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A SURVEY OF LENGTHS OF LINEAR GROUPS WITH RESPECT TO CERTAIN GENERATING SETS

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Abstract. In this paper, we summarise and present results on involution lengths and commutator lengths of certain linear groups such as special linear groups, projective linear groups, upper triangle matrix groups and Vershik-Kerov groups. Some open problems motivated by these results are also proposed.

1. Introduction

One of the classical problems in group theory is to seek and evaluate the length of a group with respect to generating sets. First, we recall the definition of the length of a group with respect to certain generating sets. Let G be a group with generator set X. For every element $g \in G \setminus \{1\}$, the smallest positive integer k such that g can be written as $g = g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_k^{\epsilon_k}$ with elements $g_1, g_2, \ldots, g_k \in X$ and $\epsilon_i = \pm 1$ is called *the length of the element* $g \in G$ with respect to X, denoted by $\ell_X(g)$. We use the convention that $\ell_X(1) = 0$. *The length* $\ell_X(G)$ *of a group* G with respect to X is the supremum of $\{\ell_X(g) : g \in G\}$.

The problem of evaluating $\ell_X(G)$ has attracted a lot of attention, especially when G is one of the classical-like groups. There have been hundreds of papers addressing this problem in the case when the generating set X is either the set of elementary transvections, the set of all transvections or ESD-transvections, the set of all unipotents, the set of all reflections or pseudo-reflections, the set of a non-central conjugacy class, or the set of all commutators. More specifically, see transvections in [32, 51, 60, 61]; reflections in [21] and [47] and dilatations in [16] and [67]. In this survey we focus on the lengths of matrices in linear groups, where X is one of the sets of involutions, commutators of involutions, or commutators. In this survey, we use the symbols $\mathcal{I}$, $\mathcal{CI}$, and $\mathcal{C}$ for the set of all involutions, commutators of involutions and commutators of the groups which are considered respect to the context, respectively.

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This article is organized as follows. In Section 2, we present results on the lengths of groups with respect to the set of involutions, which is the set of matrices of order 2. It is shown that the square of the determinant of a product of involutions is always equal to 1. Therefore, to seek and evaluate the involution lengths of matrices of the linear groups over rings, one considers matrices whose determinants are equal to 1 or satisfy property, such as the special linear groups, the upper unitriangular matrix groups. Moreover, one can also classify matrices based on the involution lengths. In particular, there are many interesting results when considering matrices over fields. One of them is the necessary and sufficient condition for a matrix with determinant ± 1 to be expressible as a product of involutions. Furthermore, a matrix is similar to its own inverse if and only if it can be written as a product of two involutions. More specifically, we also present the necessary and sufficient condition for a matrix with determinant 1 to be a product of three involutions. Some of these properties hold true for stable generalized linear groups over a field. We also mention some recent results over division rings. At the end of this section, we present some results on decompositions of matrices into products of matrices of finite order.

Note that a commutator of involutions is also a product of two involutions. In Section 3, we shall mention the lengths of the matrices with respect to the set of commutators of involutions. This topic is started by B. Zheng's results shown in 2002 (see [68]). More specifically, in [68], Zheng considered square matrices of size $n \geq 2$ over the real or complex number field. After these nice results of Zheng, some generalized results for arbitrary fields, associative rings, division rings are studied. At the end of this section are results on the lengths of commutators of matrices whose orders are finite.

In this paper, a ring R is assumed to be associative with unity $1 \neq 0$. For a positive integer n, let $\mathrm{GL}_n(R)$ be the group of invertible $n \times n$ matrices over R. By $\pm\mathrm{SL}_n(R)$ and $\mathrm{SL}_n(R)$ the subgroups of $\mathrm{GL}_n(R)$ consisting of matrices of determinant ± 1 and 1, respectively. In fact, all rings R we consider are either commutative rings or division rings. Over division rings, the determinant of a matrix of size $n \geq 2$ is understood to be the Dieudonne determinant (for a detailed definition, see [14]). If D is a division ring, then $\mathrm{SL}_n(D)$ is a commutator subgroup of $\mathrm{GL}_n(D)$. Recently, it was shown that if D is a division ring and n a positive integer greater than 1, then $\pm\mathrm{SL}_n(D)$ is a subgroup generated by involutions in $\mathrm{GL}_n(D)$ (see [6, Corrollary 2.4]). Furthermore, $\mathrm{SL}_n(D)$ is equal to the subgroup generated by commutators of involutions except in the case when $n = 2$ and D is a field of two elements. Hence, we have the inclusion

$$\mathrm{SL}_n(D) = [\mathrm{GL}_n(D), \mathrm{GL}_n(D)] \subset \pm\mathrm{SL}_n(D) = \langle \mathcal{I} \rangle$$

in which the notation $\mathcal{I}$ is the set of involutions in $\mathrm{SL}_n(D)$.

From these reasons, in Section 4 we rewrite some results concerning the commutator lengths of linear groups and focus mainly on the results that had been proven after 1989, especially those over division rings. Readers can follow the results on the commutator lengths before 1989 in a short survey by Vaserstein and Wheland (see [63]).

From these results, in Section 5, we shall present some open problems inspired by the preceding sections.

2. Involution length

2.1. Involution length of matrices over rings

We recall that an involution of a group is an element of order 2. In this section, we use the notation $\mathcal{I}$ to represent the set of all involutions of the groups which is considered respect to the context. The decompositions of matrices into products of involutions are noticed over particular rings. The first result in this direction was proved by Waterhouse in 1972 which stated that every elementary matrix over rings R can be written as a product of two involutions [64]. Namely, denote by e_{ij} a matrix of size n with only 1 in position (i,j) and the remaining positions are equal to 0. For any positive integers $i \neq j$ and $a \in R$, then

$$\mathrm{I_n} + ae_{ij} = (\mathrm{I_n} + e_{ij} - 2e_{jj})(\mathrm{I_n} + (a+1)e_{ij} - 2e_{jj}),$$

where $b \in R$ and $\mathrm{I_n} + be_{ij} - 2e_{jj}$ are involutions. In 1991, Gustafson [28, p. 251] showed that if the diagonal matrix $\mathrm{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ such that $d_m^2 = 1$ and $d_i + d_j = d_k + d_l = 0$, then

$$(\mathrm{I_n} + ae_{ij})(\mathrm{I_n} + ae_{kl}) = [(\mathrm{I_n} + ae_{ij})\mathrm{D}][\mathrm{D}(\mathrm{I_n} + be_{kl})],$$

where n, m, i, j, k, l are positive integers such that $1 \leq m \leq n$.

This leads to a product of two elementary matrices to be also a product of two involutions. This result is used by many authors such as [36, 41]. Notice that according to [10, Main Theorem], every matrix in $\mathrm{SL}_n(\mathbb{Z})$ can be written as a product of at most $\frac{1}{2}(3n^2 - n) + 36$ elementary matrices provided $n \geq 3$. In particular, this product is bounded by 48 when $n = 3$. Hence, an arbitrary matrix in $\mathrm{SL}_n(\mathbb{Z})$ is a product of at most $\frac{1}{2}(3n^2 - n) + 37$ involutions and at most 48 involutions in the case of $n = 3$ (we will find out that the number 48 is significant when we consider next results). In 1995, by using the results of decompositions of matrices into products of elementary matrices, H. Ishibashi improved this result. Observe that a matrix over an arbitrary commutative ring is invertible if and only if its determinant is invertible and invertible elements over $\mathbb{Z}$ are just ± 1, so $\mathrm{GL}_n(\mathbb{Z}) = \pm \mathrm{SL}_n(\mathbb{Z})$. H. Ishibashi proved the following theorem.

Theorem 1 ([36, Theorems 1 and 2])**.** *Let n be a positive integer greater than three. Then, the following statements are true.*

(a) $\ell_{\mathcal{I}}(\mathrm{GL}_n(\mathbb{Z})) \leq 3n + 9$.

(b) *If n is odd, then every matrix in* $\mathrm{SL}_n(\mathbb{Z})$ *can be decomposed into products of at most* $3n+9$ *involutions in* $\mathrm{SL}_n(\mathbb{Z})$. *Otherwise, the number of involutions in this decomposition is* $3n+11$.

In 1998, Laffey showed the involution length of $\mathrm{GL}_n(\mathbb{Z})$ does not depend on n, as shown the following result.

Theorem 2 ([41, Theorem 2]). *Let n be an integer. If $n \geq 3$, then* $\ell_{\mathcal{I}}(\mathrm{GL}_n(\mathbb{Z})) \leq 48$. *Furthermore, if $n \geq 82$, then* $\ell_{\mathcal{I}}(\mathrm{GL}_n(\mathbb{Z})) \leq 41$.

Next, we mention some results on linear groups over rings satisfying the first Bass stable range condition. Recall that a ring R satisfies the *first Bass stable range condition* provided that whenever $Ra + Rb = R$ for $a, b \in R$, there exists $c \in R$ such that $R(a + cb) = R$. In this case, we write $\mathrm{sr}(R) \leq 1$. For basic properties of rings satisfying the first Bass stable range condition, we refer to [8] and [25]. In particular, it is shown that if R is either a Noetherian ring or a semilocal ring, then $\mathrm{sr}(R) \leq 1$.

For a ring R, a matrix A in $\mathrm{GL}_n(R)$ is called *cyclic* if there exists a column vector v such that $v, Av, \ldots, A^{n-1}v$ form a basis of the vector space R^n of all n-tuples over R written as columns. Assume that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$ is a monic polynomial in one variable x with coefficients in R. A *companion matrix* of $f(x)$ is an $n \times n$ square matrix with ones along the line just below and parallel to the main diagonal, the last column given by the coefficients $a_0, a_1, \ldots, a_{n-1}$ and zeros elsewhere. It is known that a matrix is cyclic if and only if it is similar to a companion matrix (see [2] and [63]).

We consider the results on the decompositions of matrices by R. Dennis and L. Vasersteins [12] in 1988. Although these authors do not explicitly address involution lengths and commutator lengths, their findings have been used to estimate these lengths. In [12, Lemma 9], R. Dennis and L. Vaserstein showed that if the ring R satisfies the property $\mathrm{sr}(R) \leq 1$, then every matrix in $\mathrm{GL}_n(R)$ can be written as a product of four triangular matrices. This result was extended in 1990 by L. N. Vaserstein and E. Wheland [63], and they evaluated the number of triangular matrices in such decompositions to be three. Moreover, every matrix in $\mathrm{GL}_n(R)$ is similar to a product of two triangular matrices. For the convenience of the readers, we restate the following two results.

Lemma 3 ([63, Theorem 1]). *Let R be a ring such that* $\mathrm{sr}(R) \leq 1$ *and* $A \in \mathrm{GL}_n(R)$. *Then, there exist lower triangular matrices B, C and an upper triangular matrix X such that $A = BXC$. Therefore, A is similar to a product of an upper triangular matrix and a lower triangular matrix.*

Lemma 4 ([63, Theorem 2]). *Let R be a ring such that* $\mathrm{sr}(R) \leq 1$, *A and* $B \in \mathrm{GL}_n(R)$, *where B is an arbitrary companion matrix. Then*:

(a) *A is a product of two cyclic matrices.*
(b) *There exists a matrix X that is similar to B and a cyclic matrix Y such that $A = XY$.*

Using the above decompositions, many authors can seek and evaluate the involution lengths of matrices. We shall start with a result of F. Knüppel published in 1991.

Theorem 5 ([38, Theorem A]). *Let R be a commutative ring such that* $\mathrm{sr}(R) \leq 1$*. Then, every matrix in* $\pm\mathrm{SL}_n(R)$ *is a product of five involutions in* $\pm\mathrm{SL}_n(R)$*.*

Theorem 5 was extended by F. A. Arlinghaus et al. in 1995 [2].

Theorem 6 ([2, Theorems 7 and 8]). *Let R be a commutative ring such that* $\mathrm{sr}(R) \leq 1$*. Let $A \in \mathrm{GL}_n(R)$ such that* $(\det A)^2 = 1$*. Then*:

(a) *A can be written as a product of at most five involutions.*
(b) *In addition, if $n \leq 3$ or $n = 4$ and* $\det A = -1$*, then A can be written as a product of at most four involutions.*

Moreover, F. A. Arlinghaus et al. in [2, Theorem 13] showed that if $R = F[x]$ in which F is the field of complex numbers or an arbitrary field of infinite transcendence degree over its prime field of characteristic different from 2, then $\ell_{\mathcal{I}}(\mathrm{GL}_n(R)) = \infty$ provided $n \geq 2$. It means that there exists a ring R such that the involution length of $\ell_{\mathcal{I}}(\mathrm{GL}_n(R))$ is not bounded.

In 1998, H. You [66] studied involution lengths of cyclic matrices.

Theorem 7 ([66, Theorems 1 and 3]). *Let R be a commutative ring and $n \geq 3$. Then*:

(a) *Every cyclic matrix in* $\mathrm{SL}_n(R)$ *can be written as a product of three involutions in* $\mathrm{SL}_n(R)$*.*
(b) *If* $\mathrm{sr}(R) \leq 1$*, then each matrix in* $\mathrm{SL}_n(R)$ *can be written as a product of the six involutions in* $\mathrm{SL}_n(R)$*. Furthermore, this result is reduced to five involutions if* $n \neq 2 \pmod 4$*.*

In fact, the version of Theorem 7(a) over fields was stated by Ballantine [3, Theorem 2] in 1977.

Next, we consider the involution lengths of subgroups of $\pm\mathrm{SL}_n(R)$. Denote by $\mathrm{T}_n(R)$ (resp., $\mathrm{T}_\infty(R)$) the groups of upper triangular matrices of size n (resp., the groups of upper triangular matrices of infinite size). Let us denote by $\mathrm{UT}_n(R)$ (resp., $\mathrm{UT}_\infty(R)$) the subgroups of $\mathrm{T}_n(R)$ (resp., $\mathrm{T}_\infty(R)$) consisting of all upper triangular matrices whose diagonal entries are 1. The subgroups of $\mathrm{T}_n(R)$ and $\mathrm{T}_\infty(R)$ consisting of all upper triangular matrices whose diagonal entries are ± 1 is denoted by $\pm\mathrm{UT}_n(R)$ and $\pm\mathrm{UT}_\infty(R)$, respectively. The subgroup of $\mathrm{GL}_n(R)$ consisting of all lower triangular matrices whose diagonal entries are 1 is denoted by $\mathrm{LT}_n(R)$.

In 1998, Laffey [41, Theorem 1] showed that over an arbitrary ring R, every matrix in $\mathrm{UT}_n(R)$ is a product of ten involutions. In 2013, Słowik [54] showed that if F is a field, then each matrix in $\pm\mathrm{UT}_\infty(F)$ can be written as a product of at most five involutions. Moreover, if the characteristic of F is different from 2, then such decomposition is just a product of four involutions (see [54, Theorems 1.1 and 1.2]). The same year, Słowik showed the necessary and sufficient condition for matrices in $\mathrm{T}_\infty(F)$ and $\mathrm{T}_n(F)$ to be an involution (see [53, Theorem 1.1]). The same result is showed for lower triangular matrices (see [57, p. 250]). In 2017, X. Hou showed that Słowik's decompositions contained an identity matrix in the proof of [54, Theorem 1.1] for the case the characteristic of F is 2. Therefore, if F is a field, then every matrix in $\pm\mathrm{UT}_\infty(F)$ can be written as a product of at most four involutions. Moreover, in the case when R is an arbitrary ring, X. Hou et al. [35] evaluated the involution lengths of matrices in $\mathrm{UT}_n(R)$.

Theorem 8 ([35, Theorem 1.1]). *Let R be a ring. Then, every matrix in the groups $\pm\mathrm{UT}_n(R)$ (resp. $\pm\mathrm{UT}_\infty(R)$) can be written as a product of at most four involutions in $\pm\, UT_n(R)$ (resp. $\pm\mathrm{UT}_\infty(R)$).*

2.2. Involution lengths of matrices over fields

We denote by F an arbitrary field throughout this subsection. We first explore the theorem considered the standard for the decompositions of matrices into a product of two involutions in the linear groups over fields.

Theorem 9. *Let F be a field and $A \in \mathrm{GL}_n(F)$. Then, A is similar to A^{-1} if and only if A can be written as a product of two involutions.*

Theorem 9 was first showed in 1966 for the case when F is the complex field (see [65, Theorem 1]). In 1967, D. Ž. Djoković [15] extended this theorem to arbitrary fields. Then, F. Hoffman and E. Paige [33] also proved this result in 1971, by using companion matrices and minimal polynomials. In 2007, Theorem 9 was also proven by Ishibashi in [37, Theorem 2] by using the uniqueness of the invariant factors (the detailed definition below). This result does not hold for arbitrary rings, even if for division rings (see [20]). For example, consider the following matrices over the real quaternion division ring $\mathbb{H}$. Recall that $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$, with multiplication defined by $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Put $A = \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $P = \left(\begin{smallmatrix} j & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it is easy to see that $A = PA^{-1}P^{-1}$ and $\det A = i$. Because $(\det A)^2 = i^2 = -1 \neq 1$, so A can not be written as a product of two involutions.

Next, we present results on the group $\pm\mathrm{SL}_n(F)$. In 1974, Sampson showed that every matrix over the real number field can be written as a product of involutions provided its determinant is ± 1 (see [48, Theorem 1]). In 1976, Gustafson, Halmos, and Radjavi [29] extended the results for arbitrary fields.

Theorem 10 ([29, Theorem]). *Let F be a field and $A \in \pm\mathrm{SL}_n(F)$. Then, A is a product of at most 4 involutions, i.e., $\ell_{\mathcal{I}}(A) \leq 4$. Furthermore, 4 is the smallest number that satisfies.*

In fact, Gustafson, Halmos, and Radjavi knew that the number 4 is the smallest number satisfying Theorem 10 in 1958 (see the proof of [31, Theorem 2]). Namely, they showed that if u is in the center of G such that $u = xyz$ in which $x^2 = y^2 = z^2 = 1$, then $u^4 = uxuyuz = u(xu)y(uz) = u(yz)y(xy) = y(uz)y(xy) = yxyyxy = 1$. Many authors have also proved Theorem 10 in different ways. In 1986, Sourour [59, Theorem 5] showed a shorter proof for the case when F is a field containing at least $n+2$ elements and n is the order of matrices. In 1991, Laffey also showed this theorem, where F is a field of characteristic different from 2 (see [40, Corollary 2.2]). In 2009, Botha also proved this theorem without limiting the number of elements of fields (see [9, Theorem 6]).

To explore results on the decompositions of matrices into products of three involutions we recall the following definitions. The *geometric multiplicity* is defined as the dimension of the subspace spanned by the eigenvectors associated with the eigenvalue. Let A be a matrix in $\mathrm{GL}_n(F)$. If there exist matrices U and V in $\mathrm{GL}_n(F)$ such that $UAV = \mathrm{diag}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$, where $\alpha_n \mid \alpha_{n-1} \mid \cdots \mid \alpha_1$, then $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ are the *invariant factors* of A. Starting from Ballantine's results [3] published in 1977.

Proposition 11 ([3, Fact 4])**.** *Let F be a field. Assume that A is a product of three involutions in $\mathrm{GL}_n(F)$. Then, the geometric multiplicity of λ is at most $\frac{3}{4}n$, in which λ is an eigenvalue of A satisfying $\lambda^4 \neq 1$.*

Furthermore, Ballantine [3] also showed the necessary and sufficient condition for the involution length of $\pm\mathrm{SL}_n(F)$ to be three.

Theorem 12 ([3, Fact 5])**.** *Let F be a field and n be a positive integer. Then $\ell_{\mathcal{I}}(\pm\mathrm{SL}_n(F)) = 3$ if and only if at least one of the followings is satisfied.*

(a) $n \leq 2$.
(b) *The order of F is* 2, 3, *or* 5.
(c) $n = 3$ *and either characteristic of F is three or $x^2 + x + 1$ is irreducible in $F[x]$.*
(d) $n = 4$ *and characteristic of F is* 2.

Moreover, Ballantine [3] proved that every matrix in $\pm\mathrm{SL}_n(F)$ having exactly two invariant factors is a product of three involutions. In 2019, this result was extended by reducing one invariant factor (see [11, Proposition 3.7]). Proposition 11 was extended by Liu for the case when F is the complex number field in 1988 (see [42, Theorems 2.5 and 3.1]).

In 1991, F. Knüppel and K. Nielsen strengthened Theorem 12 by showing the necessary and sufficient condition for matrices in $\mathrm{SL}_n(F)$ to be a product of three involutions in $\mathrm{SL}_n(F)$.

Theorem 13 ([39, Theorem B])**.** *Let F be a field and $n \neq 2$. Then, each matrix in $\mathrm{SL}_n(F)$ is a product of three involutions in $\mathrm{SL}_n(F)$ if and only if at least one of the following cases is satisfied.*

(a) $n = 1$ *or* $n = 4$.
(b) *The order of F is* $2, 3,$ *or* 5 *and* $n \neq 2 \pmod 4$.
(c) $n = 3$ *and either characteristic of F is* 3 *or* $x^2 + x + 1$ *are irreducible in* $F[x]$.

Next, we present results on the involution lengths of matrices of the projective special linear groups. We denote by $\mathrm{PSL}_n(F)$ the projective special linear group, which obtained from the special linear group $\mathrm{SL}_n(F)$ on factoring by the scalar matrices contained in this group.

Starting in 1999, Ambrosievicz [1, Theorem 5(a)] showed that if F is a field of characteristic different from 2 and -1 is a square, then every matrix in $\mathrm{PSL}_2(F)$ is a product of two involutions. Then, Malcolm evaluated the involution length of $\mathrm{PSL}_n(F)$ in case F is a finite field.

Theorem 14 ([43, Corollary 3.7]). *Let F be a finite field of order m and n be an integer greater than one, except $n = 2$ and m is equal to either 2 or 3. Then,*

(a) $\ell_{\mathcal{I}}(\mathrm{PSL}_n(F)) = 2$ *if and only if* $n = 2$ *and* $m \neq 3 \pmod 4$.
(b) $\ell_{\mathcal{I}}(\mathrm{PSL}_n(F)) \leq 4$.

There are interesting results on the decompositions of augmented matrices into products of three involutions over fields. Recall that every augmented matrix has the form

$$A \oplus \mathrm{I}_m := \begin{pmatrix} A & 0 \\ 0 & \mathrm{I}_m \end{pmatrix}$$

for $A \in \mathrm{GL}_n(F)$ and $m \geq 1$. The following lemma is easy to see, we omit the proof.

Lemma 15. *Let F be a field and $A \in \mathrm{GL}_n(F)$. Then, $\ell_{\mathcal{I}}(A \oplus \mathrm{I}_m) \leq \ell_{\mathcal{I}}(A)$ with m and n are positive integers.*

We consider some cases with "=" sign. The case $m = 0$ is obvious. The sign "=" also holds for the case when A is similar to A^{-1} according to Theorem 9. Next, we find a case to show that $\ell_{\mathcal{I}}(A \oplus \mathrm{I}_m) < \ell_{\mathcal{I}}(A)$. Consider $A = 5\mathrm{I}_3$ over the field $\mathbb{Z}_7$. Because $\det A = -\bar{1}$, according to Theorem 10 the matrix A can be rewritten as a product of at most four involutions. Since $A^{-1} = 3\mathrm{I}_3 \neq A$, A is not an involution. Furthermore, A is not similar to A^{-1}, so A is not a product of two involutions by Theorem 9. Moreover, $A^4 = 2\mathrm{I}_3 \neq \mathrm{I}_n$ and A is noncentral, by [31] the matrix A is not a product of three involutions. Therefore, A is a product of exactly four involutions. On the other hand, according to [11, Lemma 5.9(ii)] $A \oplus \mathrm{I}_3$ is a product of three involutions.

Next, we shall explore a result on the involution lengths of augmented matrices.

Theorem 16 ([11, Theorem 1.7]). *Let F be a field and $A \in \pm\mathrm{SL}_n(F)$. Then, $A \oplus \mathrm{I}_n$ is a product of at most three involutions in $\mathrm{GL}_{2n}(F)$.*

According to Lemma 15 and Theorem 16, we can deduce that if $A \in \pm\mathrm{SL}_n(F)$, then $A \oplus \mathrm{I}_m \in \mathrm{GL}_{n+m}(F)$ is a product of at most three involutions for $m \geq n$. This does not hold provided $m < n$. For example, consider $B = 2\mathrm{I}_3 \in \mathrm{SL}_3(\mathbb{Z}_7)$. By the same arguments as above, we obtain $B \oplus \mathrm{I}_2 \in \mathrm{SL}_5(\mathbb{Z}_7)$ is neither an involution nor a product of two involutions.

Furthermore, by Theorem 13 the matrix $B \oplus \mathrm{I}_2 \in \mathrm{SL}_5(\mathbb{Z}_7)$ is not a product of three involutions. Therefore, the matrix $B \oplus \mathrm{I}_2$ can be written as a product of exactly four involutions.

The following are some results on the decompositions of matrices in the stable general linear groups. For convenience, we recall the definition of this group. Let R be an arbitrary ring, the direct limit $\mathrm{GL}_\infty(R) = \varinjlim \mathrm{GL}_n(R)$ with respect to the transition homomorphisms $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is called the *stable general linear group* over R.

The following shows that Theorem 9 and Theorem 10 also hold for augmented matrices.

Theorem 17 ([11, Theorem 1.1])**.** *Let F be a field and $A \in \mathrm{GL}_\infty(F)$. Then*:

(a) *A is a product of two involutions if and only if A is similar to its inverse.*
(b) *A is a product of three involutions if and only if* $\det(A) = \pm 1$.

We shall explore the involution lengths of matrices of a Vershik-Kerov subgroup over fields. Recall that the Vershik-Kerov group consisting of all matrices in which the set of nonzero entries below the main diagonal is finite, denoted by $\mathrm{GL}_{VK,\infty}(F)$. That means if $A \in \mathrm{GL}_{VK,\infty}(F)$, then $A = \left(\begin{smallmatrix} A_1 & A_2 \\ 0 & A_3 \end{smallmatrix}\right)$, where $A_1 \in \mathrm{GL}_n(F)$, $A_3 \in \mathrm{T}_\infty(F)$ and A_2 has the size $n \times \mathbb{N}$ (see [4] for more details).

In 2013, Słowik studied the subgroup of the Vershik-Kerov group consisting of matrices of the form $A = \left(\begin{smallmatrix} A_1 & A_2 \\ 0 & A_3 \end{smallmatrix}\right)$ in which $A_1 \in \pm\mathrm{SL}_n(F)$ and $A_3 \in \pm\mathrm{UT}_\infty(F)$, is denoted by $\pm\mathrm{SL}_{VK,\infty}(F)$. According to [54, Corollary 4.1], each matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$ is a product of at most six involutions over arbitrary fields, and a product of at most five involutions when characteristic of F is different from 2. Based on Słowik's arguments in [54] and X. Hou's result in [35], we have the following results.

Theorem 18. *Let F be a field. Then, each matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$ can be written as a product of at most five involutions in $\pm\mathrm{SL}_{VK,\infty}(F)$.*

Proof. Let A be an arbitrary matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$. We have $A = \left(\begin{smallmatrix} A_1 & A_2 \\ 0 & A_3 \end{smallmatrix}\right)$, with $A_1 \in \pm\mathrm{SL}_n(F)$ and $A_3 \in \pm\mathrm{UT}_\infty(F)$. Put $x = \begin{pmatrix} -\mathrm{I}_n & A_2A_3^{-1} \\ 0 & \mathrm{I}_\infty \end{pmatrix}$ and $y = \left(\begin{smallmatrix} -A_1 & 0 \\ 0 & A_3 \end{smallmatrix}\right)$, then $A = xy$. By Theorem 8, A_3 is a product of at most four involutions in $\pm\mathrm{UT}_\infty(F)$ and by Theorem 10, $-A_1$ is also a product of at most four involutions in $\pm\mathrm{SL}_n(F)$, so is y. Furthermore, x is an involution. Therefore, A can be written as a product of at most five involutions in $\pm\mathrm{SL}_{VK,\infty}(F)$. $\square$

Note that a commutator of involutions is a product of two involutions. By [34, Theorem 1.3], we see the decomposition of matrices into involutions in $\mathrm{SL}_{VK,\infty}(F)$.

Theorem 19. *Assume F is a field of characteristic different from 2. Then, every matrix in $\mathrm{SL}_{VK,\infty}(F)$ can be expressed as a product of at most four involutions.*

Can each matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$ be written as a product of four involutions? Will Theorem 19 be true for a field of characteristic 2? We can expect continued efforts in this area and new directions to be explored.

2.3. Involution lengths of matrices over division rings

Let D be a division ring. We write $D^* = D \setminus \{0\}$ and $D' = [D^*, D^*]$, which is a commutator subgroup of D throughout the paper.

In 1974, the decompositions of matrices of $\pm\mathrm{SL}_n(D)$ into a product of involutions are verified by W. C. Waterhouse in [64]. Recently M. H. Bien et al. [6, Lemma 2.6] showed that $\mathrm{SL}_n(D)$ is generated by matrices which are similar to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \oplus \mathrm{I}_{n-2}$. Furthermore, in [6, Corollary 2.4] the authors showed that $\langle\mathcal{I}\rangle = \{A \in \mathrm{GL}_n(D) : \det A = \overline{\pm 1}\}$. In addition, the involution lengths of matrices in $\pm\mathrm{UT}_n(D)$ and $\pm\mathrm{LT}_n(D)$ are also shown. The following are specific results.

Theorem 20 ([6, Lemma 4.3])**.** *If D is a division ring and $n \geq 2$, then every matrix in $\pm\mathrm{UT}_n(D)$ and $\pm\mathrm{LT}_n(D)$ can be written as a product of at most two involutions in $\pm\mathrm{UT}_n(D)$ and $\pm\mathrm{LT}_n(D)$, respectively.*

Furthermore, in [6, Section 3] the authors showed that there exists a division ring D such that $\ell_{\mathcal{I}}(\mathrm{SL}_n(D)) = \infty$. This result was built over the Mal'cev-Newmann division ring. Moreover, the version of Theorem 10 was shown over division rings.

Theorem 21 ([6, Theorem 4.5])**.** *Let D be a division ring such that $\ell_C(D') < \infty$. Assume that $A \in \pm\mathrm{SL}_n(D)$. Then, $\ell_{\mathcal{I}}(A) \leq 4 + 4\ell_{\mathcal{C}}(D')$.*

Moreover, there was a better evaluation for a finite-dimensional division ring.

Theorem 22 ([6, Theorem 5.3])**.** *Let D be a non-commutative finite-dimensional division ring and $n \geq 2$. Assume that $\ell_C(D') < \infty$.*

(a) *If A is noncentral and* $\det A = \overline{1}$, *then $\ell_{\mathcal{I}}(A) \leq 4\ell_{\mathcal{C}}(D')$.*
(b) *If* $\det A = \pm\overline{1}$, *then $\ell_{\mathcal{I}}(A) \leq 2 + 4\ell_{\mathcal{C}}(D')$.*

In particular, in the case of division rings with dimensions less than five.

Theorem 23 ([6, Corollary 5.5])**.** *Let D be a division ring with center F such that* $\dim_F D < 5$. *Assume that $A \in \mathrm{GL}_n(D)$. If* $\det A = \overline{\pm 1}$, *then $\ell_{\mathcal{I}}(A) \leq 4$.*

Recently, the decompositions of matrices into products of matrices whose orders are finite has been of interest. Next, we consider these results.

2.4. Decomposition of matrices into products of matrices whose orders are finite

First, we explore groups generated by matrices of fixed prime order p over a field F containing a primitive pth root of 1. In 1999, Grunenfelder et al. proved that the groups $\mathrm{PSL}_n(F)$ and $\mathrm{SL}_n(F)$ are generated by their matrices of order p (see in [26, Examples 2.8]). According to [26, Theorem 5.1] if every matrix

A satisfying $(\det A)^p = 1$, then $A = B_1B_2B_3B_4$, where B_1, B_2, B_3, B_4 are matrices of order p with $(\det B_i)^p = 1$. In Section 1, we knew this was also true for arbitrary fields if $p = 2$. In addition, Grunenfelder [26] decomposed matrices into products of matrices of finite order in the special linear groups.

Theorem 24 ([26, Theorem 5.4])**.** *Let p be a prime number and F be a field containing a primitive pth root of 1. Then, every matrix in $\mathrm{SL}_n(F)$ can be expressed as a product of four matrices of order p in $\mathrm{SL}_n(F)$.*

In particular, Theorem 24 holds if matrices are in the group $\mathrm{T}_n^{(p)}(F)$, the group consists of upper triangular matrices with main diagonal entries in the set $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$.

In 2004, Słowik studied decompositions of upper triangle matrices into products of matrices whose orders are finite (see [56]). Especially, if F is a field of order k, then $\mathrm{T}_\infty(F)$ and $\mathrm{T}_n(F)$ are generated by all their matrices of order $k-1$ where $k \geq 3$.

Theorem 25 ([56, Theorems 1.2 and 1.3])**.** *Assume F is a field containing k elements and $k \geq 3$. Then:*

(a) *Every matrix in $\mathrm{T}_\infty(F)$ and $\mathrm{T}_n(F)$ can be written as a product of at most four triangular matrices whose orders are divisors of $k-1$.*
(b) *Every matrix in $\mathrm{GL}_n(F)$ can be written as a product of at most twelve triangular matrices whose orders divide by $k-1$.*

Recall that, let a and b be arbitrary two elements in group G. *Commutator* $[a, b]$ is defined by $[a, b] = aba^{-1}b^{-1}$. If a, b are involutions, then $[a, b]$ is called *a commutator of involutions.* It is known that a commutator of involutions is a product of two involutions. Hence, we shall consider the decompositions of matrices into commutators of involutions.

3. Length of linear groups with respect to the set of commutators of involutions

3.1. Decompositions of matrices in linear groups over fields into commutators of involutions

Starting in 2002, B. Zheng studied decompositions of matrices in linear groups over fields whose characteristics are different from 2.

Lemma 26 ([68, Lemma 7])**.** *If F is a field of characteristic different from 2, then every matrix in $\mathrm{UT}_n(F)$ (resp., $\mathrm{LT}_n(F)$) is a commutator of involutions.*

According to [59, Theorem 1] every nonscalar matrix in $\mathrm{SL}_n(F)$ can be written as a product BC such that B and C are similar to matrices in $\mathrm{UT}_n(F)$ and $\mathrm{LT}_n(F)$, respectively. Hence, if the characteristic of F is different from 2, then every nonscalar matrix in $\mathrm{SL}_n(F)$ is a product of two involutions, and two is the smallest such number. Because every scalar matrix can be written as a product of a nonscalar matrix and a commutator of involutions, the decompositions of arbitrary matrices in $\mathrm{SL}_n(F)$ into commutators of involutions can be evaluated.

Theorem 27 ([68]). *Assume F is a field.*

(a) *If F has characteristic different from 2, then $\ell_{\mathcal{CI}}(\mathrm{SL}_n(F)) \leq 3$.*
(b) *If F is the real or complex number field, then $\ell_{\mathcal{CI}}(\mathrm{SL}_n(F)) \leq 2$, and two are the smallest such number.*

In 2018, X. Hou improved B. Zheng's results in [68]. Namely, in [34, Theorem 2.8], the author showed that if F is a field of characteristic different from 2, then every matrix in $\mathrm{SL}_n(F)$ is a product of at most two commutators of involutions. In 2022, Son et al. showed that the result also holds for fields of characteristic 2 (see [58, Theorem 1]). Hence, the following is the general result.

Theorem 28. *Let F be an arbitrary field containing more than two elements and $n \geq 2$. Then, $\ell_{\mathcal{CI}}(\mathrm{SL}_n(F)) \leq 2$.*

Then, X. Hou [34] improved Słowik's results, which we mentioned in Theorem 18.

Theorem 29 ([34, Theorem 1.3]). *Let F be a field of characteristic different from 2. Then, every matrix in $\mathrm{SL}_{VK,\infty}(F)$ can be written as a product of at most two commutators of involutions.*

Moreover, X. Hou [34] mentioned a result on the upper triangular matrices group over arbitrary rings.

Theorem 30 ([34, Theorem 1.1]). *Let R be a ring and 2 be an invertible element. Then, every matrix in $\mathrm{UT}_\infty(R)$ (resp., $\mathrm{UT}_n(R)$) can be written as a product of at most two commutators of involutions in $\mathrm{T}_\infty(R)$ (resp., $\mathrm{T}_n(R)$).*

3.2. Decomposition of matrices in linear groups over division rings into commutators of involutions

Let D be a division ring. Recall that $D^* = D \setminus \{0\}$ and $D' = [D^*, D^*]$, where D' is the commutator subgroup of D. Recently, in [5] and [6], M. H. Bien et al. evaluated the length of matrices with respect to the set of commutators of involutions over division rings, by starting with a finite-dimensional division ring D, then decomposing every matrix in $\mathrm{SL}_n(D)$ into a product XYZ, where $X \in \mathrm{LT}_n(D)$, $Y \in \mathrm{UT}_n(D)$ and a matrix diagonal Z has its (n,n)-th element in D'. According to [5], XY is a product of at most two commutators of involutions, and Z is a product of at most six commutators of involutions.

Theorem 31 ([5, Theorem 4.6]). *Let D be a finite-dimensional division ring such that $\ell_{\mathcal{C}}(D') < \infty$ and $n \geq 2$ be a positive integer. Then,*

(a) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) \leq 2 + 3\ell_{\mathcal{C}}(D')$ *if $char D \neq 2$ or $char D = 2$ and $n \geq 3$.*
(b) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) \leq 2 + 6\ell_{\mathcal{C}}(D')$ *if $char D = 2$ and $n = 2$.*

Then, in [6, Proposition 3.5] the authors showed that there exists a division ring D, where $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D))$ is infinite. In [6] the decompositions of matrices in $\mathrm{UT}_n(D)$ into commutators of involutions are estimated. Namely, if the characteristic of D is different from 2, then every matrix in $\mathrm{UT}_n(D)$ is a commutator of involutions in $\mathrm{GL}_n(D)$. Otherwise, the number of commutators of involutions in this decomposition is two if the characteristic of D is 2. Furthermore, the authors sought and evaluated this decomposition over an arbitrary division rings.

Theorem 32 ([6, Theorem 6.3])**.** *Let D be a division ring and $n \geq 2$ be a positive integer. Assume that $\ell_C(D') < \infty$. Then,*

(a) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) \leq 6\ell_C(D') + 4$ *if* $\mathit{char} D = 2$ *and* $n = 2$.
(b) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) \leq 3\ell_C(D') + 4$ *if* $\mathit{char} D = 2$ *and* $n > 2$.
(c) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) \leq 3\ell_C(D') + 2$ *if* $\mathit{char} D \neq 2$.

As we have mentioned, some authors have generalized decompositions of matrices into commutators of involutions by decompositions of matrices into products of commutators of matrices whose orders are finite, such as [23], [24] and [56]. Next, we shall restate these results.

3.3. Decompositions of matrices into commutators of matrices whose orders are finite

In 2021, I. Gargate and M. Gargate studied decompositions of matrices in $\mathrm{UT}_\infty(R)$ and $\mathrm{UT}_n(R)$ into commutators of matrices of finite orders (see [24]). For the convenience of readers, we shall use notations used in original articles. Put

$$\mathrm{UT}_n^{(k)}(R) = \{A \in \mathrm{T}_n(R) : a_{ii}^k = 1 \text{ for all } 1 \leq i \leq n\},$$
$$\mathrm{UT}_\infty^{(k)}(R) = \{A \in \mathrm{T}_\infty(R) : a_{ii}^k = 1 \text{ for all } 1 \leq i \leq n\}.$$

In Theorem 30, X. Hou decomposed matrices in $\mathrm{UT}_n(R)$ (resp., $\mathrm{UT}_\infty(R)$) into a product of commutators of involutions for the case R is a ring containing 2 which is the invertible element. The following is decompositions of matrices in $\mathrm{UT}_n(R)$ (resp., $\mathrm{UT}_\infty(R)$) into a product of commutator of matrices whose order are finite.

Theorem 33 ([24, Theorem 1.1])**.** *Let R be a commutative ring and $k \geq 2$ be a positive integer. Assume that $1 + 1 + \cdots + 1 = k$ is an invertible element of R. Then, every matrix in* $\mathrm{UT}_\infty(R)$ *(resp.* $\mathrm{UT}_n(R)$*) can be written as a product of at most $4k - 6$ commutators of matrices, which depend on two matrices of order k in* $\mathrm{UT}_\infty^{(k)}(R)$ *(resp.* $\mathrm{UT}_n^{(k)}(R)$*).*

For the real or complex number field F, B. Zheng in [68] decomposed matrices in $\mathrm{SL}_n(F)$ into a product of commutators of involutions. [23, Theorem 1.5.3] and [24, Theorem 1.2] are an extension of Zheng's result.

Theorem 34 ([23, 24]). *Let F be the real or complex number field and $k \geq 2$ be a positive integer. Then*:

(a) *If F is the complex number field, then every matrix in $\mathrm{SL}_n(F)$ can be written as a product of at most $4k-6$ commutators of matrices of order k in $\mathrm{UT}_n^{(k)}(F)$.*

(b) *Every every matrix in $\mathrm{SL}_n(F)$ can be represented as a product of at most $4k-6$ commutators of matrices, which depend on two matrices of order k in $\mathrm{GL}_n(F)$.*

The next result is an extension of Theorem 18 for the case when F is the real or complex number field.

Theorem 35 ([24, Theorem 1.3]). *Let F be the real or complex number field and $k \geq 2$. Then, every matrix in $\mathrm{SL}_{VK,\infty}(F)$ can be written as a product of at most $4k-6$ commutator, all depending on two matrices of order k in $\mathrm{GL}_{VK,\infty}(F)$.*

As we mentioned, we shall explore results on the commutator lengths of linear groups in Section 4.

4. Commutator lengths of matrices

4.1. Commutator lengths of linear groups over fields

It is known that a product of commutators in an arbitrary group is not necessarily a commutator. An interesting example of this was presented by W. B. Fite [22] in 1902.

In particular, the author built a group G such that $|G| = 256$ and $|G'| = 16$ and only 15 elements in G' are commutators. In 1936, Shoda studied the decompositions of matrices into commutators in linear groups. Namely, Shoda showed that over algebraically closed fields F every matrix in $\mathrm{SL}_n(F)$ is a commutator in [49, Theorem 1]. In 1951, Shoda proved that if F is an infinite field, then there exists an upper bound of $\ell_C(\mathrm{SL}_n(F))$ (see [50]). Then, Thompson [62] showed remarkable results on the commutator lengths of $\mathrm{SL}_n(F)$ and $\mathrm{PSL}_n(F)$ in 1960.

Theorem 36 ([62]). *Let F be a field containing at least four elements and $n \geq 2$ be an integer. Then, $\ell_C(\mathrm{SL}_n(F)) = 1$ and $\ell_C(\mathrm{PSL}_n(F)) = 1$.*

4.2. Commutator lengths of linear groups over rings and infinite fields

In 1986, Djoković proved that $\ell_C(\mathrm{SL}_n(\mathbb{H})) = 1$ with the real quaternion division ring $\mathbb{H}$ (see [17, Theorem 3]). In 1987, Newman [45] showed that over a principal ideal ring R if there exists a positive integer k such that $\ell_C(\mathrm{SL}_3(R)) \leq 3$, then $\ell_C(\mathrm{SL}_n(R)) \leq c \log n + k - 3$ for $n \geq 3$, $c = 2\log\frac{3}{2}$ and log is the symbol of logarithms to the base 10. Moreover, Newman posted a question that if the group $\mathrm{SL}_n(R)$ is perfect; that means this group is equal to its own commutator subgroup; whether there exists an absolute constant k such that every element of $\mathrm{SL}_n(R)$ is a product of at most k commutators. Not long after that, Dennis and Vaserstein ([12]) showed this question is not true in 1988. Namely, the authors showed $\ell_C(\mathrm{SL}_n(\mathbb{C})) = \infty$ for $n \geq 2$. Furthermore, they also showed $\ell_C(\mathrm{SL}_n(\mathbb{Z})) \leq 6$ with n large enough and evaluated the commutator lengths of $\mathrm{UT}_n(R)$ and $\mathrm{LT}_n(R)$ for $n \geq 3$.

Theorem 37 ([12, Lemma 13])**.** *Let R be a ring and $n \geq 3$. Then, every matrix in $\mathrm{UT}_n(R)$ (resp., $\mathrm{LT}_n(R)$) can be expressed as a product of two commutators in $\mathrm{SL}_n(R)$.*

For the case $n = 2$, $\mathrm{UT}_2(R)$ and $\mathrm{LT}_2(R)$ are commutative groups, so their commutator lengths is zero.

Moreover, Dennis and Vaserstein showed that if every matrix in $\mathrm{GL}_n(R)$ is a product of t triangular matrices, then it can be written as a product of $3+[t/2]$ commutators with $n \geq 3$ and $t > 1$ (see [12, Corollary 14]).

Later, the problem of classifying matrices in linear groups whose the commutator lengths are bounded drew interested. We start with Vaserstein and Wheland's results [63] in 1990 for the case R is a commutative ring such that $\mathrm{sr}(R) \leq 1$.

Theorem 38 ([63, Theorem 3])**.** *Let R be a commutative ring such that $\mathrm{sr}(R) \leq 1$. If either $n \geq 3$ or $n = 2$ and 1 is the sum of two invertible elements, then $\ell_C(\mathrm{SL}_n(R)) \leq 2$.*

In 2012, Słowik [52] evaluated the commutator lengths of upper triangular matrix groups.

Theorem 39 ([52, Lemma 3.2])**.** *Suppose R is a commutative ring such that $\mathrm{sr}(R) \leq 1$ and $n \geq 3$. If R contains θ such that $\theta, (1-\theta)$ are invertible, then $\mathrm{UT}_n(R) = [\mathrm{T}_n(R), \mathrm{T}_n(R)]$ and $\ell_C(\mathrm{UT}_n(R)) \leq 2$.*

Moreover, Słowik also estimated the commutator lengths of matrices in $\mathrm{T}_n(F)$ for the case F is an infinite field.

Theorem 40 ([52, Theorem 3.3])**.** *Let F be an infinite field and $n \geq 1$. If $A \in \mathrm{T}_n(F)$, then $\ell_C(A) = 1$.*

In [52] Bier and Holubowski evaluated the commutator lengths of matrices $\mathrm{UT}_n(R)$ for some rings in 2015.

Theorem 41 ([7, Theorem 1.5])**.** *Let R be a ring such that the set of invertible elements is commutative and 1 is the sum of two invertible elements. Then*:

(a) $\mathrm{UT}_n(R) = [\mathrm{T}_n(R), \mathrm{T}_n(R)]$ *for all $n \geq 2$.*
(b) $\mathrm{UT}_n(R) = [\mathrm{T}_n(R), \mathrm{T}_n(R)]$ *and $\ell_C(\mathrm{UT}_n(R)) \leq 2$ for $n \geq 2$.*

Next, we consider the commutators lengths of infinite triangular matrices in linear groups. In 2012, Gupta and Holubowski showed that if F is an infinite field, then $[\mathrm{T}_\infty(F), \mathrm{T}_\infty(F)] = \mathrm{UT}_\infty(F)$ and $\ell_C(\mathrm{UT}_\infty(F) \leq 2$ (see [27, Theorem 1.2]). In 2015, Bier and Holubowski improved this result in [7].

Theorem 42 ([7, Theorem 3.1])**.** *Assume that F is an infinite field. Then, $\ell_{\mathcal{C}}(\mathrm{UT}_\infty(F)) = 1$. Moreover, if there exists an infinite diagonal matrix B whose diagonal entries are distinct, then every matrix in $\mathrm{UT}_\infty(F)$ can be written as a commutator of B and a matrix in $\mathrm{UT}_\infty(F)$.*

Theorem 42 holds for some class of finite fields, which were showed by Słowik [55] in 2013. In particular, if F is a field containing at least three elements, then $[\mathrm{T}_\infty(F), \mathrm{T}_\infty(F)] = [\mathrm{UT}_\infty(F), \mathrm{T}_\infty(F)] = \mathrm{UT}_\infty(F)$ and every matrix in $\mathrm{UT}_\infty(F)$ can be written as $[A, B]$ in which $A \in \mathrm{UT}_\infty(F)$ and $B \in \mathrm{T}_\infty(F)$ (see [55, Lemma 3.1]). To sum up, we present the following result.

Theorem 43. *Let F be a finite field containing at least three elements or an infinite field. Assume that $A \in \mathrm{UT}_\infty(F)$. Then, there exist a matrix $B \in \mathrm{UT}_\infty(F)$ and a matrix $C \in \mathrm{T}_\infty(F)$ such that $A = [B, C]$.*

Bier and Holubowski extended Theorem 43 in [7] for some class of rings.

Theorem 44 ([7, Theorem 1.3])**.** *Assume that R is a ring such that the set of invertible elements is commutative and 1 is the sum of two invertible elements. Then, $[\mathrm{T}_\infty(R), \mathrm{T}_\infty(R)] = \mathrm{UT}_\infty(R)$ and every matrix in $\mathrm{UT}_\infty(R)$ can be written as a product of at most two commutators.*

Next, we explore the commutator lengths of linear groups over division rings.

4.3. Commutator lengths of linear groups over division rings

First, we present the commutator lengths of finite matrices in linear groups. Starting in 2018, Egorchenkova et al. [19] evaluated the commutator lengths of any noncentral matrices.

Theorem 45 ([19])**.** *Let D be a noncommutative finite-dimensional division ring and $n \geq 2$. Assume that $\ell_C(D') < \infty$. Then:*

(a) *$\ell_C(A) \leq \ell_C(D')$, where A is a noncentral matrix in $\mathrm{SL}_n(D)$.*
(b) *$\ell_C(A) \leq \ell_C(D')$, where $A \in \mathrm{PSL}_n(D)$.*

Significantly, in 2020 Gvozdevskii [30] showed that $\ell_C(\mathrm{SL}_n(D)) < \infty$ if and only if $\ell_C(D') < \infty$ provided that D is a finite-dimensional division ring.

Theorem 46 ([30, Corollaries 1 and 2])**.** *Let D be a division ring and $n \geq 2$ be a positive integer. Assume that $\ell_C(D') \geq 2$. Then, for every noncentral matrix A in $\mathrm{SL}_n(D)$, the following statements are true.*

(a) *$\ell_C(A) \geq \frac{\ell_C(D')+2n^2-3n+1}{8n^2-13n+8}$.*
(b) *If $\ell_C(D') \leq 6n^2 - 10n + 7$, then A is a commutator in $\mathrm{GL}_n(D)$.*

Moreover, Gvozdevskii showed better results over a noncommutative finite-dimensional division ring.

Theorem 47 ([30, Theorem 2])**.** *Let D be a noncommutative finite-dimensional division ring and $n > 2$ be a positive integer. Then,*

(a) *Every noncentral matrix in* $\mathrm{SL}_n(D)$ *can be written as a products of at most* $\lceil \frac{\ell_C(D')}{n} \rceil$ *commutators in* $\mathrm{GL}_n(D)$.
(b) *Every noncentral matrix in* $\mathrm{SL}_n(D)$ *can be written as a products of at most* $\lceil \frac{\ell_C(D')}{n-2} \rceil$ *commutators in* $\mathrm{SL}_n(D)$.

Where $\lceil x \rceil$ *denotes the ceiling function of* x.

In addition, Gvozdevskii also estimated the commutator length of every noncentral matrix in $\mathrm{SL}_n(D)$ for the case when $\ell_C(D') \leq n$.

Theorem 48 ([30, Corollaries 3 and 4])**.** *Let* D *be a noncommutative finite-dimensional division ring. Then*:

(a) *Every noncentral matrix* A *in* $\mathrm{SL}_n(D)$ *is a commutator in* $\mathrm{GL}_n(D)$ *if* $n \geq 2$ *and* $\ell_C(D') \leq n$.
(b) *Every noncentral matrix* A *in* $\mathrm{SL}_n(D)$ *is a commutator in* $\mathrm{SL}_n(D)$ *if* $n \geq 3$ *and* $\ell_C(D') \leq n-2$.

Note that the above results evaluated the commutator lengths of noncentral matrices in $\mathrm{SL}_n(D)$. Moreover, Gvozdevskii also showed that every central matrix in $\mathrm{SL}_n(D)$ can be written as a product of a commutator and a noncentral matrix in $\mathrm{SL}_n(D)$. Therefore, we can evaluate the commutator lengths of all matrices in $\mathrm{SL}_n(D)$.

Next, we present the commutator lengths of infinite matrices in linear groups. In 2022, M. H. Bien et al. [4] estimated the commutator lengths of matrices in $\mathrm{UT}_\infty(D)$ over finite-dimensional division rings.

Theorem 49 ([4, Theorem 1.1])**.** *Let* D *be a finite-dimensional division ring and* B *be an infinite diagonal matrix with pairwise nonconjugate diagonal entries, then every matrix in* $\mathrm{UT}_\infty(D)$ *can be written as a commutators of* B *and a matrix in* $\mathrm{UT}_\infty(D)$.

Now, we shall continue with the commutator lengths of matrices in $\mathrm{SL}_{VK,\infty}(F)$. In 2012, Gupta and Holubowski showed that if F is an infinite field, then $\ell_C(\mathrm{SL}_{VK,\infty}(F)) \leq 3$ (see [27, Theorem 1.1]). Then, this result is extended by reducing $\ell_C(\mathrm{SL}_{VK,\infty}(F)) \leq 2$ for the case when F is a field containing at least four elements (see [7, Theorem 1.4]). In this line, Bien et al. [4, Theorem 1.4] showed that $\ell_C(\mathrm{SL}_{VK,\infty}(D)) = 1$, where D is an infinite commutative division ring or D is a quaternion division ring. Therefore, based on these papers we present the following theorem.

Theorem 50. *Let* D *be a division ring. Then,*

$$[\mathrm{GL}_{VK,\infty}(D), \mathrm{GL}_{VK,\infty}(D)] = \mathrm{SL}_{VK,\infty}(D).$$

Moreover,

(a) $\ell_C(\mathrm{SL}_{VK,\infty}(D)) = 1$ *if* D *is an infinite field or a quaternion division ring.*
(b) $\ell_C(\mathrm{SL}_{VK,\infty}(D)) \leq 2$ *if* D *is a field containing at least four elements.*

Let R be an arbitrary ring. We consider the subgroup $\mathrm{SL}_\infty(R)$ of the stable general linear group $\mathrm{GL}_\infty(R)$, generated by elementary transvections. In 1989, Dennis and Vaserstein [13, Corollary 4] proved that every matrix in $\mathrm{SL}_\infty(R)$ can be written as a product of two commutators in $\mathrm{SL}_\infty(R)$. In 2022, Gvozdevskii evaluated the commutator length of matrices in $\mathrm{SL}_\infty(R)$ for the case R is a noncommutative finite-dimensional division ring.

Theorem 51. *If D is a finite-dimensional division ring, then every matrix in $\mathrm{SL}_\infty(D)$ is a commutator in $\mathrm{SL}_\infty(D)$.*

Recently Bien et al. [5] answered the problem which Draxl asked in [18, Problem 1, p. 102]. In particular, if D is a finite-dimensional division ring which is tame and its center is Heselian, then $\ell_C(D')$ is bounded from above by a positive integer depending on $\dim_F D$.

Finally, we present results showing the correlation of commutator lengths of $\mathrm{SL}_n(D)$, commutator of involutions lengths of $\mathrm{SL}_n(D)$ and commutator lengths of D.

Theorem 52 ([5, Corollary])**.** *Let D be a finite-dimensional division ring and $n \geq 2$. Then, the followings are equivalent.*

(a) $\ell_C(\mathrm{SL}_n(D)) < \infty$.
(b) $\ell_{\mathcal{CI}}(\mathrm{SL}_n(D)) < \infty$.
(c) $\ell_C(D') < \infty$.

5. Open problems

In this section, we propose some open problems that have been collected from some references and motivated by results in previous sections.

According to [19], Egorchenkova and Gordeev showed that every noncentral matrix in $\mathrm{SL}_n(D)$ can be seen as a commutator of matrices in $\mathrm{SL}_n(D)$ provided that D is a noncommutative finite-dimensional division ring and $\ell_C(D') = 1$. Moreover, the authors also proposed a question about the commutator lengths of matrices in $\mathrm{PSL}_n(D)$ (see [19, p. 563]).

Problem 1. *Let D be a noncommutative finite-dimensional division ring and $n \geq 2$. For a noncentral matrix $A \in \mathrm{PSL}_n(D)$, whether there exists matrices P and Q in $\mathrm{PSL}_n(D)$ such that $A = [P, Q]$?*

Note that M. H. Bien et al. gave an example about an infinite division D for which there does not exist a positive integer d such that every matrix in $\mathrm{PSL}_n(D)$ can be written as d commutators of involutions (see [6, Proposition 3.5]).

The following problem is posted by Draxl in 1980 (see [18, Problem 1, p. 102]).

Problem 2. *Let D be a finite-dimensional division ring with center F such that $\dim_F D = m^2$. Does there exist an positive integer d depending on m such that $\ell_C(D') \leq d$?*

In [30, Theorems 1 and 4], we see that $\ell_C(\mathrm{SL}_n(D)) < \infty$ if and only if $\ell_C(D') < \infty$ for every finite-dimensional division ring D. From this, the following problem is similar to Problem 2.

Problem 3. *Let D be a finite-dimensional division ring with center F such that $\dim_F D = m^2$ and $n \geq 1$. Does there exist an integer d depending on m such that $\ell_C(\mathrm{GL}_n(D)) \leq d$?*

Note that Problem 2 and Problem 3 are solved if the finite-dimensional division ring D satisfies one of the following conditions.

(a) D is a quaternion division ring (see [4, Lemma 2.5]).
(b) The center of D is a p-adic number field ([44]).
(c) D is a tame or totally ramifield finite-dimensional division ring with Henselian center (see [5, Section 2]).

When we wrote this paper, we realized that the commutator lengths of linear groups equaling 1 is a problem of interest.

Problem 4. *Classify rings which satisfy the property that the commutator lengths of its multiplicative subgroup equals* 1*.*

Problem 4 is inspired by [12]. Note that this problem is solved for the case when R is a quaternion division ring (see [4, Lemma 2.5]).

Let R be a ring and $n \geq 3$. Assume that the group $\mathrm{SL}_n(R)$ is perfect; that means this group equals its own commutator subgroup.

Problem 5. *Classify rings R in which $\mathrm{SL}_n(R)$ are perfect and its commutator lengths are bounded.*

Note that Problem 5 is solved for rings satisfying the first Bass stable range condition and still remains open in case when R is a division ring.

The following is a question of Grunenfelder et al. in [26, Question 4.3].

Problem 6 ([26, Question 4.3])**.** *Let p be a prime number and F be a field containing a primitive pth root of* 1*. Does there exist an positive integer k less than four such that every matrix in $\mathrm{T}_n^{(p)}$ can be written as a product of at most k matrices of order p in $\mathrm{T}_n^{(p)}$?*

According to Theorem 18, every matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$ can be written as a product of at most five involutions. May this number be smaller?

Problem 7. *Classify fields F such that every matrix in $\pm\mathrm{SL}_{VK,\infty}(F)$ is a product of at most four involutions in $\pm\mathrm{SL}_{VK,\infty}(F)$.*

Does Theorem 29 hold for the case when the characteristic of the field F is equal 2?

Problem 8. *Let F be a field of characteristic* 2*. Can every matrix in $\mathrm{SL}_{VK,\infty}(F)$ be written as a product of at most four involutions?*

It is shown that if an arbitrary matrix is a product of two involutions, then it is similar to its inverse. However, the converse is not true, particularly for matrices over non-commutative rings.

Problem 9. *Classify rings R which satisfy the property that if every matrix in $\mathrm{GL}_n(R)$ is similar to its inverse, then it is a product of two involutions.*

Note that Problem 9 was solved for the case when R is a field (see Theorem 9).

Next, let G be a group, we call an element A is *reversible* if there exists $B \in G$ such that $B^{-1}AB = A^{-1}$. Let F be a field, it is known that every matrix in $\mathrm{GL}_n(F)$ is reversible if and only if it is a product of two involutions (see Theorem 9). Does this hold for the reversible elements in the special linear group $\mathrm{SL}_n(F)$?

Problem 10. *Classify fields F which satisfy the property that every reversible matrix in $\mathrm{SL}_n(F)$ is a product of two involutions.*

This problem was solved for $\mathrm{SL}_n(\mathbb{C})$ with $n \neq 2 \pmod 4$ (see [46, p. 3]).

Inspired by the article [27], we consider the group $\mathrm{GL}_c(\infty, D)$ to be the group of all infinite dimensional column-finite invertible matrices over division rings D. It is known that $\mathrm{GL}_c(\infty, D)$ is its commutator subgroup, this means $\mathrm{GL}_c(\infty, D) = [\mathrm{GL}_c(\infty, D), \mathrm{GL}_c(\infty, D)]$. However, the commutator lengths of $\mathrm{GL}_c(\infty, D)$ are still open.

Problem 11. *Let D be a division ring. There exists a number k such that the commutator length of $\mathrm{GL}_c(\infty, D)$ is bounded by k.*

Also, the following problem is Bogopolskii's conjecture mentioned in [27, p. 1].

Problem 12. *The commutator subgroup $[\mathrm{GL}_c(\infty, \mathbb{Z}), \mathrm{GL}_c(\infty, \mathbb{Z})]$ has uncountable index in $\mathrm{GL}_c(\infty, \mathbb{Z})$.*

Conclusion and acknowledgments

There are some interesting results that the author does not have a clear source to cite, so they are not mentioned in the survey. Specifically, in the case of matrices of order two in $\mathbb{Z}$. In [36] it is mentioned that every natural number m, every matrix in $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ cannot be represented less than m involutions in $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$. However, the author could not find the original document of this result even though it is classical. Historically, this survey reminds of results for the involution lengths and the commutator lengths of matrices in linear groups. One original intention was to better understand the lengths of matrices with respect to a set of involutions and a set of commutators. In doing so, the author discovered interesting problems that are the stated open problems. The author would be grateful if readers could provide more information related to this topic, as well as participate in pursuing open problems.

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