI) = 0$.
(ii) $\Psi(-iI) = 0$*, where i is the imaginary unit.*

Proof. (i) $\Psi(iI) = 0$.
On one way, applying Lemma 2.2, we have

$$\Psi(I \diamond I \diamond I \diamond \cdots \diamond I \diamond iI \bullet \frac{I}{2^{n-2}}) = 0.$$

On other way, utilizing Lemmas 2.8-2.9, we have

$$\begin{aligned}\Psi(I \diamond I \diamond I \diamond \cdots \diamond I \diamond iI \bullet \frac{I}{2^{n-2}}) &= I \diamond I \diamond I \diamond \cdots \diamond I \diamond \Psi(iI) \bullet \frac{I}{2^{n-2}} \\ &= 2^{n-3} I \diamond \Psi(iI) \bullet \frac{I}{2^{n-2}} \\ &= \Psi(iI) + \Psi(iI)^*.\end{aligned}$$

From above two relations, we get

$$\Psi(iI) + \Psi(iI)^* = 0.$$

Now using given hypothesis, we get $\Psi(iI) = 0$.
Analogously, we can show that $\Psi(-iI) = 0$. □

Lemma 2.12. (i) $\Psi(iE) = i\Psi(E)$.
(ii) $\Psi(-iE) = -i\Psi(E)$*, where i is the imaginary unit.*

Proof. (i) On one hand, using Lemma 2.8, we have

$$\Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{iE}{2^{n-2}} \bullet iI) = \Psi(E)^* - \Psi(E).$$

On the other hand, using Lemmas 2.9 and 2.11, we have

$$\Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{iE}{2^{n-2}} \bullet iI) = i\Psi(iE) + i\Psi(iE)^*.$$

From the above two relations, we have

$$-\Psi(iE) - \Psi(iE)^* = i\Psi(E)^* - i\Psi(E). \tag{2.3}$$

Similarly, on one way, we have

$$\Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{E}{2^{n-2}} \bullet iI) = \Psi(iE) + \Psi(iE^*).$$

On other way, using Lemmas 2.9-2.11, we arrive at

$$\Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{E}{2^{n-2}} \bullet iI) = i\Psi(E) + i\Psi(E)^*.$$

From the last two relations, we have

$$\Psi(iE) - \Psi(iE)^* = i\Psi(E) + i\Psi(E)^*. \tag{2.4}$$

From (2.3) and (2.4), we get

$$\Psi(iE) = i\Psi(E).$$

Analogously, we can prove the other part also. □

Lemma 2.13. *Ψ is a derivation, i.e., $\Psi(EK) = \Psi(E)K + E\Psi(K)$ for all $E, K \in \mathfrak{R}$.*

Proof. For every $E, K \in \mathfrak{R}$, invoking Lemmas 2.8-2.12, we have

$$\begin{aligned}
&2\{\Psi(EK + E^*K + K^*E^* + K^*E)\}\\
&= \Psi(I \diamond I \diamond \cdots \diamond I \diamond \frac{E^*}{2^{n-4}} \diamond K \bullet I)\\
&= I \diamond \cdots \diamond I \diamond \Psi(\frac{E^*}{2^{n-4}}) \diamond K \bullet I + I \diamond I \diamond \cdots \diamond I \diamond \frac{E^*}{2^{n-4}} \diamond \Psi(K) \bullet I\\
&= (2^{n-4}I) \diamond \Psi(\frac{E^*}{2^{n-4}}) \diamond K \bullet I + (2^{n-4}I) \diamond \frac{E^*}{2^{n-4}} \diamond \Psi(K) \bullet I\\
&= \{\Psi(E)^* + \Psi(E)\} \diamond K \bullet I + (E^* + E) \diamond \Psi(K) \bullet I\\
&= 2\{\Psi(E)^*K + \Psi(E)K + K^*\Psi(E)^* + K^*\Psi(E) + E^*\Psi(K) + E\Psi(K)\\
&\quad + \Psi(K)^*E^* + \Psi(K)^*E\}.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
&\Psi(EK + E^*K + K^*E^* + K^*E)\\
&= \Psi(E)^*K + \Psi(E)K + K^*\Psi(E)^* + K^*\Psi(E)\\
&\quad + E^*\Psi(K) + E\Psi(K) + \Psi(K)^*E^* + \Psi(K)^*E.
\end{aligned} \tag{2.5}$$

Also, we have

$$\begin{aligned}
&2\{\Psi(iEK - iE^*K - iK^*E^* + iK^*E)\}\\
&= \Psi(I \diamond I \diamond \cdots \diamond I \diamond (\frac{iE}{2^{n-4}}) \diamond K \bullet I)\\
&= I \diamond I \diamond \cdots \diamond I \diamond \Psi(\frac{iE}{2^{n-4}}) \diamond K \bullet I + I \diamond \cdots \diamond I \diamond (\frac{iE}{2^{n-4}}) \diamond \Psi(K) \bullet I
\end{aligned}$$

$$= (2^{n-4}I) \diamond \Psi(\frac{iE}{2^{n-4}}) \diamond K \bullet I + (2^{n-4}I) \diamond \frac{iE}{2^{n-4}} \diamond \Psi(K) \bullet I$$
$$= \{\Psi(iE) + \Psi(iE)^*\} \diamond K \bullet I + (iE - iE^*) \diamond \Psi(K) \bullet I$$
$$= 2\{i\Psi(E)K - i\Psi(E)^*K + iK^*\Psi(E) - iK^*\Psi(E)^* + iE\Psi(K) - iE^*\Psi(K) + i\Psi(K)^*E - i\Psi(K)^*E^*\}.$$

Thus, we obtain

$$\Psi(iEK - iE^*K - iK^*E^* + iK^*E)$$
$$= i\Psi(E)K - i\Psi(E)^*K + iK^*\Psi(E) - iK^*\Psi(E)^* + iE\Psi(K) - iE^*\Psi(K) + i\Psi(K)^*E - i\Psi(K)^*E^*.$$

Multiplying by $-iI$ in the last relation, we get

$$\Psi(EK - E^*K - K^*E^* + K^*E)$$
$$= \Psi(E)K - \Psi(E)^*K + K^*\Psi(E) - K^*\Psi(E)^* + E\Psi(K) - E^*\Psi(K) + \Psi(K)^*E - \Psi(K)^*E^*. \tag{2.6}$$

Now from (2.5) and (2.6), we get

$$\Psi(EK + K^*E) = \Psi(E)K + E\Psi(K) + \Psi(K)^*E + K^*\Psi(E). \tag{2.7}$$

From (2.7), we have

$$\Psi(EK - K^*E) = \Psi((-iE)(iK) + (iK)^*(-iE))$$
$$= \Psi(-iE)(iK) + (-iE)\Psi(iK) + \Psi(iK)^*(-iE) + (iK)^*\Psi(-iE)$$
$$= \Psi(E)K + E\Psi(K) - \Psi(K)^*(E) - K^*\Psi(E). \tag{2.8}$$

Adding (2.7) and (2.8), we get $\Psi(EK) = \Psi(E)K + E\Psi(K)$. Hence, Ψ is a derivation. □

Thus, Ψ is an additive $*$-derivation. This concludes the proof of Theorem 2.1.

3. Corollaries

Let $\mathfrak{R}$ be an algebra, we say that it is prime if for each $E, K \in \mathfrak{R}$, $E\mathfrak{R}K = \{0\}$, implies either $E = 0$ or $K = 0$. So, it can be easily seen that every prime $*$-algebra satisfies conditions (2.1) and (2.2) in Theorem 2.1. Therefore the following corollary follows.

Corollary 3.1. *Let $\mathfrak{R}$ be a unital prime $*$-algebra containing a non-trivial projection. Let a map $\Psi : \mathfrak{R} \to \mathfrak{R}$ satisfies*

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ *and* $S_i = I$ *for all* $i \in \{1, 2, \dots, n-3\}$*, where* $n \geq 3$*. Then* Ψ *is additive. Moreover, if* $\Psi(\alpha I)$ *is self-adjoint for* $\alpha \in \{i, 1\}$*, then* Ψ *is an additive* $*$*-derivation.*

Consider $\mathcal{H}$, as a complex Hilbert space. Assume $\mathcal{B}(\mathcal{H})$ and $\mathbb{T}(\mathcal{H})$ denote the algebra of all bounded linear operators and the subalgebra of bounded operators of finite rank respectively. It is well known that $\mathbb{T}(\mathcal{H})$ forms a $*$-closed ideal of $\mathcal{B}(\mathcal{H})$. A subalgebra $\mathcal{L}$ of $\mathcal{B}(\mathcal{H})$ is called a standard operator algebra if $\mathbb{T}(\mathcal{H}) \subseteq \mathcal{L}$. As a result, the following immediate corollary follows.

Corollary 3.2. *Let* $\mathcal{H}$ *be an infinite dimensional complex Hilbert space and* $\mathcal{L}$ *be a unital standard operator algebra on* $\mathcal{H}$ *such that* $\mathcal{L}$ *is closed under adjoint operation. Suppose that a map* $\Psi : \mathcal{L} \to \mathcal{L}$ *satisfies*

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathcal{L}$ *and* $S_i = I$ *for all* $i \in \{1, 2, \dots, n-3\}$*, where* $n \geq 3$*. Then* Ψ *is additive. Moreover, if* $\Psi(\alpha I)$ *is self-adjoint for* $\alpha \in \{i, 1\}$*, then* Ψ *is an additive* $*$*-derivation.*

A von Neumann algebra $\mathcal{Z}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator. Also it is well known that if a von Neumann algebra $\mathcal{Z}$ has no central summands of type I_1, then $\mathcal{Z}$ satisfies conditions (2.1) and (2.2) of Theorem 2.1. As a result, we have the following immediate corollary:

Corollary 3.3. *Let* $\mathcal{Z}$ *be a von Neumann algebra with no central summands of type* I_1 *and consider the map* $\Psi : \mathcal{Z} \to \mathcal{Z}$ *satisfying*

$$\Psi(S_1 \diamond S_2 \diamond \cdots \diamond S_{n-1} \bullet S_n) = \sum_{k=1}^{n} S_1 \diamond S_2 \diamond \cdots \diamond S_{k-1} \diamond \Psi(S_k) \diamond S_{k+1} \diamond \cdots \diamond S_{n-1} \bullet S_n$$

for all $S_{n-2}, S_{n-1}, S_n \in \mathfrak{R}$ *and* $S_i = I$ *for all* $i \in \{1, 2, \dots, n-3\}$*, where* $n \geq 3$*. Then* Ψ *is additive. Moreover, if* $\Psi(\alpha I)$ *is self-adjoint for* $\alpha \in \{i, 1\}$*, then* Ψ *is an additive* $*$*-derivation.*

□

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Raof Ahmad Bhat
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email address: raofbhat1211@gmail.com

Abbas Hussain Shikeh
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email address: abbasnabi94@gmail.com

Mohammad Aslam Siddeeque
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email address: aslamsiddeeque@gmail.com