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PROPERTIES OF POSITIVE SOLUTIONS FOR THE FRACTIONAL LAPLACIAN SYSTEMS WITH POSITIVE-NEGATIVE MIXED POWERS

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Abstract. In this paper, by establishing the direct method of moving planes for the fractional Laplacian system with positive-negative mixed powers, we obtain the radial symmetry and monotonicity of the positive solutions for the fractional Laplacian systems with positive-negative mixed powers in the whole space. We also give two special cases.

1. Introduction

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and fractional nonlinear equations (systems) have been applied to research physical phenomena, such as anomalous diffusion, quasi-geostrophic flows, water waves, molecular dynamics, and relativistic quantum mechanics of stars, probability and finance. For fractional order equations (systems), we can see for example [1, 3, 4, 7, 14–17, 19, 20, 28].

Symmetry and monotonicity for solutions play significant roles in understanding fractional nonlinear equations (systems). Many authors have studied the symmetry and monotonicity of solutions for fractional nonlinear equations (systems).

For the symmetry and monotonicity of solutions of fractional Laplacian equations (systems), several systematic methods are currently available to study these properties, such as the moving plane method [10, 11, 13, 22, 24], the moving spherical method [12] and the sliding method [14, 26, 33]. Here we only consider the moving plane method.

For the fractional p-Laplacian equations (systems) and the fully nonlinear fractional order equations (systems), we can see for example [8, 27, 34, 36] and

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[9, 29, 31, 37], respectively. In what follows, we mainly consider the fractional Laplacian systems.

Jarohs and Weth [18] established the direct method for the moving plane to suitable only for bounded domain, W. Chen, C. Li and Y. Li [10] generalized the direct method for the moving plane to unbounded domain by giving the narrow domain lemma and infinite decay theorem for fractional Laplacian equation and prove the symmetry and monotonicity of positive solutions for the fractional Laplacian equation. Later Cao and Wang [6], Cheng, Lü and Lü [13], B. Liu and L. Ma [23], Wang and Ren [30], Wu and Xu [35], and Zhuo, Chen, Cui and Yuan [38] gave the symmetry, monotonicity and nonexistence of positive solutions to fractional Laplacian equations (systems) with positive powers.

For fractional Laplacian equations with negative powers, Cai and Ma [5] proved the symmetry of the positive solution of the fractional Laplacian equation with negative powers

$$(-\Delta)^{\frac{\alpha}{2}}u(x) + u^{-\beta}(x) = 0, \ x \in \mathbb{R}^n,$$

where $\alpha \in (0, 2)$, $\beta > 0$.

For fractional Laplacian equation with positive-negative mixed powers, Cao and Wang [6] proved the symmetry or monotonicity of positive solutions to nonlocal fractional Laplacian equation with a singular nonlinearity.

$$(-\Delta)^s u(x) = \lambda u^{\beta}(x) + a_0 u^{-\gamma}(x), \ x \in \mathbb{R}^n,$$

where $0 < s < 1$, $\gamma > 0$, $1 < \beta \leq \frac{n+2s}{n-2s}$, $\lambda > 0$ are constants and $a_0 \geq 0$.

For the existence of fractional Laplacian equation with positive-negative mixed powers, we can see for example Wang and Zhang [32], T. Mukherjee and K. Sreenadh [25] in critical cases and J. Giacomoni, T. Mukherjee and K. Sreenadh [17], B. Barrios, I. De Bonis, M. Medina and I. Peral [2] in subcritical cases.

For the symmetry and monotonicity of solutions of fractional Laplacian system with positive-negative mixed powers, as far as we know, we haven't seen it in the literature. We only see the symmetry of positive solutions to fractional p-Laplacian systems in [21] by P. Le. But these results are similar to those of positive indicators.

Inspired by these articles, we consider the following general fractional Laplacian systems with positive-negative mixed powers in the whole space.

$$\begin{cases} (-\Delta)^s u(x) = f(u(x), v(x)), & x \in \mathbb{R}^n, \\ (-\Delta)^t v(x) = g(u(x), v(x)), & x \in \mathbb{R}^n, \\ u(x) > 0, \ v(x) > 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

with

$$(-\Delta)^s u(x) = C_{n,s} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} dy = C_{n,s} PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} dy,$$

where $C_{n,s}$ is a normalization positive constant depending on n, s and $s \in (0,1)$. PV stands for the Cauchy principle value.

In order that the fractional Laplacian is well-defined, we require that

$$u \in C^{1,1}_{loc} \cap L_{2s},\ v \in C^{1,1}_{loc} \cap L_{2t}$$

with

$$L_{2s} = \left\{ u \in L^1_{loc}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < +\infty \right\},$$
$$L_{2t} = \left\{ u \in L^1_{loc}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2t}} dx < +\infty \right\}.$$

Our main results are as follows:

Theorem 1.1. *Assume that $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ are positive solutions of* (1.1) *satisfying*

$$u(x) \sim |x|^{m_1}, \quad v(x) \sim |x|^{m_2} \text{ as } |x| \text{ sufficiently large,} \tag{1.2}$$

with $m_1, m_2 > 0$, and f, g are continuous functions satisfying

$$\begin{aligned} &f_x(u,v) < 0, \quad 0 < g_x(u,v) \le C_1 u^{r-1}; \\ &0 < f_y(u,v) \le C_2 v^{q-1}, \quad g_y(u,v) < 0; \end{aligned} \tag{1.3}$$

where $C_i > 0$, $i = 1, 2$, and C_1 is independent of v, C_2 is independent of u, and $q, r < 0$ satisfying

$$2s < m_2 - m_2 q,\ 2t < m_1 - m_1 r. \tag{1.4}$$

Then $u(x)$ and $v(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.

Theorem 1.2. *Assume that $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ are positive solutions of* (1.1) *satisfying*

$$u(x) \sim |x|^{m_1}, \quad v(x) \sim |x|^{m_2} \text{ as } |x| \text{ sufficiently large,} \tag{1.5}$$

with $m_1, m_2 > 0$, and f, g are continuous functions satisfying

$$\begin{aligned} &-C_1 u^{p^*-1} v^q \le f_x(u,v) < 0, \quad 0 < g_x(u,v) \le C_2 u^{r-1} v^{s^*}; \\ &0 < f_y(u,v) \le C_3 u^{p^*} v^{q-1}, \quad -C_4 u^r v^{s^*-1} \le g_y(u,v) < 0, \end{aligned} \tag{1.6}$$

where $C_i > 0$, $i = 1, 2, 3, 4$ and $p^*, s^* < 0$, $0 < q, r < 1$ satisfying

$$2s < m_2 - m_1 p - m_2 q^*,\ 2t < m_1 - m_1 r^* - m_2 s. \tag{1.7}$$

Then $u(x)$ and $v(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$.

Remark 1. Regarding conditions (1.2) and (1.5), we are inspired by Theorem 1 and Remark 1 in [5]. Compared with the result of Cai and Ma [5], we generalize condition $0 < m_1, m_2 < 1$ to a more general case $m_1, m_2 > 0$.

Remark 2. Conditions (1.4) and (1.7) are to make sure 'Decay at infinity'. If rewriting (1.4) as follows:

$$2s - m_2 < -m_2 q,\ 2t - m_1 < -m_1 r,$$

one can see that the right sides of the above two inequalities have the same structure, namely, they are all the power of negative indicator.

If rewriting (1.7) as follows:

$$2s - m_2 < -m_1 p^* - m_2 q,\ 2t - m_1 < -m_2 s^* - m_1 r,$$

one can see that the right sides of the above two inequalities have the same structure, namely, the power of negative indicator minus the power of positive indicator.

Remark 3. Theorem 1.2 contrasts sharply with the case $p = q = 2$ of Theorems 1.2 and 1.4 in [21]. They are entirely different results.

By Theorem 1.1 and Theorem 1.2, we consider the following special cases of (1.1):

System 1:

$$\begin{cases} (-\Delta)^s u(x) = u^{-p_1}(x) + v^{q_1}(x), & x \in \mathbb{R}^n, \\ (-\Delta)^t v(x) = v^{-s_1}(x) + u^{r_1}(x), & x \in \mathbb{R}^n, \\ u(x) > 0,\ v(x) > 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where $p_1, s_1 > 0$, $0 < q_1, r_1 < 1$.

System 2:

$$\begin{cases} (-\Delta)^s u(x) = -u^{p_2}(x) v^{-q_2}(x), & x \in \mathbb{R}^n, \\ (-\Delta)^t v(x) = -v^{s_2}(x) u^{-r_2}(x), & x \in \mathbb{R}^n, \\ u(x) > 0,\ v(x) > 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.9}$$

where $p_2, s_2, q_2, r_2 > 0$.

Specially, if we set $u = v$, System 2 will degenerate into fractional Laplacian systems with negative powers similar to (1) in Cai and Ma [5].

Corollary 1.1. *Assume that* $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ *are positive solutions of* (1.8) *satisfying*

$$u(x) \sim |x|^{m_1}, \quad v(x) \sim |x|^{m_2} \text{ as } |x| \text{ sufficiently large},$$

with $m_1, m_2 > 0$, $p_1, s_1 > 0$, $0 < q_1, r_1 < 1$, *satisfying*

$$2s < m_2 - m_2 q_1, \quad 2t < m_1 - m_1 r_1.$$

Then $u(x)$ *and* $v(x)$ *must be radially symmetric and monotone increasing about some point in* $\mathbb{R}^n$.

Corollary 1.2. *Assume that* $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ *are positive solutions of* (1.9) *satisfying*

$$u(x) \sim |x|^{m_1}, \quad v(x) \sim |x|^{m_2} \text{ as } |x| \text{ sufficiently large},$$

with $m_1, m_2 > 0$, $p_2, q_2, s_2, r_2 > 0$, *satisfying*

$$2s < m_2 + m_2q_2 - m_1p_2, \quad 2t < m_1 + m_1r_2 - m_2s_2.$$

Then $u(x)$ *and* $v(x)$ *must be radially symmetric and monotone increasing about some point in* $\mathbb{R}^n$.

The paper is organized as follows. In Section 2, in order to prove Theorems 1.1-1.2, we establish the *Decay at infinity* and *Boundary Estimate* theorems. In Section 3, we present the proofs of Theorems 1.1-1.2. Finally, we consider the symmetry and monotonicity of the positive solutions to Systems 1-2 in Section 4.

2. Basic tools

In order to prove Theorems 1.1-1.2, we need following notations and some lemmas:

Let $T_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 = \lambda\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^n \,|\, x_1 < \lambda\}$, $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$, and

$$u_\lambda(x) = u(x^\lambda),\ v_\lambda(x) = v(x^\lambda),\ U_\lambda(x) = u(x) - u(x^\lambda),\ V_\lambda(x) = v(x) - v(x^\lambda).$$

Lemma 2.1. *Let* $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ *be positive solutions of* (1.1). *If there are* $\bar{x}, \tilde{x} \in \Sigma_\lambda$ *with* $\lambda \le 0$ *such that* $U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0$, $V_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0$, *then*

$$(-\Delta)^s U_\lambda(\bar{x}) \le \frac{C}{|\bar{x}|^{2s}} U_\lambda(\bar{x}); \quad (-\Delta)^s V_\lambda(\tilde{x}) \le \frac{C}{|\tilde{x}|^{2t}} V_\lambda(\tilde{x}).$$

Proof. Assume that there exists a point $\bar{x} \in \Sigma_\lambda$ such that

$$U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0. \tag{2.1}$$

Then

$$\begin{aligned}
(-\Delta)^s U_\lambda(\bar{x}) &= C_{n,s} PV \int_{\mathbb{R}^n} \frac{U_\lambda(\bar{x}) - U_\lambda(y)}{|\bar{x} - y|^{n+2s}} dy \\
&= C_{n,s} PV \int_{\Sigma_\lambda} \frac{U_\lambda(\bar{x}) - U_\lambda(y)}{|\bar{x} - y|^{n+2s}} dy + C_{n,s} PV \int_{\Sigma_\lambda} \frac{U_\lambda(\bar{x}) + U_\lambda(y)}{|\bar{x} - y^\lambda|^{n+2s}} dy \\
&= C_{n,s} PV \int_{\Sigma_\lambda} [\frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - y^\lambda|^{n+2s}}][U_\lambda(\bar{x}) - U_\lambda(y)] dy \qquad (2.2) \\
&\quad + C_{n,s} PV \int_{\Sigma_\lambda} \frac{2U_\lambda(\bar{x})}{|\bar{x} - y^\lambda|^{n+2s}} dy \\
&= C_{n,s}\{I_1 + I_2\}.
\end{aligned}$$

For I_1, from (2.1) and the fact that $\frac{1}{|\bar{x}-y|} > \frac{1}{|\bar{x}-y^\lambda|}$ for any $\bar{x}, y \in \Sigma_\lambda$, one can get

$$I_1 < 0. \tag{2.3}$$

Because $B^{\lambda}_{|\bar{x}|}(\tilde{x}) \subset \mathbb{R}^n \backslash \Sigma_\lambda$ for each fixed $\lambda < 0$, where $\bar{B}^{\lambda}_{|\bar{x}|}(\tilde{x}) \subset \Sigma_\lambda$ is the reflection of $B_{|\bar{x}|}(\tilde{x})$ about the plane T_λ with $\tilde{x} = (3|\bar{x}| + \bar{x}_1, (\bar{x})')$, here $(\bar{x})' = (\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n)$, then

$$
\begin{aligned}
I_2 &= C_{n,s} PV \int_{\Sigma_\lambda} \frac{2U_\lambda(\bar{x})}{|\bar{x} - y^\lambda|^{n+2s}} dy \\
&\leq C_{n,s} U_\lambda(\bar{x}) PV \int_{\bar{B}^{\lambda}_{|\bar{x}|}(\tilde{x})} \frac{1}{|\bar{x} - y^\lambda|^{n+2s}} dy \\
&\leq C_{n,s} U_\lambda(\bar{x}) PV \int_{\bar{B}^{\lambda}_{|\bar{x}|}(\tilde{x})} \frac{1}{4^{n+2s} |\bar{x}|^{n+2s}} dy \\
&\leq \frac{C_1}{|\bar{x}|^{2s}} U_\lambda(\bar{x}).
\end{aligned} \tag{2.4}
$$

Combining (2.2), (2.3) and (2.4), one has

$$
(-\Delta)^s U_\lambda(\bar{x}) \leq \frac{C_1}{|\bar{x}|^{2s}} U_\lambda(\bar{x}). \tag{2.5}
$$

Similarly, one can prove

$$
(-\Delta)^s V_\lambda(\tilde{x}) \leq \frac{C}{|\tilde{x}|^{2t}} V_\lambda(\tilde{x}).
$$

□

Lemma 2.2 (Decay at infinity I)**.** *Let* $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ *be positive solutions of* (1.1). *Assume that* (1.5)-(1.7) *hold. Then, there exists a sufficiently large constant* $R_0 > 0$*,*

(a) *if there is* $\bar{x} \in \Sigma_\lambda$ *with* $\lambda \leq 0$, $|\bar{x}| > R_0$ *such that* $U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0$*, then*

$$
V_\lambda(\bar{x}) < 2U_\lambda(\bar{x}) < 0;
$$

(b) *if there is* $\tilde{x} \in \Sigma_\lambda$ *with* $\lambda \leq 0$, $|\tilde{x}| > R_0$ *such that* $V_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0$*, then*

$$
U_\lambda(\tilde{x}) < 2V_\lambda(\tilde{x}) < 0.
$$

Lemma 2.3 (Decay at infinity II)**.** *Let* $u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2s}(\mathbb{R}^n)$, $v \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_{2t}(\mathbb{R}^n)$ *be positive solutions of* (1.1). *Assume that* (1.2)-(1.4) *hold. Then, there exists a sufficiently large constant* $R_0 > 0$*,*

(a) *if there is* $\bar{x} \in \Sigma_\lambda$ *with* $\lambda \leq 0$, $|\bar{x}| > R_0$ *such that* $U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0$*, then*

$$
V_\lambda(\bar{x}) < 2U_\lambda(\bar{x}) < 0;
$$

(b) *if there is* $\tilde{x} \in \Sigma_\lambda$ *with* $\lambda \leq 0$, $|\tilde{x}| > R_0$ *such that* $V_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0$*, then*

$$
U_\lambda(\tilde{x}) < 2V_\lambda(\tilde{x}) < 0.
$$

Proof of Lemma 2.2. Assume that there exists a point $\bar{x} \in \Sigma_\lambda$ such that

$$
U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0. \tag{2.6}
$$

From Lemma 2.1, one has

$$(2.7)\qquad (-\Delta)^s U_\lambda(\bar{x}) \leq \frac{C_1}{|\bar{x}|^{2s}} U_\lambda(\bar{x}).$$

Moreover, applying the mean value theorem yields

$$(2.8)\qquad \begin{aligned}(-\Delta)^s U_\lambda(\bar{x}) &= f(u(\bar{x}), v(\bar{x})) - f(u_\lambda(\bar{x}), v_\lambda(\bar{x})) \\ &= f_x(\xi_\lambda(\bar{x}), v_\lambda(\bar{x}))U_\lambda(\bar{x}) + f_y(u(\bar{x}), \eta_\lambda(\bar{x}))V_\lambda(\bar{x}),\end{aligned}$$

where $\xi_\lambda(\bar{x})$ is valued between $u(\bar{x})$ and $u_\lambda(\bar{x})$, and $\eta_\lambda(\bar{x})$ is valued between $v(\bar{x})$ and $v_\lambda(\bar{x})$.

Combining (2.7) and (2.8), one also get

$$(2.9)\qquad f_y(u(\bar{x}), \eta_\lambda(\bar{x}))V_\lambda(\bar{x}) \leq [\frac{c_1}{|\bar{x}|^{2s}} - f_x(\xi_\lambda(\bar{x}), v_\lambda(\bar{x}))]U_\lambda(\bar{x}).$$

From (2.9) and the assumption (1.6), it follows that

$$(2.10)\qquad V_\lambda(\bar{x}) < 0.$$

Conclusion (a) will be proved if we show $\frac{\frac{c_1}{|\bar{x}|^{2s}} - f_x(\xi_\lambda(\bar{x}), v_\lambda(\bar{x}))}{f_y(u(\bar{x}), \eta_\lambda(\bar{x}))} > 2$ for sufficiently large $|\bar{x}|$.

In fact, from $U_\lambda(\bar{x}) < 0$, $V_\lambda(\bar{x}) < 0$, then $u(\bar{x}) < \xi_\lambda(\bar{x}) < u_\lambda(\bar{x})$, $v(\bar{x}) < \eta_\lambda(\bar{x}) < v_\lambda(\bar{x})$.

Hence by (1.7) and (1.6), it follows that for sufficiently large $|\bar{x}|$,

$$(2.11)\qquad \frac{\frac{c_1}{|\bar{x}|^{2s}} - f_x(\xi_\lambda(\bar{x}), v_\lambda(\bar{x}))}{f_y(u(\bar{x}), \eta_\lambda(\bar{x}))} \geq c|\bar{x}|^{-m_1p - m_2(q^*-1) - 2s} > 2.$$

Then conclusion (a) holds.

By the same token, we can give the proof of (b), which will omit it here. This completes the proof of Lemma 2.2. □

Similar to the proof of Lemma 2.2, we can prove Lemma 2.3. We omit it here.

Lemma 2.4 (Boundary Estimate)**.** *Assume that $U_{\lambda_0}(x) > 0$, $x \in \Sigma_{\lambda_0}$. Suppose $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that*

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k}(x) \leq 0 \text{ and } x^k \to x^0 \in \partial\Sigma_{\lambda_0},$$

let $\delta_k = dist(x^k, \partial\Sigma_{\lambda_k}) \equiv |\lambda_k - x_1^k|$. Then

$$(2.12)\qquad \overline{\lim_{\delta_k \to 0}} \frac{(-\Delta)^s U_{\lambda_k}(x^k)}{\delta_k} < 0.$$

Proof. Similar to (2.2), we derive that

$$\begin{aligned}
&\frac{(-\Delta)^s U_{\lambda_k}(x^k)}{\delta_k}\\
&= \frac{C_{n,s}}{\delta_k} PV \int_{\Sigma_{\lambda_k}} [\frac{1}{|x^k-y|^{n+2s}} - \frac{1}{|x^k-y^{\lambda_k}|^{n+2s}}][U_{\lambda_k}(x^k) - U_{\lambda_k}(y)]dy\\
&\quad + \frac{2U_{\lambda_k}(x^k)}{\delta_k} C_{n,s} PV \int_{\Sigma_{\lambda_k}} \frac{1}{|x^k-y^{\lambda_k}|^{n+2s}} dy\\
&= C_{n,s}\{I_{1k} + I_{2k}\}.
\end{aligned} \tag{2.13}$$

Obviously,

$$I_{2k} \le 0. \tag{2.14}$$

Next, we estimate I_{1k}. Applying the mean value theorem yields

$$\begin{aligned}
&\frac{1}{\delta_k}\left[\frac{1}{|x^k-y|^{n+2s}} - \frac{1}{|x^k-y^{\lambda_k}|^{n+2s}}\right]\\
&= \frac{1}{\delta_k}\left[-\frac{n+2s}{2\eta_k(y)^{n+2s+2}}\right](|x^k-y|^2 - |x^k-y^{\lambda_k}|^2)\\
&= \frac{1}{\delta_k}\left[-\frac{n+2s}{2\eta_k(y)^{n+2s+2}}\right](-4)(y_1-\lambda_k)(x_1^k-\lambda_k)\\
&= \frac{2(n+2s)(\lambda_k-y_1)}{\eta_k(y)^{n+2s+2}}\\
&\to \frac{2(n+2s)(\lambda_0-y_1)}{\eta_0(y)^{n+2s+2}} \text{ as } k\to\infty,
\end{aligned} \tag{2.15}$$

here $|x^k-y| \le \eta_k(y) \le |x^k-y^{\lambda_k}|$, $|x^0-y| \le \eta_0(y) \le |x^0-y^{\lambda_0}|$, and $\eta_k(y) = |x^k-y| + \theta(|x^k-y^{\lambda_k}| - |x^k-y|)$, $\eta_0(y) = |x^0-y| + \theta(|x^0-y^{\lambda_0}| - |x^0-y|)$.

Obviously, the last term in (2.15) is strictly positive in Σ_{λ_0} (it may be $+\infty$ at some point y). Therefore, by the fact that $U_{\lambda_k}(x^k) - U_{\lambda_0}(y) \le 0$ for all $y \in \Sigma_{\lambda_0}$, one can get

$$\lim_{k\to\infty} I_{1k} < 0. \tag{2.16}$$

Combining (2.13), (2.14) and (2.16), we arrive at (2.12).

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Assume $U_{\lambda_0}(x) \ge 0$, $V_{\lambda_0}(x) \ge 0$, $x \in \Sigma_{\lambda_0}$, if $U_{\lambda_0}(x) \not\equiv 0$ or $V_{\lambda_0}(x) \not\equiv 0$, $x \in \Sigma_{\lambda_0}$. Then $U_{\lambda_0}(x) > 0$ and $V_{\lambda_0}(x) > 0$, $x \in \Sigma_{\lambda_0}$.*

Remark 4. According to the definition of fractional Laplacian and assumption (1.3) or (1.6), one can easily get if $U_{\lambda_0}(x) \equiv 0$, then $V_{\lambda_0}(x) \equiv 0$. If $V_{\lambda_0}(x) \equiv 0$, then $U_{\lambda_0}(x) \equiv 0$.

Proof of Lemma 2.5. Without loss of generality, we assume that $U_{\lambda_0}(x) \not\equiv 0$, $x \in \Sigma_{\lambda_0}$. If not, then there exists a point $x^0 \in \Sigma_{\lambda_0}$ such that

$$U_{\lambda_0}(x^0) = 0. \tag{2.17}$$

On one hand, according to the definition of fractional Laplacian, one has

$$\begin{aligned} &(-\Delta)^s U_{\lambda_0}(x^0) \\ &= C_{n,s} PV \int_{\Sigma_{\lambda_0}} [\frac{1}{|x^0 - y|^{n+2s}} - \frac{1}{|x^0 - y^{\lambda_0}|^{n+2s}}][U_{\lambda_0}(x^0) - U_{\lambda_0}(y)]dy \\ &\quad + C_{n,s} PV \int_{\Sigma_{\lambda_0}} \frac{2U_{\lambda_0}(x^0)}{|x^0 - y^{\lambda_0}|^{n+2s}} dy \\ &= C_{n,s}\{I_1 + I_2\}. \end{aligned} \tag{2.18}$$

Obviously,

$$I_2 = C_{n,s} PV \int_{\Sigma_{\lambda_0}} \frac{2U_{\lambda_0}(x^0)}{|x^0 - y^{\lambda_0}|^{n+2s}} dy = 0. \tag{2.19}$$

For I_1, one has $U_{\lambda_0}(x^0) - U_{\lambda_0}(y) \leq 0$, but $\not\equiv 0$ for $y \in \Sigma_{\lambda_0}$. And from the fact that $\frac{1}{|x^0-y|} > \frac{1}{|x^0-y^{\lambda_0}|}$, $x^0, y \in \Sigma_{\lambda_0}$, one can get

$$I_1 < 0. \tag{2.20}$$

Combining (2.18), (2.19) and (2.20), we deduce that

$$(-\Delta)^s U_{\lambda_0}(x^0) = C_{n,s}\{I_1 + I_2\} < 0.$$

On the other hand, similar to (2.8), one has

$$(-\Delta)^s U_{\lambda_0}(x^0) = f_x(\xi_{\lambda_0}(x^0), v(x^0))U_{\lambda_0}(x^0) + f_y(u_{\lambda_0}(x^0), \eta_{\lambda_0}(x^0))V_{\lambda_0}(x^0) \geq 0.$$

This contradiction shows that

$$U_{\lambda_0}(x) > 0, \ x \in \Sigma_{\lambda_0}. \tag{2.21}$$

According to (2.21) and Remark 4, repeating the above process, we can easily get $V_{\lambda_0}(x) > 0, \ x \in \Sigma_{\lambda_0}$. Therefore, we complete the proof of Lemma 2.5. □

3. Symmetry and monotonicity in $\mathbb{R}^n$

Proof of Theorem 1.1. Let x^λ, $u_\lambda(x)$, $v_\lambda(x)$, $U_\lambda(x)$, $V_\lambda(x)$, T_λ, Σ_λ as defined in Section 2.

Recall $U_\lambda(x) \sim |x|^{m_1} - |x^\lambda|^{m_1}$, $|x| > |x^\lambda|$ for $|x|$ large, we can easily show that for $x \in \Sigma_\lambda$, $\lambda \leq 0$,

$$\lim_{|x|\to\infty} U_\lambda(x) \geq 0, \quad \lim_{|x|\to\infty} V_\lambda(x) \geq 0. \tag{3.1}$$

It implies that if $U_\lambda(x), V_\lambda(x)$ are negative somewhere in Σ_λ, then the negative minimum of $U_\lambda(x), V_\lambda(x)$ must be attained in the interior of Σ_λ.

In the following, by employing the method of moving planes, we carry out the proof in two steps.

Step 1. We show that for λ sufficiently negative,

$$U_\lambda(x),\ V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.2}$$

Choose $\lambda < -R_0$, where R_0 is given by Lemma 2.3. It follows that $U_\lambda(x) \geq 0$ and $V_\lambda(x) \geq 0$ in Σ_λ for all $\lambda < -R_0$.

Assume for contradiction that there exist a $\lambda < -R_0$ and a point $\bar{x} \in \Sigma_\lambda$ such that $U_\lambda(\bar{x}) < 0$. Without loss of generality, we assume

$$U_\lambda(\bar{x}) = \min_{\Sigma_\lambda} U_\lambda(x) < 0.$$

Since $\lambda < -R_0$, we know that $|\bar{x}| > R_0$. Then by Lemma 2.3,

$$V_\lambda(\bar{x}) < 2U_\lambda(\bar{x}) < 0. \tag{3.3}$$

According to (3.1), there exists a point $\tilde{x} \in \Sigma_\lambda$ such that

$$V_\lambda(\tilde{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0.$$

In view of $|\tilde{x}| > R_0$, it follows from Lemma 2.3 that

$$U_\lambda(\tilde{x}) < 2V_\lambda(\tilde{x}) < 0. \tag{3.4}$$

Combining (3.3) and (3.4), one can get

$$V_\lambda(\bar{x}) < 2U_\lambda(\bar{x}) \leq 2U_\lambda(\tilde{x}) \leq 4V_\lambda(\tilde{x}) < 4V_\lambda(\bar{x}).$$

This is a contradiction for $V_\lambda(\bar{x}) < 0$. This completes Step 1.

Step 2. Step 1 provides a starting point to move the plane T_λ. Now we keep moving the plane to the limiting position T_{λ_0} as long as (3.2) holds to its limiting position. More precisely, define

$$\lambda_0 = \sup\{\lambda \leq 0 \,|\, U_\mu(x),\ V_\mu(x) \geq 0,\ x \in \Sigma_\mu,\ \mu \leq \lambda\}.$$

Now we want to prove u and v are symmetric about the limiting plane T_{λ_0}, or

$$U_{\lambda_0}(x) \equiv 0,\ V_{\lambda_0}(x) \equiv 0,\ x \in \Sigma_{\lambda_0}. \tag{3.5}$$

If (3.5) is false, then $U_{\lambda_0}(x) \not\equiv 0$ or $V_{\lambda_0}(x) \not\equiv 0$, $x \in \Sigma_{\lambda_0}$. Furthermore, through Lemma 2.5 and Remark 4, we derive that

$$U_{\lambda_0}(x) > 0,\ V_{\lambda_0}(x) > 0,\ x \in \Sigma_{\lambda_0}. \tag{3.6}$$

By the definition of λ_0, there exist a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that either $U_{\lambda_k}(x^k) < 0$ or $V_{\lambda_k}(x^k) < 0$. Without loss of generality, assume that $U_{\lambda_k}(x^k) < 0$ and regard x^k as its minimum point, then

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k}(x) < 0, \quad \nabla U_{\lambda_k}(x^k) = 0, \tag{3.7}$$

and

$$\begin{aligned} &(-\Delta)^s U_{\lambda_k}(x^k) \\ &= f_x(\xi_{\lambda_k}(x^k), v(x^k))U_{\lambda_k}(x^k) + f_y(u_{\lambda_k}(x^k), \eta_{\lambda_k}(x^k))V_{\lambda_k}(x^k). \end{aligned} \tag{3.8}$$

For the sequence $\{x^k\}_{k=1}^\infty$, there are two possible cases:

Case 1. The sequence $\{x^k\}_{k=1}^{\infty}$ contains a bounded subsequence. Then there exists a subsequence of $\{x^k\}$ (for convenience, we still denote $\{x^k\}$) that converges to some point x^0, i.e., $\lim_{k\to\infty} x^k = x^0$.

Dividing both sides of (3.8) by $\delta_k = |\lambda_k - x_1^k| = \lambda_k - x_1^k$, we derive that

$$\begin{aligned}&\overline{\lim_{k\to\infty}} \frac{(-\Delta)^s U_{\lambda_k}(x^k)}{\delta_k}\\ &= \overline{\lim_{k\to\infty}} \frac{f_x(\xi_{\lambda_k}(x^k), v(x^k))U_{\lambda_k}(x^k) + f_y(u_{\lambda_k}(x^k), \eta_{\lambda_k}(x^k))V_{\lambda_k}(x^k)}{\delta_k}.\end{aligned} \tag{3.9}$$

Noticing that $f_x(\xi_{\lambda_k}(x^k), v(x^k)) < 0$, $U_{\lambda_k}(x^k) < 0$, $\delta_k > 0$, and hence

$$\overline{\lim_{k\to\infty}} \frac{f_x(\xi_{\lambda_k}(x^k), v(x^k))U_{\lambda_k}(x^k)}{\delta_k} \geq 0. \tag{3.10}$$

Furthermore, one can obtain

$$\overline{\lim_{k\to\infty}} \frac{V_{\lambda_k}(x^k)}{\delta_k} = \overline{\lim_{k\to\infty}} \frac{\frac{\partial v}{\partial x_1}(\zeta^k, (x^k)') \cdot 2(x_1^k - \lambda_k)}{\lambda_k - x_1^k} = -2\frac{\partial v}{\partial x_1}(x^0), \tag{3.11}$$

here ζ^k is valued between x_1^k and $2\lambda_k - x_1^k$.

Since $V_{\lambda_0}(x) > 0$ if $x \in \Sigma_{\lambda_0}$ and $V_{\lambda_0}(x) = 0$ if $x \in T_{\lambda_0}$, then $\frac{\partial V_{\lambda_0}}{\partial x_1}(x^0) \leq 0$. It implies that

$$0 \geq \frac{\partial V_{\lambda_0}}{\partial x_1}(x^0) = \frac{\partial v}{\partial x_1}(x_1^0, (x^0)') + \frac{\partial v}{\partial x_1}(2\lambda_0 - x_1^0, (x^0)') = 2\frac{\partial v}{\partial x_1}(x^0). \tag{3.12}$$

Combining (1.3), (3.9)-(3.12), one has

$$\overline{\lim_{k\to\infty}} \frac{(-\Delta)^s U_{\lambda_k}(x^k)}{\delta_k} \geq 0,$$

which contradicts with Lemma 2.4. Thus, $U_{\lambda_0}(x) \equiv 0,\ V_{\lambda_0}(x) \equiv 0,\ x \in \Sigma_{\lambda_0}$.

Case 2. The sequence $\{x_k\}_{k=1}^{\infty}$ is an unbounded subsequence. We can express it as $\lim_{k\to\infty} |x^k| = \infty$. Namely, for sufficiently large $R_0 > 0$, there exists $N_1 > 0$ such that for all $k > N_1$, $|x^k| > R_0$.

Then by Lemma 2.3 and (3.7), for any $k > N_1$,

$$V_{\lambda_k}(x^k) < 2U_{\lambda_k}(x^k) < 0. \tag{3.13}$$

Hence, for any $k > N_1$, there exists $y^k \in \Sigma_{\lambda_k}$ such that

$$V_{\lambda_k}(y^k) = \min_{\Sigma_{\lambda_k}} V_{\lambda_k}(y) < 0. \tag{3.14}$$

For the sequence $\{y^k\}_{k=1}^{\infty}$, there are two possible cases:

Case 2.1. The sequence $\{y^k\}_{k=1}^{\infty}$ contains a bounded subsequence. Repeating the process of Case 1, we can get the conclusion.

Case 2.2. The sequence $\{y^k\}_{k=1}^{\infty}$ is an unbounded subsequence. We can express it as $\lim_{k\to\infty} |y^k| = \infty$. Namely, for sufficiently large $R_0 > 0$, there exists $N_2 > 0$ such that for all $k > N_2$, $|y^k| > R_0$.

Then by Lemma 2.3 and (3.14), for $k > N_2$, one has

$$U_{\lambda_k}(y^k) < 2V_{\lambda_k}(y^k) < 0. \tag{3.15}$$

Hence, from (3.13) and (3.15), for any $k > N = \max\{N_1, N_2\}$, one can get

$$U_{\lambda_k}(y^k) < 2V_{\lambda_k}(y^k) \leq 2V_{\lambda_k}(x^k) < 4U_{\lambda_k}(x^k) \leq 4U_{\lambda_k}(y^k).$$

This contradicts with $U_{\lambda_k}(y^k) < 0$, and finishes the proof of Step 2.

Since the x_1 direction can be chosen arbitrarily, we actually indicate that $u(x)$ and $v(x)$ are radically symmetric about some point x^0. Also, the monotonicity of solutions follows easily from the argument.

Thus, this completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, we can prove Theorem 1.2 by using Lemmas 2.1, 2.2, 2.4-2.5. Here we omit it. □

4. Application to the Laplacian system

Proof of Corollary 1.1. Let

$$f(u(x), v(x)) = u^{-p_1}(x) + v^{q_1}(x),$$
$$g(u(x), v(x)) = v^{-s_1}(x) + u^{r_1}(x),$$

and

$$q = q_1, \ r = r_1.$$

One can easily conclude that f and g satisfy (1.3). Then all conditions of Theorem 1.1 are satisfied. So by Theorem 1.1, $u(x)$ and $v(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$. □

Proof of Corollary 1.2. Let

$$f(u(x), v(x)) = -u^{p_2}(x)v^{-q_2}(x), \quad g(u(x), v(x)) = -v^{s_2}(x)u^{-r_2}(x),$$

and

$$p = p_2, \ q^* = -q_2, \ r^* = -r_2, \ s = s_2.$$

One can easily conclude that f and g satisfy (1.6). Then all conditions of Theorem 1.2 are satisfied. So by Theorem 1.2, $u(x)$ and $v(x)$ must be radially symmetric and monotone increasing about some point in $\mathbb{R}^n$. □

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References

[1] D. Applebaum, *Lévy processes—from probability to finance and quantum groups*, Notices Amer. Math. Soc. **51** (2004), no. 11, 1336–1347.

[2] B. Barrios, I. De Bonis, M. Medina, and I. Peral, *Semilinear problems for the fractional laplacian with a singular nonlinearity*, Open Math. **13** (2015), no. 1, 390–407. https://doi.org/10.1515/math-2015-0038

[3] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 1, 39–71. https://doi.org/10.1017/S0308210511000175

[4] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260. https://doi.org/10.1080/03605300600987306

[5] M. Cai and L. Ma, *Moving planes for nonlinear fractional Laplacian equation with negative powers*, Discrete Contin. Dyn. Syst. **38** (2018), no. 9, 4603–4615. https://doi.org/10.3934/dcds.2018201

[6] L. Cao and X. Wang, *Radial symmetry of positive solutions to a class of fractional Laplacian with a singular nonlinearity*, J. Korean Math. Soc. **58** (2021), no. 6, 1449–1460. https://doi.org/10.4134/JKMS.j210091

[7] W. Chen, Y. Fang, and R. Yang, *Liouville theorems involving the fractional Laplacian on a half space*, Adv. Math. **274** (2015), 167–198. https://doi.org/10.1016/j.aim.2014.12.013

[8] W. Chen and C. Li, *Maximum principles for the fractional p-Laplacian and symmetry of solutions*, Adv. Math. **335** (2018), 735–758. https://doi.org/10.1016/j.aim.2018.07.016

[9] W. Chen, C. Li, and G. Li, *Maximum principles for a fully nonlinear fractional order equation and symmetry of solutions*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 29, 18 pp. https://doi.org/10.1007/s00526-017-1110-3

[10] W. Chen, C. Li, and Y. Li, *A direct method of moving planes for the fractional Laplacian*, Adv. Math. **308** (2017), 404–437. https://doi.org/10.1016/j.aim.2016.11.038

[11] W. Chen, Y. Li, and P. Ma, *The Fractional Laplacian*, World Sci. Publ., Hackensack, NJ, 2020. https://doi.org/10.1142/10550

[12] W. Chen, Y. Li, and R. Zhang, *A direct method of moving spheres on fractional order equations*, J. Funct. Anal. **272** (2017), no. 10, 4131–4157. https://doi.org/10.1016/j.jfa.2017.02.022

[13] C. Cheng, Z. Lü, and Y. Lü, *A direct method of moving planes for the system of the fractional Laplacian*, Pacific J. Math. **290** (2017), no. 2, 301–320. https://doi.org/10.2140/pjm.2017.290.301

[14] W. Dai, G. Qin, and D. Wu, *Direct methods for pseudo-relativistic Schrödinger operators*, J. Geom. Anal. **31** (2021), no. 6, 5555–5618. https://doi.org/10.1007/s12220-020-00492-1

[15] R. L. Frank and E. Lenzmann, *Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$*, Acta Math. **210** (2013), no. 2, 261–318. https://doi.org/10.1007/s11511-013-0095-9

[16] R. L. Frank, E. Lenzmann, and L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726. https://doi.org/10.1002/cpa.21591

[17] J. Giacomoni, T. Mukherjee, and K. Sreenadh, *Positive solutions of fractional elliptic equation with critical and singular nonlinearity*, Adv. Nonlinear Anal. **6** (2017), no. 3, 327–354. https://doi.org/10.1515/anona-2016-0113

[18] S. Jarohs and T. Weth, *Symmetry via antisymmetric maximum principles in nonlocal problems of variable order*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 1, 273–291. https://doi.org/10.1007/s10231-014-0462-y

[19] T. Jin, Y. Y. Li, and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 6, 1111–1171. https://doi.org/10.4171/JEMS/456

[20] T. Jin and J. Xiong, *A fractional Yamabe flow and some applications*, J. Reine Angew. Math. **696** (2014), 187–223. https://doi.org/10.1515/crelle-2012-0110

[21] P. Le, *Symmetry of positive solutions to quasilinear fractional systems*, Taiwanese J. Math. **25** (2021), no. 3, 517–534. https://doi.org/10.11650/tjm/201203

[22] C. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 491–526. https://doi.org/10.1080/03605309108820766

[23] B. Liu and L. Ma, *Radial symmetry results for fractional Laplacian systems*, Nonlinear Anal. **146** (2016), 120–135. https://doi.org/10.1016/j.na.2016.08.022

[24] Y. Lü and Z. Lü, *Some properties of solutions to the weighted Hardy-Littlewood-Sobolev type integral system*, Discrete Contin. Dyn. Syst. **36** (2016), no. 7, 3791–3810. https://doi.org/10.3934/dcds.2016.36.3791

[25] T. Mukherjee and K. Sreenadh, *Fractional elliptic equations with critical growth and singular nonlinearities*, Electron. J. Differential Equations **2016** (2016), Paper No. 54, 23 pp.

[26] Y. Niu, *Monotonicity of solutions for a class of nonlocal Monge-Ampère problem*, Commun. Pure Appl. Anal. **19** (2020), no. 11, 5269–5283. https://doi.org/10.3934/cpaa.2020237

[27] M. Qie, Z. Lü, and X. Zhang, *A direct method of moving planes for the fractional p-Laplacian system with negative powers*, Indian J. Pure Appl. Math. **54** (2023), no. 2, 344–358. https://doi.org/10.1007/s13226-022-00257-2

[28] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112. https://doi.org/10.1002/cpa.20153

[29] P. Wang and P. Niu, *A direct method of moving planes for a fully nonlinear nonlocal system*, Commun. Pure Appl. Anal. **16** (2017), no. 5, 1707–1718. https://doi.org/10.3934/cpaa.2017082

[30] G. Wang and X. Ren, *Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy-Schrödinger systems*, Appl. Math. Lett. **110** (2020), 106560, 8 pp. https://doi.org/10.1016/j.aml.2020.106560

[31] P. Wang and M. Yu, *Solutions of fully nonlinear nonlocal systems*, J. Math. Anal. Appl. **450** (2017), no. 2, 982–995. https://doi.org/10.1016/j.jmaa.2017.01.070

[32] X. Wang and L. Zhang, *Existence and multiplicity of weak positive solutions to a class of fractional Laplacian with a singular nonlinearity*, Results Math. **74** (2019), no. 2, Paper No. 81, 18 pp. https://doi.org/10.1007/s00025-019-1004-0

[33] L. Wu and W. Chen, *The sliding methods for the fractional p-Laplacian*, Adv. Math. **361** (2020), 106933, 26 pp. https://doi.org/10.1016/j.aim.2019.106933

[34] L. Wu and P. Niu, *Symmetry and nonexistence of positive solutions to fractional p-Laplacian equations*, Discrete Contin. Dyn. Syst. **39** (2019), no. 3, 1573–1583. https://doi.org/10.3934/dcds.2019069

[35] Z. Wu and H. Xu, *Symmetry properties in systems of fractional Laplacian equations*, Discrete Contin. Dyn. Syst. **39** (2019), no. 3, 1559–1571. https://doi.org/10.3934/dcds.2019068

[36] L. Zhang, B. Ahmad, G. Wang, and X. Ren, *Radial symmetry of solution for fractional p-Laplacian system*, Nonlinear Anal. **196** (2020), 111801, 16 pp. https://doi.org/10.1016/j.na.2020.111801

[37] B. Zhang and Z. Lü, *Symmetry and nonexistence of solutions for a fully nonlinear nonlocal system*, Pacific J. Math. **299** (2019), no. 1, 237–255. https://doi.org/10.2140/pjm.2019.299.237

[38] R. Zhuo, W. Chen, X. Cui, and Z. Yuan, *Symmetry and non-existence of solutions for a nonlinear system involving the fractional Laplacian*, Discrete Contin. Dyn. Syst. **36** (2016), no. 2, 1125–1141. https://doi.org/10.3934/dcds.2016.36.1125

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