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TOTALLY REAL AND COMPLEX SUBSPACES OF A RIGHT QUATERNIONIC VECTOR SPACE WITH A HERMITIAN FORM OF SIGNATURE $(n, 1)$

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Abstract. We study totally real and complex subsets of a right quaternionic vector space of dimension $n+1$ with a Hermitian form of signature $(n, 1)$ and extend these notions to right quaternionic projective space. Then we give a necessary and sufficient condition for a subset of a right quaternionic projective space to be totally real or complex in terms of the quaternionic Hermitian triple product. As an application, we show that the limit set of a non-elementary quaternionic Kleinian group Γ is totally real (resp. commutative) with respect to the quaternionic Hermitian triple product if and only if Γ leaves a real (resp. complex) hyperbolic subspace invariant.

1. Introduction

Let $\mathbb{H}$ denote the algebra of quaternions $\mathbb{H} = \mathbb{R}\oplus i\mathbb{R}\oplus j\mathbb{R}\oplus k\mathbb{R}$, where i, j and k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and let $\mathbb{H}^{n,1}$ be a right quaternionic vector space $\mathbb{H}^{n+1}$ with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. Denote by $\mathbb{PH}^{n,1}$ the set of right quaternionic lines in $\mathbb{H}^{n,1}$, that is, a right quaternionic projective n-space.

Definition 1.1. A subset S of $\mathbb{H}^{n,1}$ is said to be *totally real* (resp. *complex*) if $\langle v, w\rangle$ is real (resp. complex) for all $v, w \in S$. A subset $\mathbf{S} \subset \mathbb{PH}^{n,1}$ is said to be *totally real* (resp. *complex*) if there is a lift S of $\mathbf{S}$ to $\mathbb{H}^{n,1}$ that is totally real (resp. complex).

Note that not every lift of a totally real subset of $\mathbb{PH}^{n,1}$ is totally real. Hence, in order to check whether a given subset $\mathbf{S}$ of $\mathbb{PH}^{n,1}$ is totally real or not, either one needs to find a lift of $\mathbf{S}$ that is totally real, or needs to verify that every lift of $\mathbf{S}$ is not totally real. The same problem also happens in the totally complex

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case. For this reason, it is not easy to know directly if a given subset of $\mathbb{PH}^{n,1}$ is totally real or complex. To resolve this problem, we introduce equivalent notions for total reality and complexity in $\mathbb{PH}^{n,1}$ in terms of the quaternionic Hermitian triple product.

We introduce new notions, totally real and commutative subsets with respect to the quaternionic Hermitian triple product, in both $\mathbb{H}^{n,1}$ and $\mathbb{PH}^{n,1}$. For $v_1, v_2, v_3 \in \mathbb{H}^{n,1}$, their quaternionic Hermitian triple product is defined as

$$\langle v_1, v_2, v_3 \rangle = \langle v_2, v_1 \rangle \langle v_3, v_2 \rangle \langle v_1, v_3 \rangle.$$

It was verified in [1] that the real part of $\langle v_1, v_2, v_3 \rangle$ is negative for vectors v_1, v_2, v_3 with non-positive norm. Through the quaternionic Hermitian triple product, Apanosov and Kim [1] defined the quaternionic Cartan angular invariant of a triple of distinct points in a quaternionic hyperbolic space.

We say that $S \subset \mathbb{H}^{n,1}$ is *non-degenerate with respect to the quaternionic Hermitian triple product* if $\langle u, v, w \rangle \neq 0$ for all $(u, v, w) \in S^{(3)}$, where $S^{(n)}$ denotes the set of n-tuples of distinct points of a set S. Furthermore a subset $\mathbf{S} \subset \mathbb{PH}^{n,1}$ is said to be *non-degenerate with respect to the quaternionic Hermitian triple product* if some (hence any) lift $S \subset \mathbb{H}^{n,1}$ of $\mathbf{S}$ is non-degenerate with respect to the quaternionic Hermitian triple product. Note that the non-degeneracy of $\mathbf{S}$ with respect to the quaternionic Hermitian triple product does not depend on the choice of a lift S of $\mathbf{S}$.

Definition 1.2. A subset $S \subset \mathbb{H}^{n,1}$ is *totally real* (resp. *commutative*) *with respect to the quaternionic Hermitian triple product* if $\{\langle u_0, v, w \rangle : (u_0, v, w) \in S^{(3)}\}$ is real (resp. commutative) for some $u_0 \in S$. A subset $\mathbf{S} \subset \mathbb{PH}^{n,1}$ is *totally real* (resp. *commutative*) *with respect to the quaternionic Hermitian triple product* if there is a lift $S \subset \mathbb{H}^{n,1}$ of $\mathbf{S}$ that is totally real (resp. commutative) with respect to the quaternionic Hermitian triple product.

Obviously, if $S \subset \mathbb{H}^{n,1}$ is totally real (resp. complex), S is totally real (resp. commutative) with respect to the quaternionic Hermitian triple product. In general, the converse does not hold. However, for non-degenerate subsets of $\mathbb{PH}^{n,1}$ with respect to the quaternionic Hermitian triple product, it turns out that Definitions 1.1 and 1.2 are equivalent to each other.

Theorem 1.3. *Let $\mathbf{S} \subset \mathbb{PH}^{n,1}$ be a non-degenerate subset with respect to the quaternionic Hermitian triple product. Then,*

(i) $\mathbf{S}$ *is totally real if and only if* $\mathbf{S}$ *is totally real with respect to the quaternionic Hermitian triple product.*
(ii) $\mathbf{S}$ *is totally complex if and only if* $\mathbf{S}$ *is totally commutative with respect to the quaternionic Hermitian triple product.*

One advantage of Definition 1.2 is that the total reality and commutativity with respect to the quaternionic Hermitian triple product do not depend on either of the choices of a lift S of $\mathbf{S}$ and $u_0 \in S$. In other words, if there exist a lift S of $\mathbf{S}$ and $u_0 \in S$ such that $\{\langle u_0, v, w \rangle : (u_0, v, w) \in S^{(3)}\}$ is real (resp. commutative), then for any lift T of $\mathbf{S}$ and any $u_1 \in T$, the set $\{\langle u_1, v, w\rangle : (u_1, v, w) \in T^{(3)}\}$ is real (resp. commutative). Hence, Theorem 1.3 makes it easy to check whether a given subset $\mathbf{S}$ of $\mathbb{PH}^{n,1}$ is totally real or complex. More concretely, choose a lift $S \subset \mathbb{H}^{n,1}$ of $\mathbf{S}$ and an element $u_0 \in S$. Then by checking the reality and commutativity of $\{\langle u_0, v, w\rangle : (u_0, v, w) \in S^{(3)}\}$, one can see whether $\mathbf{S}$ is totally real or complex.

We apply our results to quaternionic hyperbolic n-space $\mathbf{H}^n_{\mathbb{H}} \subset \mathbb{PH}^{n,1}$. Quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ is defined as the set of negative lines of $\mathbb{H}^{n,1}$, and its boundary $\partial\mathbf{H}^n_{\mathbb{H}}$ is defined as the set of null lines of $\mathbb{H}^{n,1}$. The isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is $\mathrm{PSp}(n,1)$. Let Γ be a discrete subgroup of $\mathrm{PSp}(n,1)$. The limit set Λ_Γ of Γ is defined to be the smallest nonempty, closed, Γ-invariant subset of $\partial\mathbf{H}^n_{\mathbb{H}}$, or the set of accumulation points of an orbit of a point of $\mathbf{H}^n_{\mathbb{H}}$ in Γ. The group Γ is said to be *elementary* if its limit set has at most two points. From the fact that inner products of non-positive vectors in $\mathbb{H}^{n,1}$ can not be zero, it follows that every subset of $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}} \subset \mathbb{PH}^{n,1}$ is non-degenerate with respect to the quaternionic Hermitian triple product. Futhermore it turns out that totally real and complex subsets of $\mathbf{H}^n_{\mathbb{H}}$ are contained in real and complex hyperbolic subspaces of $\mathbf{H}^n_{\mathbb{H}}$, respectively. Therefore, as a corollary of Theorem 1.3, we characterize the non-elementary discrete subgroups of $\mathrm{PSp}(n,1)$ preserving a real or complex hyperbolic subspace of $\mathbf{H}^n_{\mathbb{H}}$ through the quaternionic Hermitian triple product.

Corollary 1.4. *Let Γ be a non-elementary, discrete subgroup of* $\mathrm{PSp}(n,1)$. *Then its limit set is totally real* (*resp. commutative*) *with respect to the quaternionic Hermitian triple product if and only if Γ preserves a real* (*resp. complex*) *hyperbolic subspace of $\mathbf{H}^n_{\mathbb{H}}$.*

We explore the total reality and complexity of finite points in $\mathbb{PH}^{n,1}$. Obviously, any one point in $\mathbb{PH}^{n,1}$ is totally real. Moreover, it can be easily seen that any two distinct points in $\mathbb{PH}^{n,1}$ are also totally real. The total reality first fails for 3 distinct points in $\mathbb{PH}^{n,1}$. On the other hand, total complexity holds for 3 distinct points in $\mathbb{PH}^{n,1}$ but fails for more than 3 distinct points. As an application, we prove that every quaternionic hyperbolic triangle group in $\mathbf{H}^2_{\mathbb{H}}$ leaves a complex hyperbolic 2-subspace of $\mathbf{H}^n_{\mathbb{H}}$ invariant, which generalizes the result of Cao and Huang in [2].

Theorem 1.5. *Let $\Gamma \subset$ $\mathrm{PSp}(2,1)$ be a quaternionic hyperbolic triangle group acting on $\mathbf{H}^2_{\mathbb{H}}$. Then Γ preserves a complex hyperbolic* 2*-subspace of $\mathbf{H}^2_{\mathbb{H}}$. Therefore, Γ is conjugate to a subgroup of* $\mathrm{PU}(2,1)$.

Theorem 1.5 implies that every quaternionic hyperbolic triangle group is indeed a complex hyperbolic triangle group. Goldman and Parker [4] and Schwartz [6] classified which complex hyperbolic ideal triangle groups are discrete and faithful through the Cartan angular invariant. Theorem 1.5 makes it possible to apply their theorems to the quaternionic hyperbolic ideal triangle groups.

2. Preliminaries

In this section, we collect basic definitions and facts on quaternionic hyperbolic geometry. For more details, we refer the reader to [1, 3, 5].

Let $\mathbb{H}^{n,1}$ be a right quaternionic vector space of dimension $n+1$ with a Hermitian form of signature $(n,1)$. Vectors are multiplied by quaternions from the right:

$$\mathbb{H}^{n,1} \times \mathbb{H} \to \mathbb{H}^{n,1},\ (v,\lambda) \mapsto v\lambda.$$

An element of $\mathbb{H}^{n,1}$ is a column vector $v = (v_1, \dots, v_{n+1})^t \in \mathbb{H}^{n+1}$. More precisely, set

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where I_{n-1} is the identity matrix of size $n-1$ and choose the Hermitian form on $\mathbb{H}^{n,1}$ given by

$$\langle u, v\rangle = v^* J u = \overline{v}_{n+1} u_1 + \overline{v}_2 u_2 + \cdots + \overline{v}_n u_n + \overline{v}_1 u_{n+1}$$

for $v = (v_1, \dots, v_{n+1})^t$, $u = (u_1, \dots, u_{n+1})^t \in \mathbb{H}^{n+1}$. By the definition, for $\lambda, \mu \in \mathbb{H}$,

$$\langle u\lambda, v\mu\rangle = \bar{\mu}\langle u, v\rangle\lambda.$$

The group $\mathrm{Sp}(n,1)$ is the subgroup of $\mathrm{GL}(n+1,\mathbb{H})$ which, when acting on the left, preserves the Hermitian form J given above.

Let $\mathbf{0}$ be the zero vector in $\mathbb{H}^{n+1}$ and $\mathbb{P} : \mathbb{H}^{n,1} \setminus \{\mathbf{0}\} \to \mathbb{P}\mathbb{H}^{n,1}$ be the canonical projection onto a right quaternionic projective space $\mathbb{P}\mathbb{H}^{n,1}$. Consider the following subspaces in $\mathbb{H}^{n,1}$;

$$V_0 = \{v \in \mathbb{H}^{n,1} - \{\mathbf{0}\} \mid \langle v, v\rangle = 0\},$$
$$V_- = \{v \in \mathbb{H}^{n,1} \mid \langle v, v\rangle < 0\}.$$

The n-dimensional *quaternionic hyperbolic space* $\mathbf{H}^n_{\mathbb{H}}$ is defined as $\mathbb{P}(V_-)$ and its *boundary* $\partial\mathbf{H}^n_{\mathbb{H}}$ is defined as $\mathbb{P}(V_0)$. There is a metric on $\mathbf{H}^n_{\mathbb{H}}$ called the Bergman metric and the isometry group of $\mathbf{H}^n_{\mathbb{H}}$ with respect to this metric is

$$\begin{aligned}\mathrm{PSp}(n,1) &= \left\{[A] \mid A \in \mathrm{GL}(n+1,\mathbb{H}), \langle u, v\rangle = \langle Au, Av\rangle \text{ for all } u, v \in \mathbb{H}^{n,1}\right\} \\ &= \{[A] \mid A \in \mathrm{GL}(n+1,\mathbb{H}), J = A^* J A\},\end{aligned}$$

where $[A] : \mathbb{P}\mathbb{H}^{n,1} \to \mathbb{P}\mathbb{H}^{n,1}; x\mathbb{H} \mapsto (Ax)\mathbb{H}$ for $A \in \mathrm{Sp}(n,1)$. Here we adopt the convention that $\mathrm{Sp}(n,1)$ acts on $\mathbf{H}^n_{\mathbb{H}}$ on the left and the projectivization of $\mathrm{Sp}(n,1)$ acts on the right. In fact $\mathrm{PSp}(n,1)$ is the quotient of $\mathrm{Sp}(n,1)$ by its center, which is just $\{\pm I_{n+1}\}$. Naturally, the real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$ and the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ are embedded in $\mathbf{H}^n_{\mathbb{H}}$. Indeed, the real hyperbolic m-space $\mathbf{H}^m_{\mathbb{R}}$ and the complex hyperbolic m-space for $m \leq n$ are identified with $\mathbb{P}(\mathrm{Span}_{\mathbb{R}}\{e_1, \dots, e_m, e_{n+1}\})$ and $\mathbb{P}(\mathrm{Span}_{\mathbb{C}}\{e_1, \dots, e_m, e_{n+1}\})$ in $\mathbf{H}^n_{\mathbb{H}}$, respectively. A subspace of $\mathbf{H}^n_{\mathbb{H}}$ isometric to $\mathbf{H}^m_{\mathbb{R}}$ (resp. $\mathbf{H}^m_{\mathbb{C}}$) for some $m \leq n$ is called a real (resp. complex) hyperbolic subspace of $\mathbf{H}^n_{\mathbb{H}}$.

3. Totally real and complex subspaces

In this section, we define and classify the totally real and complex vector subspaces of $\mathbb{H}^{n,1}$ instead of the totally real and complex subsets of $\mathbb{H}^{n,1}$ since the totally real and complex vector subsets span the totally real and complex subspaces, respectively.

3.1. Totally $\mathbb{F}$-vector subspaces

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and denote the set of non-zero elements of $\mathbb{F}$ by $\mathbb{F}^*$. We define the notion of a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$.

Definition 3.1. Let $V \subset \mathbb{H}^{n,1}$ be a vector space over $\mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. An $\mathbb{F}$-vector space $V \subset \mathbb{H}^{n,1}$ is called *a totally $\mathbb{F}$-vector subspace* of $\mathbb{H}^{n,1}$ if $\langle v, w\rangle \in \mathbb{F}$ for all $v, w \in V$.

Clearly, a totally real subset of $\mathbb{H}^{n,1}$ spans a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$ and a totally complex subset of $\mathbb{H}^{n,1}$ spans a totally $\mathbb{C}$-vector subspace of $\mathbb{H}^{n,1}$. Any $\mathbb{F}$-vector subspace of dimension 1 is a totally $\mathbb{R}$-vector subspace and hence a totally $\mathbb{F}$-vector subspace for any $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We begin with a simple observation.

Lemma 3.2. *Let V be a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$. Let v be a non-zero vector in V such that $\langle v, v\rangle \neq 0$. If $vq \in V$ for some quaternion $q \in \mathbb{H}$, then $q \in \mathbb{F}$.*

Proof. Suppose that $vq \in V$ for some quaternion $q \in \mathbb{H}$. By the assumption that V is a totally $\mathbb{F}$-vector subspace, we have that $\langle vq, v\rangle = \langle v, v\rangle q = |v|^2 q \in \mathbb{F}$. Since $|v|^2$ is a non-zero real number, it immediately follows that $q \in \mathbb{F}$. □

By Lemma 3.2, one can easily see that if $\langle v, v\rangle \neq 0$, then $\mathrm{Span}_{\mathbb{F}}\{v\}$ is the unique one-dimensional totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$ containing v.

If V is a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$, then $g \cdot V$ is also a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$ for any $g \in \mathrm{Sp}(n, 1)$. For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we classify all totally $\mathbb{F}$-vector subspaces of $\mathbb{H}^{n,1}$ up to the action of $\mathrm{Sp}(n, 1)$ on $\mathbb{H}^{n,1}$ as follows.

Proposition 3.3. *Let V be a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$ with dimension ≥ 1, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then V is isomorphic, up to the action of $\mathrm{Sp}(n, 1)$ on $\mathbb{H}^{n,1}$, to one of the following*:

(i) $\mathrm{Span}_{\mathbb{F}}\{e_1 q_1, \ldots, e_1 q_b\}$, *where $1 \leq b \leq \dim_{\mathbb{F}} \mathbb{H}$ and $q_i \in \mathbb{H}^*$ for each $i = 1, \ldots, b$*;
(ii) $\mathrm{Span}_{\mathbb{F}}\{e_2\}$;
(iii) $\mathrm{Span}_{\mathbb{F}}\{e_1 - e_{n+1}\}$;
(iv) $\mathrm{Span}_{\mathbb{F}}\{e_2, \ldots, e_{a+1}, e_1 q_1, \ldots, e_1 q_b\}$, *where $1 \leq a \leq n-1$ and $1 \leq b \leq \dim_{\mathbb{F}} \mathbb{H}$*;
(v) $\mathrm{Span}_{\mathbb{F}}\{e_1 + e_{n+1}, e_2, \ldots, e_a\}$, *where $2 \leq a \leq n$*;
(vi) $\mathrm{Span}_{\mathbb{F}}\{e_1, \ldots, e_a, e_{n+1}\}$, *where $1 \leq a \leq n$.*

Proof. Choose a basis $\{v_1, \dots, v_m\}$ of V. Clearly we can write $V = \mathrm{Span}_{\mathbb{F}}\{v_1, \dots, v_m\}$. We first suppose that $\dim_{\mathbb{H}}(\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_m\}) = 1$, i.e., $\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_m\} = v\mathbb{H}$ for some non-zero vector $v \in \mathbb{H}^{n,1}$. Then for each $1 \leq l \leq m$ there is a non-zero quaternion q_l such that $v_l = vq_l$. Since $v\mathbb{H}$ is an $\mathbb{F}$-vector space of dimension $\dim_{\mathbb{F}} \mathbb{H}$, the dimension of V can not exceed $\dim_{\mathbb{F}} \mathbb{H}$. Thus $1 \leq m \leq \dim_{\mathbb{F}} \mathbb{H}$. If v is a null vector, there is an element of $\mathrm{Sp}(n, 1)$ mapping v to e_1 and thus we conclude that V is isomorphic to (i). If v is not a null vector, Lemma 3.2 implies that $V = \mathrm{Span}_{\mathbb{F}}\{vq\}$ for some non-zero quaternion $q \in \mathbb{H}^*$. It can be easily seen that if $\langle v, v\rangle > 0$, $\mathrm{Span}_{\mathbb{F}}\{vq\}$ is isomorphic to $\mathrm{Span}_{\mathbb{F}}\{e_2\}$ and if $\langle v, v\rangle < 0$, $\mathrm{Span}_{\mathbb{F}}\{vq\}$ is isomorphic to $\mathrm{Span}_{\mathbb{F}}\{e_1 - e_{n+1}\}$, up to the action of $\mathrm{Sp}(n, 1)$ on $\mathbb{H}^{n,1}$.

We now suppose that $\dim_{\mathbb{H}}(\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_m\}) \geq 2$. We will apply Gram–Schmidt orthogonalization to the basis $\{v_1, \dots, v_m\}$ of V. One problem here is the existence of null vectors in $\mathbb{H}^{n,1}$. For example, if $v_1, \dots, v_m$ are all null vectors, one can not even start the Gram–Schmidt orthogonalization procedure. Assume that $v_1, \dots, v_m$ are all null vectors. By the hypothesis that $\dim_{\mathbb{H}}(\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_m\}) \geq 2$, there are two linearly independent null vectors v_c and v_d over $\mathbb{H}$ for some distinct integers $1 \leq c, d \leq m$. Recalling that two linearly independent null vectors in $\mathbb{H}^{n,1}$ can not be orthogonal, it is easy to see that both $v_c - v_d$ and $v_c + v_d$ are non-null vectors and obviously they are linearly independent. Replacing v_c and v_d by $v_c - v_d$ and $v_c + v_d$ in the basis $\{v_1, \dots, v_m\}$, we obtain a new basis of V in which there is a non-null vector. By this process, two linearly independent null vectors over $\mathbb{H}$ can be always replaced by two linearly independent non-null vectors over $\mathbb{H}$ in a basis of V. Hence we may assume that there is a non-null vector in $\{v_1, \dots, v_m\}$ whenever $\dim_{\mathbb{H}}(\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_m\}) \geq 2$.

Let v_1 be a non-null vector. Since V is a totally $\mathbb{F}$-vector subspace, one can see that $\mathrm{proj}_{v_1}(v_l) \in V$ for all $l = 2, \dots, m$, where

$$\mathrm{proj}_u(v) = u \cdot \langle v, u\rangle \langle u, u\rangle^{-1}.$$

One can easily check that

$$\begin{aligned}\langle v - \mathrm{proj}_u(v), u\rangle &= \langle v, u\rangle - \langle u \cdot \langle v, u\rangle\langle u, u\rangle^{-1}, u\rangle \\ &= \langle v, u\rangle - \langle u, u\rangle\langle v, u\rangle\langle u, u\rangle^{-1} = 0,\end{aligned}$$

which implies that $v - \mathrm{proj}_u(v)$ is orthogonal to u. Replace v_l by $v_l - \mathrm{proj}_{v_1}(v_l) \in V$ for each $l = 2, \dots, m$. For simplicity, denote $v_l - \mathrm{proj}_{v_1}(v_l) \in V$ again by v_l. Then v_l is orthogonal to v_1 for all $l = 2, \dots, m$. Applying both the process of switching two linearly independent null vectors over $\mathbb{H}$ into two linearly independent non-null vectors and the Gram–Schmidt orthogonalization procedure inductively, we finally get a basis $\{v_1, \dots, v_m\}$ such that $v_1, \dots, v_a$ are pairwise orthogonal non-null vectors and $\mathrm{Span}_{\mathbb{H}}\{v_{a+1}, \dots, v_m\}$ is a 1-dimensional vector space over $\mathbb{H}$ spanned by a null vector for some integer $1 \leq a \leq m$ and moreover it is orthogonal to $\mathrm{Span}_{\mathbb{H}}\{v_1, \dots, v_a\}$. If $a = m$, we set $\mathrm{Span}_{\mathbb{H}}\{v_{a+1},\ldots,v_m\} = \emptyset$. By normalizing $v_1,\ldots,v_a$, we can assume that $\langle v_l, v_l\rangle = \pm 1$ for all $l = 1,\ldots,a$.

Consider two cases: $a < m$ and $a = m$. If $a < m$, then v_m is a null vector and all vectors of $v_1,\ldots,v_a$ are orthogonal to the null vector v_m. The orthogonal complement of a null vector v_m includes v_m itself and has dimension n. Thus $1 \le a \le n-1$. The fact that any vector with negative norm can not be orthogonal to any null vector leads us to conclude that every non-null vector v_l must have a positive norm and hence $\langle v_l, v_l\rangle = 1$ for all $l = 1,\ldots,a$. There is an element of $\mathrm{Sp}(n,1)$ which maps $v_1,\ldots,v_a,v_m$ to $e_2,\ldots,e_{a+1},e_1$, respectively. Therefore, V is isomorphic to (iv).

If $a = m$, every vector v_l is a non-null vector for $l = 1,\ldots,m$. If every vector v_l has a positive norm, V is isomorphic to (v) since there is an element of $\mathrm{Sp}(n,1)$ which maps $v_1,\ldots,v_m$ to $\frac{e_1+e_{n+1}}{\sqrt{2}}, e_2,\ldots,e_m$, respectively. Otherwise, since no two vectors with negative norm can be orthogonal, only one vector of $v_1,\ldots,v_m$ has a negative norm. Let $\langle v_1, v_1\rangle = -1$ and $\langle v_l, v_l\rangle = 1$ for all $l = 2,\ldots,m$. Then $\{v_1,\ldots,v_m\}$ is isomorphic to $\{(e_1-e_{n+1})/\sqrt{2}, e_2,\ldots,e_{m-1},(e_1+e_{n+1})/\sqrt{2}\}$ up to the action of $\mathrm{Sp}(n,1)$ on $\mathbb{H}^{n,1}$ and thus V is isomorphic to (vi). $\square$

Note that (iii) and (vi) are the only types of totally $\mathbb{F}$-vector subspaces of $\mathbb{H}^{n,1}$ in Proposition 3.3 which contains a negative vector. From this observation, we immediately obtain the following corollary.

Corollary 3.4. *Let V be a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. If V has a vector with negative norm, then V is isomorphic to either* $\mathrm{Span}_{\mathbb{F}}\{e_1 - e_{n+1}\}$ *or* $\mathrm{Span}_{\mathbb{F}}\{e_1,\ldots,e_m,e_{n+1}\}$ *up to the action of* $\mathrm{Sp}(n,1)$ *on* $\mathbb{H}^{n,1}$ *for some integer* $1 \le m \le n$.

3.2. Totally projective $\mathbb{F}$-subspaces

In this section, we treat the notion of totally projective $\mathbb{F}$-subspace in the right quaternionic projective space $\mathbb{PH}^{n,1}$.

Definition 3.5. A subset $\mathbf{V} \subset \mathbb{PH}^{n,1}$ is said to be a *totally projective* $\mathbb{F}$-*subspace* if $\mathbf{V}$ is a projectivization of a totally $\mathbb{F}$-vector subspace V of $\mathbb{H}^{n,1}$.

In particular, if $\dim_{\mathbb{F}} V = 2$ and V is a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n,1}$, we call $\mathbb{P}V$ a *totally projective* $\mathbb{F}$-*line* in $\mathbb{PH}^{n,1}$. Now we will explore how many totally projective $\mathbb{F}$-lines pass through two given distinct points of $\mathbb{PH}^{n,1}$. Obviously, given two distinct points of $\mathbb{PH}^{n,1}$, there is exactly one $\mathbb{H}$-line through them. However, the situation is a little different when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

3.2.1. *Totally projective* $\mathbb{R}$-*lines.* Projectivizations of totally $\mathbb{R}$-vector subspaces of $\mathbb{H}^{n,1}$ make it possible that two distinct totally $\mathbb{R}$-vector subspaces have the same projectivization onto $\mathbb{PH}^{n,1}$. In order to study this issue, we begin with the following simple observation.

Lemma 3.6. *Let $v_1, v_2 \in \mathbb{H}^{n+1}$ be linearly independent vectors over $\mathbb{H}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2$ be non-zero quaternions. Then*

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\lambda_1, v_2\lambda_2) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\mu_1, v_2\mu_2)$$

if and only if $\mu_1^{-1}\lambda_1, \mu_2^{-1}\lambda_2 \in \mathbb{R}^*q$ for some non-zero quaternion $q \in \mathbb{H}$.

Proof. Suppose that $\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\lambda_1, v_2\lambda_2) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\mu_1, v_2\mu_2)$. Then there exist $r_1, r_2 \in \mathbb{R}^*$ and $q \in \mathbb{H}^*$ such that $v_1\lambda_1 + v_2\lambda_2 = (v_1\mu_1r_1 + v_2\mu_2r_2)q$. Since v_1 and v_2 are linearly independent over $\mathbb{H}$, it is obvious that $\lambda_1 = \mu_1r_1q$ and $\lambda_2 = \mu_2r_2q$. Thus $\mu_a^{-1}\lambda_a \in \mathbb{R}^*q$ for all $a = 1, 2$.

Conversely, suppose that $\lambda_1 = \mu_1r_1q$ and $\lambda_2 = \mu_2r_2q$ for some $r_1, r_2 \in \mathbb{R}^*$ and some $q \in \mathbb{H}^*$. Then, for any $s_1, s_2 \in \mathbb{R}$,

$$v_1\lambda_1s_1 + v_2\lambda_2s_2 = (v_1\mu_1(r_1s_1) + v_2\mu_2(r_2s_2))q,$$

which implies $\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\lambda_1, v_2\lambda_2) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\mu_1, v_2\mu_2)$. □

Due to Lemma 3.6, we easily get its generalization as follows.

Corollary 3.7. *Let $v_1, \ldots, v_m \in \mathbb{H}^{n+1}$ be linearly independent vectors over $\mathbb{H}$ and $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m$ be non-zero quaternions. Then*

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\lambda_1, \ldots, v_m\lambda_m) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\mu_1, \ldots, v_m\mu_m)$$

if and only if $\mu_1^{-1}\lambda_1, \ldots, \mu_m^{-1}\lambda_m \in \mathbb{R}^*q$ for some $q \in \mathbb{H}^*$.

Proof. Suppose that $\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\lambda_1, \ldots, v_m\lambda_m) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1\mu_1, \ldots, v_m\mu_m)$. From the hypothesis that $v_1, \ldots, v_m \in \mathbb{H}^{n+1}$ are linearly independent vectors over $\mathbb{H}$, it follows that for any distinct integers $1 \le a, b \le m$,

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_a\lambda_a, v_b\lambda_b) = \mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_a\mu_a, v_b\mu_b).$$

Applying Lemma 3.6, there is a quaternion $q \in \mathbb{H}^*$ such that $\mu_a^{-1}\lambda_a \in \mathbb{R}^*q$ for all $a = 1, \ldots, m$. The converse is obvious. □

Now we are ready to answer the question of how many totally projective $\mathbb{R}$-lines pass through two distinct points of $\mathbb{P}\mathbb{H}^{n,1}$.

Lemma 3.8. *Let $v_1, \ldots, v_m \in \mathbb{H}^{n,1}$ be linearly independent vectors over $\mathbb{H}$. Then $\mathrm{Span}_{\mathbb{F}}\{v_1, \ldots, v_m\}$ is a totally $\mathbb{F}$-vector subspace of $\mathbb{H}^{n+1}$ if and only if $\langle v_a, v_b\rangle \in \mathbb{F}$ for all integers $1 \le a, b \le m$.*

Proof. The lemma easily follows from the formula

$$\left\langle \sum_{a=1}^{m} v_ar_a, \sum_{b=1}^{m} v_bs_b \right\rangle = \sum_{a=1}^{m}\sum_{b=1}^{m} \bar{s}_b\langle v_a, v_b\rangle r_a.$$

The details are left to the reader. □

Proposition 3.9. *Let v_1 and v_2 be linear independent vectors in $\mathbb{H}^{n,1}$ over $\mathbb{H}$. If $\langle v_1, v_2\rangle = 0$, then $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\}$ is a totally projective $\mathbb{R}$-line for any non-zero quaternions q_1, q_2 and moreover, the set of totally projective $\mathbb{R}$-lines through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ in $\mathbb{P}\mathbb{H}^n$ is identified with $\mathbb{R}\mathbb{P}^3$. If $\langle v_1, v_2\rangle \neq 0$, then*

$\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2\langle v_1, v_2\rangle\}$ *is the unique totally projective* $\mathbb{R}$*-line through* $\mathbb{P}(v_1)$ *and* $\mathbb{P}(v_2)$ *in* $\mathbb{P}\mathbb{H}^{n,1}$.

Proof. First, suppose that $\langle v_1, v_2\rangle = 0$. By Lemma 3.8, $\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\}$ is a totally $\mathbb{R}$-vector space of $\mathbb{H}^{n,1}$ for any non-zero quaternions q_1, q_2 and hence $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\}$ is a totally projective $\mathbb{R}$-line for any $q_1, q_2 \in \mathbb{H}^*$. Noting that

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\} = \mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2q_2q_1^{-1}\},$$

any totally projective $\mathbb{R}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ can be written as $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2q\}$ for $q \in \mathbb{H}^*$. According to Lemma 3.6, $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2p\} = \mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2q\}$ if and only if $p \in \mathbb{R}^*q$, which implies that there is a one-to-one correspondence between the space of totally projective $\mathbb{R}$-lines through two distinct points $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ in $\mathbb{P}\mathbb{H}^{n,1}$ and the space of one-dimensional real vector subspaces of $\mathbb{H}$, that is, $\mathbb{R}\mathbb{P}^3$.

Now we suppose that $\langle v_1, v_2\rangle \neq 0$. By Lemma 3.8, $\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\}$ is a totally $\mathbb{R}$-vector space if and only if

$$\langle v_1q_1, v_2q_2\rangle = \bar{q}_2\langle v_1, v_2\rangle q_1 \in \mathbb{R}^*, \text{ i.e., } q_2 = r\langle v_1, v_2\rangle q_1 \text{ for some } r \in \mathbb{R}^*.$$

Applying Lemma 3.6,

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2q_2\} = \mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1q_1, v_2r\langle v_1, v_2\rangle q_1\} = \mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2\langle v_1, v_2\rangle\},$$

which implies that $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2\langle v_1, v_2\rangle\}$ is the unique totally projective $\mathbb{R}$-line passing through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$. This completes the proof. □

Let v_1 and v_2 be vectors with non-positive norm such that $\mathbb{P}(v_1) \neq \mathbb{P}(v_2)$. In other words, $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ are two distinct points in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$. Then $\langle v_1, v_2\rangle \neq 0$. By Proposition 3.9, there is a unique totally projective $\mathbb{R}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$. This $\mathbb{R}$-line can be written as $\mathbb{P}\mathrm{Span}_{\mathbb{R}}(v_1, v_2\langle v_1, v_2\rangle)$. Moreover, $\mathrm{Span}_{\mathbb{R}}(v_1, v_2\langle v_1, v_2\rangle)$ has a vector $v_1 - v_2\langle v_1, v_2\rangle$ with negative norm:

$$\langle v_1 - v_2\langle v_1, v_2\rangle, v_1 - v_2\langle v_1, v_2\rangle\rangle = \langle v_1, v_1\rangle - 2|\langle v_1, v_2\rangle|^2 + \langle v_2, v_2\rangle|\langle v_1, v_2\rangle|^2 < 0.$$

According to Corollary 3.4, $\mathrm{Span}_{\mathbb{R}}(v_1, v_2\langle v_1, v_2\rangle)$ is isomorphic to $\mathrm{Span}_{\mathbb{R}}(e_1, e_{n+1})$ up to the action of $\mathrm{Sp}(n,1)$ on $\mathbb{H}^{n,1}$ and the intersection of $\mathbb{P}\mathrm{Span}_{\mathbb{R}}(e_1, e_{n+1})$ and $\mathbf{H}^n_{\mathbb{H}}$ is $\mathbf{H}^1_{\mathbb{R}}$, which is a geodesic in $\mathbf{H}^n_{\mathbb{H}}$. Summarizing, we have the following corollary.

Corollary 3.10. *The totally projective* $\mathbb{R}$*-line passing through two given distinct points in* $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ *is unique. Furthermore its intersection with* $\mathbf{H}^n_{\mathbb{H}}$ *is the geodesic joining those two points.*

3.2.2. *Totally projective* $\mathbb{C}$*-lines.* We now turn to totally projective $\mathbb{C}$-lines through two distinct points of $\mathbb{P}\mathbb{H}^{n,1}$. The complex case is a little more complicated than the real case since complex numbers do not commute with quaternion numbers while real numbers do. To handle this issue, we begin by proving an elementary fact about quaternions.

Lemma 3.11. *Suppose that p and q are non-zero quaternions such that $p\cdot\mathbb{C} = \mathbb{C}\cdot q$. Then either both p and q are complex numbers or they are elements of $\mathbb{C}\cdot j$.*

Proof. From the hypothesis that $p\cdot\mathbb{C} = \mathbb{C}\cdot q$, we can write $p = zq$ and $pi = wq$ for some $z, w \in \mathbb{C}^*$. Then $p = zq = -wqi$ and thus $qi\bar{q} = -w^{-1}z|q|^2 \in \mathbb{C}$. Let $q = q_1 + q_2 i + q_3 j + q_4 k$ for $q_1, q_2, q_3, q_4 \in \mathbb{R}$. By a straightforward computation,

$$qi\bar{q} = (q_1^2 + q_2^2 - q_3^2 - q_4^2)i + 2(q_2q_3 + q_1q_4)j + 2(-q_1q_3 + q_2q_4)k.$$

Thus $qi\bar{q} \in \mathbb{C}$ is equivalent to

$$q_2q_3 + q_1q_4 = -q_1q_3 + q_2q_4 = 0. \tag{3.1}$$

From these two equations,

$$q_1q_2(q_3^2 + q_4^2) = (q_1q_3)(q_2q_3) + q_1q_2q_4^2 = (q_2q_4)(-q_1q_4) + q_1q_2q_4^2 = 0.$$

If $q_3^2 + q_4^2 = 0$, then $q \in \mathbb{C}$ and hence $p \in \mathbb{C}\cdot q = \mathbb{C}$. If $q_3^2 + q_4^2 \neq 0$, then $q_1q_2 = 0$ which is equivalent to $q_1 = 0$ or $q_2 = 0$. In either case, one can easily conclude that $q_1 = q_2 = 0$ by (3.1). Therefore, $q \in \mathbb{C}\cdot j$ and $p = zq \in \mathbb{C}\cdot j$. This completes the proof. □

Projectivizations of totally $\mathbb{C}$-vector subspaces of $\mathbb{H}^{n,1}$ onto $\mathbb{PH}^{n,1}$ may cause two distinct totally $\mathbb{C}$-vector subspaces to have the same projectivization. To count how many totally projective $\mathbb{C}$-lines pass through two given distinct points of $\mathbb{PH}^{n,1}$, we first prove the following lemma.

Lemma 3.12. *Let v_1 and v_2 be linearly independent vectors over $\mathbb{H}$ and λ_1, λ_2, μ_1, μ_2 be non-zero quaternions. Then*

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1, v_2\lambda_2) = \mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1, v_2\mu_2)$$

if and only if either $\lambda_a i \lambda_a^{-1} = \mu_a i \mu_a^{-1}$ for all $a = 1, 2$ or $\lambda_a i \lambda_a^{-1} = -\mu_a i \mu_a^{-1}$ for all $a = 1, 2$.

Proof. First suppose that $\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1, v_2\lambda_2) = \mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1, v_2\mu_2)$. This means that given complex numbers z_1 and z_2, there exist $w_1, w_2 \in \mathbb{C}$ and $q \in \mathbb{H}$ such that

$$v_1\lambda_1 z_1 + v_2\lambda_2 z_2 = (v_1\mu_1 w_1 + v_1\mu_2 w_2)q.$$

By the linear independence of v_1 and v_2 over $\mathbb{H}$, it follows that

$$\lambda_1 z_1 - \mu_1 w_1 q = \lambda_2 z_2 - \mu_2 w_2 q = 0$$

which gives $q = w_1^{-1}\mu_1^{-1}\lambda_1 z_1 = w_2^{-1}\mu_2^{-1}\lambda_2 z_2$ and hence $(\mu_2^{-1}\lambda_2)z_2z_1^{-1}(\mu_1^{-1}\lambda_1)^{-1} = w_2w_1^{-1} \in \mathbb{C}$. Therefore we obtain that $(\mu_2^{-1}\lambda_2)\cdot\mathbb{C} = \mathbb{C}\cdot(\mu_1^{-1}\lambda_1)$. By Lemma 3.11, we conclude that $\mu_1^{-1}\lambda_1$ and $\mu_2^{-1}\lambda_2$ are either both in $\mathbb{C}$ or both in $\mathbb{C}\cdot j$.

If $\mu_1^{-1}\lambda_1$ and $\mu_2^{-1}\lambda_2$ are in $\mathbb{C}$, then $\lambda_1 = \mu_1 z_1$ and $\lambda_2 = \mu_2 z_2$ for some $z_1, z_2 \in \mathbb{C}^*$. Then for $a = 1, 2$,

$$\lambda_a i \lambda_a^{-1} = \mu_a z_a i z_a^{-1} \mu_a^{-1} = \mu_a i \mu_a^{-1}.$$

If $\mu_1^{-1}\lambda_1$ and $\mu_2^{-1}\lambda_2$ are in $\mathbb{C}\cdot j$, then $\lambda_1=\mu_1 z_1 j$ and $\lambda_2=\mu_2 z_2 j$ for some $z_1,z_2\in\mathbb{C}^*$. Then for $a=1,2$,

$$\lambda_a i\lambda_a^{-1}=\mu_a z_a j i j^{-1} z_a^{-1}\mu_a^{-1}=-\mu_a i\mu_a^{-1}.$$

Conversely, suppose that $\lambda_a i\lambda_a^{-1}=\mu_a i\mu_a^{-1}$ for all $a=1,2$ or $\lambda_a i\lambda_a^{-1}=-\mu_a i\mu_a^{-1}$ for all $a=1,2$. If $\lambda_a i\lambda_a^{-1}=\mu_a i\mu_a^{-1}$ for all $a=1,2$, it follows from Lemma 3.11 that $\mu_a^{-1}\lambda_a\in\mathbb{C}$ for all $a=1,2$, which means that $\lambda_1=\mu_1 z_1$ and $\lambda_2=\mu_2 z_2$ for some $z_1,z_2\in\mathbb{C}^*$. Then it is obvious that

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1,v_2\lambda_2)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1 z_1,v_2\mu_2 z_2)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1,v_2\mu_2).$$

If $\lambda_a i\lambda_a^{-1}=-\mu_a i\mu_a^{-1}$ for all $a=1,2$, then by Lemma 3.11, $\lambda_1=\mu_1 z_1 j$ and $\lambda_2=\mu_2 z_2 j$ for some $z_1,z_2\in\mathbb{C}^*$. Noting that $zj=j\bar{z}$ for any complex number z, we have that for any complex numbers $w_1,w_2\in\mathbb{C}$,

$$\begin{aligned}v_1\lambda_1 w_1+v_2\lambda_2 w_2&=v_1\mu_1 z_1 j w_1+v_2\mu_2 z_2 j w_2\\&=v_1\mu_1 z_1\bar{w}_1 j+v_2\mu_2 z_2\bar{w}_2 j\\&=(v_1\mu_1 z_1\bar{w}_1+v_2\mu_2 z_2\bar{w}_2)j,\end{aligned}$$

which implies $\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1,v_2\lambda_2)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1,v_2\mu_2)$. □

As a corollary, we have a generalized version of Lemma 3.12 as follows.

Corollary 3.13. *Let $v_1,\dots,v_m\in\mathbb{H}^{n,1}$ be linearly independent vectors over $\mathbb{H}$ and $\lambda_1,\dots,\lambda_m,\mu_1,\dots,\mu_m$ be non-zero quaternions. Then*

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1,\dots,v_m\lambda_m)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1,\dots,v_m\mu_m)$$

if and only if $\lambda_a i\lambda_a^{-1}=\mu_a i\mu_a^{-1}$ for all $a=1,\dots,m$ or $\lambda_a i\lambda_a^{-1}=-\mu_a i\mu_a^{-1}$ for all $a=1,\dots,m$.

Proof. First of all, observe that by the linear independence of $v_1,\dots,v_m\in\mathbb{H}^{n,1}$ over $\mathbb{H}$, the condition that $\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\lambda_1,\dots,v_m\lambda_m)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_1\mu_1,\dots,v_m\mu_m)$ implies that for any integers $1\le a,b\le m$,

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_a\lambda_a,v_b\lambda_b)=\mathbb{P}\mathrm{Span}_{\mathbb{C}}(v_a\mu_a,v_b\mu_b).$$

Applying Lemma 3.12, we conclude that $\lambda_a i\lambda_a^{-1}=\mu_a i\mu_a^{-1}$ and $\lambda_b i\lambda_b^{-1}=\mu_b i\mu_b^{-1}$ for any distinct integers $1\le a,b\le m$ or $\lambda_a i\lambda_a^{-1}=-\mu_a i\mu_a^{-1}$ and $\lambda_b i\lambda_b^{-1}=-\mu_b i\mu_b^{-1}$ for any distinct integers $1\le a,b\le m$. This leads us to conclude that $\lambda_a i\lambda_a^{-1}=\mu_a i\mu_a^{-1}$ for all $a=1,\dots,m$ or $\lambda_a i\lambda_a^{-1}=-\mu_a i\mu_a^{-1}$ for all $a=1,\dots,m$.

The converse easily follows by a similar argument as in the proof of Lemma 3.12. □

Now we are ready to count how many totally projective $\mathbb{C}$-lines passing through two distinct points of $\mathbb{P}\mathbb{H}^{n,1}$.

Proposition 3.14. *Let* v_1 *and* v_2 *be linearly independent vectors over* $\mathbb{H}$*. If* $\langle v_1, v_2\rangle = 0$*, then* $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2q_2\}$ *is a totally projective* $\mathbb{C}$*-line for any non-zero quaternions* q_1, q_2 *and thus the set of totally projective* $\mathbb{C}$*-lines through* $\mathbb{P}(v_1)$ *and* $\mathbb{P}(v_2)$ *is identified with the quotient space of* $S^2 \times S^2$ *under the following equivalence relation*

$$(u_1, u_2) \sim (-u_1, -u_2).$$

If $\langle v_1, v_2\rangle \neq 0$*, any totally projective* $\mathbb{C}$*-line through* $\mathbb{P}(v_1)$ *and* $\mathbb{P}(v_2)$ *can be written as*

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q, v_2\langle v_2, v_1\rangle q\}$$

for some $q \in \mathbb{H}^*$ *and thus the set of totally projective* $\mathbb{C}$*-lines through* $\mathbb{P}(v_1)$ *and* $\mathbb{P}(v_2)$ *is identified with* $\mathbb{RP}^2$*.*

Proof. First suppose that $\langle v_1, v_2\rangle = 0$. Then $\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2q_2\}$ is a totally $\mathbb{C}$-vector space for any $q_1, q_2 \in \mathbb{H}^*$. Define a map from the set of all totally projective $\mathbb{C}$-lines through $\mathbb{P}v_1$ and $\mathbb{P}v_2$ into the quotient space of $S^2\times S^2$ under the following equivalence relation

$$(u_1, u_2) \sim (-u_1, -u_2)$$

by mapping $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2q_2\}$ to $[(q_1iq_1^{-1}, q_2iq_2^{-1})]$. Here note that S^2 is identified with $\{qiq^{-1} : q \in \mathbb{H}^*\} = \{xi+yj+zk : x^2+y^2+z^2 = 1\}$. By Lemma 3.12, it can be easily seen that this is well defined and moreover bijective.

We now suppose that $\langle v_1, v_2\rangle \neq 0$. According to Lemma 3.8, $\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2q_2\}$ is a totally $\mathbb{C}$-vector space if and only if

$$\langle v_1q_1, v_2q_2\rangle = \bar{q}_2\langle v_1, v_2\rangle q_1 \in \mathbb{C}^*, \text{ i.e., } q_2 = \langle v_1, v_2\rangle q_1 z \text{ for some } z \in \mathbb{C}^*.$$

Applying Lemma 3.12,

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2\langle v_1, v_2\rangle q_1z\} = \mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q_1, v_2\langle v_1, v_2\rangle q_1\}$$

and furthermore,

$$\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q, v_2\langle v_1, v_2\rangle q\} = \mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1p, v_2\langle v_1, v_2\rangle p\}$$
$$\text{if and only if } qiq^{-1} = \pm pip^{-1}.$$

By mapping $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1q, v_2\langle v_1, v_2\rangle q\}$ to $[qiq^{-1}] = \{qiq^{-1}, -qiq^{-1}\}$, there is a one-to-one correspondence between the set of all totally projective $\mathbb{C}$-lines through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$, and $\{\{\pm(xi + yj + zk)\} : x^2 + y^2 + z^2 = 1\}$, which is naturally identified with $\mathbb{RP}^2$. This completes the proof. □

One can see that each totally projective $\mathbb{C}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ contains a totally projective $\mathbb{R}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$. However, it may be not unique. For example, consider a totally projective $\mathbb{C}$-line $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{e_1, e_2\}$ through $\mathbb{P}(e_1)$ and $\mathbb{P}(e_2)$ for $n \geq 3$. Then since $\langle e_1, e_2\rangle = 0$, it is easy to see that $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{e_1, e_2z\}$ is a totally projective $\mathbb{R}$-line contained in $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{e_1, e_2\}$ for any $z \in \mathbb{C}^*$. By Lemma 3.6,

$$\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{e_1, e_2z\} = \mathbb{P}\mathrm{Span}_{\mathbb{R}}\{e_1, e_2w\} \text{ if and only if } w^{-1}z \in \mathbb{R}^*.$$

Therefore, the space of totally projective $\mathbb{R}$-lines which are contained in the totally projective $\mathbb{C}$-line $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{e_1, e_2\}$ and pass through $\mathbb{P}(e_1)$ and $\mathbb{P}(e_2)$ is identified with $\mathbb{RP}^1$. This can be generalized to the case of $\langle v_1, v_2\rangle = 0$.

In the case of $\langle v_1, v_2\rangle \neq 0$, as seen in Proposition 3.14, a totally projective $\mathbb{C}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ can be written as $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_1, v_2\rangle q\}$. Then it is not difficult to see that $\mathbb{P}\mathrm{Span}_{\mathbb{R}}\{v_1, v_2\langle v_1, v_2\rangle\}$ is the unique totally projective $\mathbb{R}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ which is contained in $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_1, v_2\rangle q\}$ for all $q \in \mathbb{H}^*$. If $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ are two distinct points in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$, then $\langle v_1, v_2\rangle \neq 0$ and thus any totally projective $\mathbb{C}$-line through $\mathbb{P}(v_1)$ and $\mathbb{P}(v_2)$ can be written as $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_2, v_1\rangle q\}$ for some $q \in \mathbb{H}^*$. Futhermore any $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_2, v_1\rangle q\}$ contains the projectivization of a vector with negative norm in $\mathbb{H}^{n,1}$, as seen before. By Corollary 3.4, $\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_2, v_1\rangle q\}$ is isomorphic to $\mathrm{Span}_{\mathbb{C}}\{e_1, e_{n+1}\}$ up to the action of $\mathrm{Sp}(n,1)$ on $\mathbb{H}^{n,1}$. Thus the intersection of $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{v_1 q, v_2\langle v_2, v_1\rangle q\}$ and $\mathbf{H}^n_{\mathbb{H}}$ is isometric to $\mathbf{H}^1_{\mathbb{C}}$, which is a complex geodesic in $\mathbf{H}^n_{\mathbb{H}}$.

Corollary 3.15. *Given two distinct points in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$, there are $\mathbb{RP}^2$ totally projective $\mathbb{C}$-lines passing through them. Each totally projective $\mathbb{C}$-line through them has a unique totally projective $\mathbb{R}$-line through them and furthermore its intersection with $\mathbf{H}^n_{\mathbb{H}}$ is isometric to the complex geodesic $\mathbf{H}^1_{\mathbb{C}}$ in $\mathbf{H}^n_{\mathbb{H}}$.*

4. The quaternionic Hermitian triple product

In this section, we will characterize the totally real and complex subsets of $\mathbb{H}^{n,1}$ in terms of the quaternionic Hermitian triple product. For $v_1, v_2, v_3 \in \mathbb{H}^{n,1}$, their quaternionic Hermitian triple product is defined as

$$\langle v_1, v_2, v_3\rangle = \langle v_2, v_1\rangle\langle v_3, v_2\rangle\langle v_1, v_3\rangle.$$

Let v_3' be the orthogonal projection of v_3 onto $\mathrm{Span}_{\mathbb{H}}\{v_1, v_2\}$. Since $v_3 - v_3'$ is orthogonal to both v_1 and v_2, we have that $\langle v_1, v_3\rangle = \langle v_1, v_3'\rangle$ and $\langle v_3, v_2\rangle = \langle v_3', v_2\rangle$. From these equalities, we easily get

$$\langle v_1, v_2, v_3'\rangle = \langle v_2, v_1\rangle\langle v_3', v_2\rangle\langle v_1, v_3'\rangle = \langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle = \langle v_1, v_2, v_3\rangle.$$

Proposition 4.1. *Let S be a nonempty subset of $\mathbb{H}^{n,1}$. Suppose that $\langle u, v, w\rangle \neq 0$ for every $(u, v, w) \in S^{(3)}$. Then the following are equivalent.*

(i) *There is an element $u_0 \in S$ such that $\langle u_0, v, w\rangle \in \mathbb{R}$ for every $(u_0, v, w) \in S^{(3)}$.*

(ii) *$\langle u, v, w\rangle \in \mathbb{R}$ for every $(u, v, w) \in S^{(3)}$.*

(iii) *There is a function $\lambda : S \to \mathbb{H}^*$ such that the $\mathbb{R}$-linear span of $\{v\lambda_v \mid v \in S\}$ is a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$.*

Proof. To prove that (i) $\Longrightarrow$ (iii), suppose that there is an element $u_0 \in S$ such that $\langle u_0, v, w\rangle \in \mathbb{R}$ for every $(u_0, v, w) \in S^{(3)}$. Set $\lambda_v = \langle u_0, v\rangle$ for each $v \neq u_0$ and $\lambda_{u_0} = 1$. By the assumption that $\langle u, v, w\rangle \neq 0$ for every $(u, v, w) \in S^{(3)}$, it follows that $\lambda_v = \langle u_0, v\rangle \neq 0$ for all $v \in S$. We now claim that the $\mathbb{R}$-linear span of $\{v\lambda_v \mid v \in S\}$ is a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$. First, we can check that $\langle u_0\lambda_{u_0}, v\lambda_v\rangle = \langle v, u_0\rangle\langle u_0, v\rangle = |\langle u_0, v\rangle|^2 \in \mathbb{R}$ for all $v \neq u_0$ and for any $v, w \neq u_0$,

$$\langle v\lambda_v, w\lambda_w\rangle = \langle w, u_0\rangle\langle v, w\rangle\langle u_0, v\rangle = \langle u_0, w, v\rangle \in \mathbb{R}.$$

This means that the inner product of any two vectors of $\{v\lambda_v \,|\, v \in S\}$ is a real number, which implies (iii), i.e., the $\mathbb{R}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$.

Next we will prove that (iii) $\implies$ (ii). If the $\mathbb{R}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$, then $\langle u\lambda_u, v\lambda_v, w\lambda_w\rangle \in \mathbb{R}$ for any $u, v, w \in S$. From the following equality

$$\langle u\lambda_u, v\lambda_v, w\lambda_w\rangle = \bar{\lambda}_u\langle u, v, w\rangle\lambda_u|\lambda_v|^2|\lambda_w|^2,$$

one can easily deduce that $\langle u, v, w\rangle \in \mathbb{R}$ for any $u, v, w \in S$.

Since (ii) $\implies$ (i) is obvious, this completes the proof. □

Before giving a complex version of Proposition 4.1, we need a maximal abelian subfield of $\mathbb{H}$. Let F be a maximal abelian subfield of $\mathbb{H}$. Then it is well known that $F = \mathbb{R} \oplus q\mathbb{R}$ for some non-real quaternion $q \in \mathbb{H}$. Every quaternion is conjugate to a complex number and thus there is a non-zero quaternion $p \in \mathbb{H}^*$ such that $\bar{p}qp \in \mathbb{C}$. Then $\bar{p}Fp = \mathbb{C}$. In other words, any maximal abelian subfield of $\mathbb{H}$ is conjugate to $\mathbb{C}$.

Proposition 4.2. *Let S be a subset of $\mathbb{H}^{n,1}$ such that $\langle u, v, w\rangle \neq 0$ for all $(u, v, w) \in S^{(3)}$. Then the following are equivalent*:

(i) *For some $u_0 \in S$, there is a maximal abelian subfield F_0 of $\mathbb{H}$ such that $\langle u, v, w\rangle \in F_0$ for every $(u, v, w) \in S^{(3)}$.*
(ii) *For each $u \in S$, there is a maximal abelian subfield F_u of $\mathbb{H}$ such that $\langle u, v, w\rangle \in F_u$ for every $(u, v, w) \in S^{(3)}$.*
(iii) *There is a function $\lambda : S \to \mathbb{H}^*$ such that the $\mathbb{C}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{C}$-vector subspace of $\mathbb{H}^{n,1}$.*

Proof. We first prove that (i) $\implies$ (iii). Suppose that for an element $u_0 \in S$, there is a maximal abelian subfield F_0 of $\mathbb{H}$ such that $\langle u_0, v, w\rangle \in F_0$ for every $(u_0, v, w) \in S^{(3)}$. As mentioned above, there is a non-zero quaternion p_0 such that $\bar{p}_0F_0p_0 = \mathbb{C}$. Set $\lambda_{u_0} = p_0$ and $\lambda_v = \langle u_0, v\rangle p_0$ for each $v \neq u_0$. Then by the same argument as in the proof of Proposition 4.1, we have that $\langle u_0\lambda_{u_0}, v\lambda_v\rangle = \bar{p}_0\langle v, u_0\rangle\langle u_0, v\rangle p_0 = |p_0|^2|\langle u_0, v\rangle|^2 \in \mathbb{R}$ and for any $v, w \neq u_0$,

$$\langle v\lambda_v, w\lambda_w\rangle = \bar{p}_0\langle w, u_0\rangle\langle v, w\rangle\langle u_0, v\rangle p_0 = \bar{p}_0\langle u_0, w, v\rangle p_0 \in \bar{p}_0F_0p_0 = \mathbb{C}.$$

Thus the $\mathbb{C}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{C}$-vector subspace of $\mathbb{H}^{n,1}$.

Next we prove that (iii) $\implies$ (ii). Suppose that there is a function $\lambda : S \to \mathbb{H}^*$ such that the $\mathbb{C}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{C}$-vector subspace of $\mathbb{H}^{n,1}$. Then for any $u, v, w \in S$,

$$\langle u\lambda_u, v\lambda_v, w\lambda_w\rangle = \bar{\lambda}_u\langle u, v, w\rangle\lambda_u|\lambda_v|^2|\lambda_w|^2 \in \mathbb{C},$$

which is equivalent to $\langle u, v, w\rangle \in \lambda_u \mathbb{C}\bar{\lambda}_u$. For each $u \in S$, we set $F_u = \lambda_u \mathbb{C}\bar{\lambda}_u$. Then obviously, F_u is a maximal abelian subfield of $\mathbb{H}$ and $\langle u, v, w\rangle \in F_u$ for every $(u, v, w) \in S^{(3)}$, which completes the proof for (iii) $\implies$ (ii).

Lastly, (ii) $\implies$ (i) is obvious. Therefore we can conclude that (i), (ii) and (iii) are all equivalent. □

Propositions 4.1 and 4.2 provide a tool to give an answer to the question: Is there a way to know whether a given set $\mathbf{S}$ in $\mathbb{PH}^{n,1}$ is totally real or complex? More concretely, lift $\mathbf{S} \subset \mathbb{PH}^{n,1}$ to a set $S \subset \mathbb{H}^{n,1}$. Then fix an element $u_0 \in S$ and investigate the reality or commutativity of the set $\{\langle u_0, v, w\rangle \,|\, (u_0, v, w) \in S^{(3)}\}$. If $\{\langle u_0, v, w\rangle \,|\, (u_0, v, w) \in S^{(3)}\}$ is real (resp. commutative) and has no the zero element, we can conclude that $\mathbf{S}$ is totally real (resp. complex). It is worth pointing out that this conclusion does not depend on the choices of a lift S of $\mathbf{S}$ and $u_0 \in S$. This can be seen from Propositions 4.1(iii) and 4.2(iii), which do not depend on the choice of a lift of $\mathbf{S}$. Hence the following definition, which is equivalent to Definition 1.2, makes sense.

Definition 4.3. We say that a subset $\mathbf{S} \subset \mathbb{PH}^n$ is *totally real* (resp. *totally commutative*) *with respect to the quaternionic Hermitian triple product* if some (hence any) lift S of $\mathbf{S}$ to $\mathbb{H}^{n,1}$ satisfies the following property: For some (hence any) $u \in S$, the associated set $\{\langle u, v, w\rangle \,|\, (u, v, w) \in S^{(3)}\}$ is real (resp. commutative).

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We first prove (i). Suppose that $\mathbf{S}$ is totally real. Then there is a lift S of $\mathbf{S}$ such that S is totally real. Clearly S is totally real with respect to the quaternionic Hermitian triple product and so is $\mathbf{S}$.

Conversely, suppose that $\mathbf{S}$ is totally real with respect to the quaternionic Hermitian triple product. By Proposition 4.1, there exist a lift S of $\mathbf{S}$ and a function $\lambda : S \to \mathbb{H}^*$ such that the $\mathbb{R}$-linear span of $\{v\lambda_v \,|\, v \in S\}$ is a totally $\mathbb{R}$-vector subspace of $\mathbb{H}^{n,1}$. Noting that $\{v\lambda_v : v \in S\}$ is also a lift of $\mathbf{S}$ to $\mathbb{H}^{n,1}$, it follows that $\mathbf{S}$ is totally real.

The proof of (ii) is similar and is left to the reader. □

Let Γ be a discrete subgroup of $\mathrm{PSp}(n, 1)$ acting on $\mathbf{H}^n_{\mathbb{H}}$. Then its limit set Λ_Γ is the unique minimal non-empty closed Γ-invariant subset of $\partial\mathbf{H}^n_{\mathbb{H}}$. More concretely, Λ_Γ is obtained by the intersection of $\partial\mathbf{H}^n_{\mathbb{H}}$ and the closure of $\Gamma \cdot x$ in $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ for some $x \in \mathbf{H}^n_{\mathbb{H}}$. Since no two null-vectors in $\mathbb{H}^{n,1}$ can be orthogonal, $\langle u, v, w\rangle \neq 0$ for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \Lambda^{(3)}_\Gamma$, where u, v and w are the lifts of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ to $\mathbb{H}^{n,1}$, respectively. As an application of Propositions 4.1 and 4.2 to quaternionic hyperbolic space, we have the following corollary.

Proof of Corollary 1.4. Suppose that $\Lambda_\Gamma \subset \partial\mathbf{H}^n_{\mathbb{H}}$ is totally real with respect to the quaternionic Hermitian triple product. By Theorem 1.3, Λ_Γ is totally real, which implies that a lift of Λ_Γ spans a totally $\mathbb{R}$-vector subspace V of

$\mathbb{H}^{n,1}$. Since V is spanned by null vectors, it has a vector with negative norm. Hence, by Corollary 3.4, V is isomorphic, up to the action of $\mathrm{Sp}(n,1)$ on $\mathbb{H}^{n,1}$, to $\mathrm{Span}_{\mathbb{R}}\{e_1,\dots,e_m,e_{n+1}\}$ for some integer $m \le n$. This means that $\mathbb{P}V$ is isometric to the real hyperbolic subspace $\mathbf{H}^m_{\mathbb{R}} \subset \mathbf{H}^n_{\mathbb{H}}$. Moreover it is obvious that Γ leaves $\mathbb{P}V$ invariant since $\mathbb{P}V$ is the smallest totally projective $\mathbb{R}$-subspace of $\mathbb{PH}^{n,1}$ containing Λ_Γ which is Γ-invariant. The converse is trivial.

A similar proof works for the totally commutative case with respect to the quaternionic Hermitian triple product. □

5. Quaternionic hyperbolic triangle groups

Section 3.2 implies that given two distinct points of $\mathbb{PH}^{n,1}$, there always exist both a totally projective $\mathbb{R}$-line and a totally projective $\mathbb{C}$-line passing through them. However, for three distinct points of $\mathbb{PH}^{n,1}$, it turns out that there exists a totally projective $\mathbb{C}$-plane passing through them but there may not exist a totally projective $\mathbb{R}$-plane passing through them. Furthermore, for more than three distinct points, there may be no totally projective $\mathbb{C}$-subspace of $\mathbb{PH}^{n,1}$ passing through them. In this section we will prove these statements.

First of all, we prove that there always exists a totally projective $\mathbb{C}$-plane passing through given three distinct points of $\mathbb{PH}^{n,1}$.

Lemma 5.1. *Let u, v and w be linearly independent vectors of $\mathbb{H}^{n,1}$ over $\mathbb{H}$. Suppose that $\langle u,v,w\rangle \neq 0$. Then there are $\lambda_u,\lambda_v,\lambda_w \in \mathbb{H}^*$ such that the $\mathbb{C}$-linear span of $\{u\lambda_u, v\lambda_v, w\lambda_w\}$ is a totally complex vector subspace of $\mathbb{H}^{n,1}$.*

Proof. Every quaternion is conjugate to a complex number and thus there exists a quaternion $\lambda_u \in \mathbb{H}^*$ such that $\bar\lambda_u\langle u,w,v\rangle\lambda_u \in \mathbb{C}$. Set $\lambda_v = \langle u,v\rangle\lambda_u$ and $\lambda_w = \langle u,w\rangle\lambda_u$. Then

$$\langle u\lambda_u, v\lambda_v\rangle = \bar\lambda_u\langle v,u\rangle\langle u,v\rangle\lambda_u = |\lambda_u|^2|\langle u,v\rangle|^2 \in \mathbb{R},$$

and, moreover,

$$\langle u\lambda_u, w\lambda_w\rangle = \bar\lambda_u\langle w,u\rangle\langle u,w\rangle\lambda_u = |\lambda_u|^2|\langle u,w\rangle|^2 \in \mathbb{R}.$$

Lastly, we get

$$\langle v\lambda_v, w\lambda_w\rangle = \bar\lambda_u\langle w,u\rangle\langle v,w\rangle\langle u,v\rangle\lambda_u = \bar\lambda_u\langle u,w,v\rangle\lambda_u \in \mathbb{C}.$$

Therefore, by Proposition 4.2, the $\mathbb{C}$-linear span of $\{u\lambda_u, v\lambda_v, w\lambda_w\}$ is a totally complex subspace of $\mathbb{H}^{n,1}$. □

Example 5.2. We here give an example of three distinct points of $\mathbb{PH}^{n,1}$ which are not contained in any totally real projective subspace of $\mathbb{PH}^{n,1}$. Let $v_1 = e_1 + ie_2, v_2 = e_2$ and $v_3 = e_2 + e_{n+1}$. Then by a straightforward computation,

$$\langle v_1, v_2, v_3\rangle = 1 - i.$$

This implies that $\langle v_1\lambda_1, v_2\lambda_2, v_3\lambda_3\rangle$ can not be real for any non-zero quaternions λ_1, λ_2 and λ_3. According to Proposition 4.1, this means that $\mathbb{P}(v_1)$, $\mathbb{P}(v_2)$ and $\mathbb{P}(v_3)$ can not be contained in any totally real projective subspace of $\mathbb{PH}^{n,1}$.

Example 5.3. We now give an example of four distinct points of $\mathbb{PH}^{n,1}$ which are not contained in any totally real and complex projective subspace of $\mathbb{PH}^{n,1}$. Let $v_1 = e_1 + ie_2$, $v_2 = e_2 + e_3$, $v_3 = e_2 + ke_{n+1}$ and $v_4 = je_3 + e_{n+1}$ for $n \geq 3$. Then,

$$\langle v_1, v_2, v_3 \rangle = 1 - j, \ \langle v_1, v_2, v_4 \rangle = -k.$$

Note that $\langle v_1, v_2, v_3 \rangle$ and $\langle v_1, v_2, v_4 \rangle$ do not commute and hence by Proposition 4.2, the four distinct points $\mathbb{P}(v_1), \mathbb{P}(v_2), \mathbb{P}(v_3), \mathbb{P}(v_4)$ can not be contained in any totally complex projective subspace of $\mathbb{PH}^{n,1}$.

As an application of Lemma 5.1, we study quaternionic hyperbolic triangle groups in $\mathbf{H}^2_{\mathbb{H}}$. Given two distinct points $\mathbf{u}, \mathbf{v} \in \mathbf{H}^2_{\mathbb{H}} \cup \partial \mathbf{H}^2_{\mathbb{H}}$, there is a unique quaternionic projective line L spanned by $\mathbf{u}, \mathbf{v}$. There is a unique vector $c \in \mathbb{H}^{2,1}$ up to scaling by non-zero quaternions such that the line $\mathbf{L}$ is the projection of a 2-dimensional quaternionic subspace $\{z \in \mathbb{H}^{2,1} \mid \langle z, c \rangle = 0\}$. The vector c is called the *polar vector* of the quaternionic projective line $\mathbf{L}$. Indeed, the polar vector c is determined by two equations $\langle c, u \rangle = 0$ and $\langle c, v \rangle = 0$, where $u, v \in \mathbb{H}^{2,1}$ are lifts of $\mathbf{u}, \mathbf{v}$, respectively. Then the quaternionic inversion $\tau_{\mathbf{L}}$ in $\mathbf{L}$ is defined by

$$\tau_{\mathbf{L}}(z) = \mathbb{P}\left(-z + c \cdot 2\langle c, c \rangle^{-1} \langle z, c \rangle\right).$$

Then it is easy to check that $\tau_{\mathbf{L}}$ is an isometry of $\mathbf{H}^2_{\mathbb{H}}$ of order 2 and fixes $\mathbf{L}$ pointwise.

For three distinct points $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^2_{\mathbb{H}} \cup \partial \mathbf{H}^2_{\mathbb{H}}$, there are three quaternionic projective lines $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ passing through two points of them and three quaternionic inversions τ_1, τ_2, τ_3 associated to $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$. The group generated by τ_1, τ_2, τ_3 is called a *quaternionic hyperbolic triangle group* in $\mathbf{H}^2_{\mathbb{H}}$.

As seen before, every quaternionic hyperbolic triangle group is determined by three distinct points $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^{\mathbf{2}}_{\mathbb{H}} \cup \partial \mathbf{H}^{\mathbf{2}}_{\mathbb{H}}$. By Lemma 5.1, there are lifts $u, v, w \in \mathbb{H}^{2,1}$ of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively, such that $\mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ is totally complex. Let $c_1 \in \mathbb{H}^{2,1}$ be the polar vector of the quaternionic projective line $\mathbf{L}_1$ through $\mathbf{u}$ and $\mathbf{v}$. Then $\langle c_1, u \rangle = 0$ and $\langle c_1, v \rangle = 0$. Since $\{u, v, w\}$ is a basis of $\mathbb{H}^{2,1}$, we can write $c_1 = u \cdot q_1 + v \cdot q_2 + w \cdot q_3$ for some quaternions $q_1, q_2, q_3 \in \mathbb{H}$. Applying $\langle c_1, u \rangle = 0$ and $\langle c_1, v \rangle = 0$, we have the following two equations:

$$\langle u, u \rangle \cdot q_1 + \langle v, u \rangle \cdot q_2 + \langle w, u \rangle \cdot q_3 = 0,$$
$$\langle u, v \rangle \cdot q_1 + \langle v, v \rangle \cdot q_2 + \langle w, v \rangle \cdot q_3 = 0.$$

All the coefficients in these two equations are complex numbers and thus there are complex number solutions q_1, q_2, q_3. This means that $c_1 \in \mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ and τ_1 leaves $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ invariant. Similarly, one can prove that the other quaternionic inversions τ_1 and τ_2 also leave $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ invariant. Therefore, the quaternionic hyperbolic triangle group generated by τ_1, τ_2, τ_3 leaves the totally $\mathbb{C}$-subspace $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ of $\mathbb{PH}^{2,1}$ invariant. Furthermore, the intersection of $\mathbb{P}\mathrm{Span}_{\mathbb{C}}\{u, v, w\}$ and $\mathbf{H}^2_{\mathbb{H}}$ is isometric to $\mathbf{H}^2_{\mathbb{C}}$, which implies that every quaternionic hyperbolic triangle group in $\mathrm{PSp}(2, 1)$ is conjugate to a subgroup of $\mathrm{PU}(2, 1)$. Therefore, we get Theorem 1.5.

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