J. Korean Math. Soc. **61** (2024), No. 3, pp. 409–425
https://doi.org/10.4134/JKMS.j230039
pISSN: 0304-9914 / eISSN: 2234-3008

SELF-PAIR HOMOTOPY EQUIVALENCES RELATED TO CO-VARIANT FUNCTORS

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Abstract. The category of pairs is the category whose objects are maps between two based spaces and morphisms are pair-maps from one object to another object. To study the self-homotopy equivalences in the category of pairs, we use covariant functors from the category of pairs to the group category whose objects are groups and morphisms are group homomorphisms. We introduce specific subgroups of groups of self-pair homotopy equivalences and put these groups together into certain sequences. We investigate properties of these sequences, in particular, the exactness and split. We apply the results to two special functors, homotopy and homology functors and determine the suggested several subgroups of groups of self-pair homotopy equivalences.

1. Introduction

The category of pairs is the category whose objects are maps between two based spaces and morphisms are pair-maps from one object to another. In [4], the category of pairs was described in detail. In this category, objects are maps between two based spaces and a morphism from one object $\alpha\colon (X_1, *) \to (X_2, *)$ to another object $\beta\colon (Y_1, *) \to (Y_2, *)$ is a pair of maps (f_1, f_2) such that the

Received January 19, 2023; Revised January 2, 2024; Accepted January 23, 2024.
2020 *Mathematics Subject Classification.* Primary 55P10, 55Q05, 55N20.
Key words and phrases. Category of pairs, self homotopy equivalence, co-variant functor.
The first-named author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2022R1I1A1A01053856).
The second-named author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (2021R1F1A1048686).
The third-named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1A6A3A01100186).

diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[\beta]{} & Y_2 \end{array}$$

is commutative, i.e., $f_2 \circ \alpha = \beta \circ f_1$.

Two maps (f_1, f_2) and (g_1, g_2) from α to β are called pair homotopic if the diagram commutes:

$$\begin{array}{ccc} X_1 \times I & \xrightarrow{\alpha \times I} & X_2 \times I \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle H_2} \\ Y_1 & \xrightarrow[\beta]{} & Y_2 \end{array}$$

where H_1 and H_2 are homotopies $f_1 \simeq g_1$ and $f_2 \simeq g_2$, respectively. In this case, we denote by $(H_1, H_2)\colon (f_1, f_2) \simeq (g_1, g_2)$. The homotopy class of (f_1, f_2) is denoted by $[f_1, f_2]$ and the set of those homotopy classes from α to β is denoted by $\Pi(\alpha, \beta)$. A morphism (f_1, f_2) is called a homotopy equivalence in the category of pairs, if there is a morphism (g_1, g_2) such that $(g_1, g_2) \circ (f_1, f_2) \simeq (id_{X_1}, id_{X_2})$ and $(f_1, f_2) \circ (g_1, g_2) \simeq (id_{Y_1}, id_{Y_2})$. In this case, (g_1, g_2) is called a homotopy inverse of (f_1, f_2). Furthermore, (f_1, f_2) is called a self homotopy equivalence in the category of pairs, if $\alpha = \beta$. Moreover, it is called a self-pair homotopy equivalence in the category of pairs, if $\alpha = \beta = i\colon A \to X$ is the inclusion. Groups of self homotopy equivalences were introduced in [5, Lee]. For a given object $\alpha\colon X_1 \to X_2$, $\mathcal{E}(\alpha)$ is the group whose elements are self homotopy equivalences from α to itself.

In [7], we introduce specific subgroups of the group of self-pair homotopy equivalences that induce the identity map on homotopy groups up to dimension n. In this paper, we extend these subgroups of self-pair homotopy equivalences to a more general case using covariant functors. We readily demonstrate that all results in [7] hold. Moreover, we show that the results related to homology groups. Consequently, we can compute the specific subgroups of self-pair homotopy equivalences that induce the identity map on homotopy and homology groups up to dimension n.

In Section 3, we introduce definitions of specific subgroups $\mathcal{E}_{\mathcal{F}}(\alpha)$ and $\mathcal{E}_{\mathcal{F}}(\alpha; id_A)$, see Definition 3.5. Using these subgroups, we have proven the following theorem.

Theorem 3.7. *Let $X \diamond Y$ denote the product (resp. wedge) of spaces and $i\colon Y \to X \diamond Y$ be the inclusion map. Then for a given ordered family $\mathcal{F}$ of covariant functors $F_k\colon HoTop_* \to Gr$ such that $F_k(X \diamond Y) = F_k(X) \times F_k(Y)$ for all $k \leq n$, there exists a split short exact sequence*

$$1 \longrightarrow \mathcal{E}^n_{\mathcal{F}}(i; id_Y) \xrightarrow{\textbf{inc.}} \mathcal{E}^n_{\mathcal{F}}(i) \underset{s}{\overset{\pi_Y}{\rightleftarrows}} \mathcal{E}^n_{\mathcal{F}}(Y) \longrightarrow 1.$$

In Section 4, for give two abelian groups G_1 and G_2, we let $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$ are Moore spaces, respectively. Let $X = M_1 \vee M_2$ and $i_k \colon M_k \to X$ be the inclusion map for $k = 1, 2$.

Theorem 4.6. *Let $H = F \oplus T$ be a finitely generated abelian group such that F is a free abelian group and T is a finite abelian group. If $M_1 = M(F, n)$, $M_2 = M(T, n)$, and $X = M_1 \vee M_2$, then we have*

$$\mathcal{E}_*^{\infty}(i_1) \cong \oplus^{(r+s)s}\mathbb{Z}_2 \quad \text{and} \quad \mathcal{E}_*^{\infty}(i_2) \cong \oplus^{s^2}\mathbb{Z}_2,$$

where $i_k \colon M_k \to X$ is an inclusion map for $k \in \{1, 2\}$, and s is the number of 2-torsion summands and r is the rank of H.

In Theorem 4.8, we calculate the homology case of wedge product of Moore spaces which have different dimension. The following theorem is the result related to the homotopy case.

Theorem 4.13. *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$ and let $X = M_1 \vee M_2$. Then*

$$\mathcal{E}_\sharp^{s}(i_1) \cong \begin{cases} \mathbb{Z}_q \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \text{ or } q : \text{odd}, \\ \mathbb{Z}_q \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 0 \pmod 4 \end{cases}$$

for $s \leq n$ and

$$\mathcal{E}_\sharp^{n+1}(i_1) \cong \begin{cases} \oplus^{s_q^2}\mathbb{Z}_2 & \text{if } p \text{ or } q : \text{odd}, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_4 & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 0 \pmod 4. \end{cases}$$

And

$$\mathcal{E}_\sharp^{s}(i_2) \cong \begin{cases} \mathbb{Z}_q \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } (p,q) = 1, \\ \mathbb{Z}_q \oplus \mathbb{Z}_d \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } (p,q) = d \neq 1 \end{cases}$$

for $s \leq n$ and

$$\mathcal{E}_\sharp^{n+1}(i_2) \cong \begin{cases} \oplus^{s_q^2}\mathbb{Z}_2 & \text{if } (p,q) = 1 \text{ or } p : \text{odd}, \\ \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus \mathbb{Z}_d & \text{if } (p,q) = d \neq 1 \text{ and } p : \text{even}. \end{cases}$$

Throughout this paper, all topological spaces are based on connected CW complexes and all maps and homotopies are base point preserving. The set of based homotopy classes of based maps from X to Y is denoted by $[X, Y]$. A map $f \colon X \to Y$ and its homotopy class $[f]$ in $[X, Y]$ will not be distinguished.

2. Preliminaries

In this section, we review several results introduced in previous studies to make it easier to read this paper.

Definition 2.1 ([5]). For an object $\alpha\colon X_1 \to X_2$, we define $\mathcal{E}(\alpha)$ as

$$\mathcal{E}(\alpha) = \{[f_1, f_2] \,|\, (f_1, f_2) \text{ is a homotopy equivalence in } \Pi(\alpha, \alpha)\}.$$

It was shown that $\mathcal{E}(\alpha)$ has a group structure in [5, Theorem 2.1].

Proposition 2.2 ([1]).
(1) $\pi_n(M(Z_q, n)) \cong \mathbb{Z}_q$ *for all* q.
(2) $\pi_{n+1}(M(Z_q, n)) \cong \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \mathbb{Z}_2 & \text{if } q \text{ is even.} \end{cases}$
(3) $\pi_{n+2}(M(Z_q, n)) \cong \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \mathbb{Z}_4 & \text{if } q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } q \equiv 0 \pmod 4. \end{cases}$

Proposition 2.3 ([2]). *If X is $(r-1)$-connected, Y is $(\ell-1)$-connected and, further, if $r, \ell \geq 2$ and $\dim P < r + \ell - 1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection*:

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

Theorem 2.4 ([2]). *Let $M(G, n)$ be a Moore space. Then*

$$\mathcal{E}_*^{\infty}(M(G, n)) \cong \oplus^{(r+s)s}\mathbb{Z}_2,$$

where r is the rank of G and s is the number of 2-torsion summands in G.

Theorem 2.5 ([2]). *Let $M(G, n)$ be a Moore space. Then*

$$\mathcal{E}_\sharp^{n}(M(G, n)) \cong \mathcal{E}_*^{\infty}(M(G, n)),$$
$$\mathcal{E}_\sharp^{n+1}(M(G, n)) \cong 1 \text{ if } n > 3.$$

Theorem 2.6 ([3, Lemmas 3.2, 3.3, 3.4, 3.5]). *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$. Then*

$$[M_2, M_1] \cong \begin{cases} 0 & \text{if } (p, q) = 1, \\ \mathbb{Z}_d\,\{\pi_2^*(i_1)\} & \text{if } (p, q) = d \neq 1 \end{cases}$$

and

	$[M_1, M_2]$	*Generator*
p *or* q : *odd*	0	-
$p \equiv 2, q \equiv 0 \pmod 4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\alpha, \pi_1^*(\eta_2)$
$p \equiv 0, q \equiv 2 \pmod 4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\pi_1^*(\bar{\eta}), \beta$
$p \equiv q \equiv 2 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\pi_1^*(\bar{\eta}), \beta$
$p \equiv q \equiv 0 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\pi_1^*(\eta_1), \pi_1^*(\eta_2), \alpha$

where $\pi_1\colon M_1 \to S^{n+2}$ and $\pi_2\colon M_2 \to S^{n+1}$ are projections and $i_1\colon S^{n+1} \to M_1$ and $i_2\colon S^n \to M_2$ are inclusions and α and β satisfies the relations that

$\pi_1^*(\eta_1) = 2\alpha$, $i_1^*(\alpha) = i_{2\sharp}(\eta)$, $i_1^*(\beta) = i_{2\sharp}(\eta)$, where η is the generator of $\pi_{n+1}(S^n)$ and η_1 is a generator of $\pi_2(M(\mathbb{Z}_q, n))$ such that $i_\sharp(\eta^2) = \eta_1$ in [8] *and* [3].

3. Self-pair homotopy equivalences related to covariant functors

In this section, we introduce specific subgroups of $\mathcal{E}(\alpha)$ and give an exact sequence related to covariant functors.

Definition 3.1. Let $\alpha\colon A \to X$ be an object in category of pair and $F\colon HoTop_* \to Gr$ be a covariant functor, where $HoTop_*$ is the category of based topological spaces and base preserving homotopy class and Gr is the category of groups and homomorphisms. We define

$$\mathcal{E}_F(\alpha) = \left\{[f_1, f_2] \in \mathcal{E}(\alpha) \,|\, F(f_1) = id_{F(A)},\ F(f_2) = id_{F(X)}\right\}.$$

In particular, we define a subset $\mathcal{E}_F(\alpha; id_A)$ of $\mathcal{E}_F(\alpha)$ as

$$\mathcal{E}_F(\alpha; id_A) = \{[f_1, f_2] \in \mathcal{E}_F(\alpha) \,|\, f_1 = id_A\}.$$

Furthermore, if $\alpha\colon A \to *$, then $\mathcal{E}_F(A) = \mathcal{E}_F(\alpha)$.

Proposition 3.2. *$\mathcal{E}_F(\alpha)$ is a subgroup of $\mathcal{E}(\alpha)$.*

Proof. Let $(f_1, f_2), (f_1', f_2') \in \mathcal{E}_F(\alpha)$. Then

$$\begin{aligned} F(f_1 \circ f_1') &= F(f_1) \circ F(f_1') \\ &= id_{F(A)} \circ id_{F(A)} \\ &= id_{F(A)} \end{aligned}$$

and similarly,

$$F(f_2 \circ f_2') = id_{F(X)}.$$

Thus $(f_1, f_2) \circ (f_1', f_2') \in \mathcal{E}_F(\alpha)$.

Let $(f_1, f_2) \in \mathcal{E}_F(\alpha)$. Since $\mathcal{E}(\alpha)$ is a group, there is an inverse $(h_1, h_2) \in \mathcal{E}(\alpha)$ of (f_1, f_2). Then

$$\begin{aligned} id_{F(A)} &= F(id_A) \\ &= F(f_1 \circ h_1) \\ &= F(f_1) \circ F(h_1) \\ &= id_{F(A)} \circ F(h_1) \\ &= F(h_1) \end{aligned}$$

and similarly,

$$id_{F(X)} = F(h_2).$$

Thus, $(h_1, h_2) \in \mathcal{E}_F(\alpha)$. Therefore, $\mathcal{E}_F(\alpha)$ is a subgroup of $\mathcal{E}(\alpha)$. □

Proposition 3.3. *Let $\alpha\colon A \to X$ be an object. Then there is an exact sequence*

$$1 \longrightarrow \mathcal{E}_F(\alpha; id_A) \xrightarrow{\textbf{inc.}} \mathcal{E}_F(\alpha) \xrightarrow{\pi_A} \mathcal{E}_F(A),$$

where π_A is the projection to the first factor.

Proof. It is sufficient to show that $\mathbf{inc}.(\mathcal{E}_F(\alpha; id_A)) = \ker(\pi_A)$. Since each element in $\mathcal{E}_F(\alpha; id_A)$ is of the form $[id_A, g]$, $\pi_A(\mathbf{inc}.([id_A, g])) = [id_A]$. Thus $\mathbf{inc}.(\mathcal{E}_F(\alpha; id_A)) \subseteq \ker(\pi_A)$.

Conversely, if $[f_1, f_2] \in \ker(\pi_A)$, then $\pi_A([f_1, f_2]) = [id_A]$. Hence $[f_1] = [id_A]$. By the definition of $\mathcal{E}_F(\alpha; id_A)$, $[f_1, f_2] \in \mathcal{E}_F(\alpha; id_A)$. □

It is easy to prove the following corollary.

Corollary 3.4. *Let F_1 and F_2 be covariant functors. If $\alpha\colon A \to X$ is an object, then there is an exact sequence*

$$1 \longrightarrow \mathcal{E}_{F_1}(\alpha; id_A) \cap \mathcal{E}_{F_2}(\alpha; id_A) \xrightarrow{\mathbf{inc.}} \mathcal{E}_{F_1}(\alpha) \cap \mathcal{E}_{F_2}(\alpha) \xrightarrow{\pi_A} \mathcal{E}_{F_1}(A) \cap \mathcal{E}_{F_2}(A).$$

Definition 3.5. Let α be an object, and $\mathcal{F} = \{F_k\colon HoTop_* \to Gr \mid k \in \mathbb{Z}\}$ be an ordered family of functors. Then we define a subset $\mathcal{E}^n_{\mathcal{F}}(\alpha)$ of $\mathcal{E}(\alpha)$ as

$$\mathcal{E}^n_{\mathcal{F}}(\alpha) = \{[f_1, f_2] \in \mathcal{E}(\alpha) \mid F_k(f_1) = F_k(id_A),\ F_k(f_2) = F_k(id_X) \text{ for } 0 \le k \le n\}.$$

Since the finite intersection of subgroups is also a subgroup, by Proposition 3.2, $\mathcal{E}^n_{\mathcal{F}}(\alpha)$ is also a subgroup of $\mathcal{E}(\alpha)$.

Corollary 3.6. *Let $\mathcal{F} = \{F_k\colon HoTop_* \to Gr \mid k \in \mathbb{Z}\}$ be an ordered family of functors. If $\alpha\colon A \to X$ is an object, then there is an exact sequence*

$$1 \longrightarrow \mathcal{E}^n_{\mathcal{F}}(\alpha; id_A) \xrightarrow{\mathbf{inc.}} \mathcal{E}^n_{\mathcal{F}}(\alpha) \xrightarrow{\pi_A} \mathcal{E}^n_{\mathcal{F}}(A)\ .$$

Remark. (1) Let $\mathcal{F}$ be an ordered family of homology functors and $\alpha\colon A \to X$ be an object. Then

$$\mathcal{E}^n_{\mathcal{F}}(\alpha; id_A) = \mathcal{E}^n_{*}(\alpha; id_A),\ \mathcal{E}^n_{\mathcal{F}}(\alpha) = \mathcal{E}^n_{*}(\alpha), \text{ and } \mathcal{E}^n_{\mathcal{F}}(A) = \mathcal{E}^n_{*}(A).$$

Thus we have an exact sequence

$$1 \longrightarrow \mathcal{E}^n_{*}(\alpha; id_A) \xrightarrow{\mathbf{inc.}} \mathcal{E}^n_{*}(\alpha) \xrightarrow{\pi_A} \mathcal{E}^n_{*}(A)\ .$$

(2) Let $\mathcal{F}$ be an ordered family of homotopy functors and $\alpha\colon A \to X$ be an object. Then

$$\mathcal{E}^n_{\mathcal{F}}(\alpha; id_A) = \mathcal{E}^n_{\sharp}(\alpha; id_A),\ \mathcal{E}^n_{\mathcal{F}}(\alpha) = \mathcal{E}^n_{\sharp}(\alpha), \text{ and } \mathcal{E}^n_{\mathcal{F}}(A) = \mathcal{E}^n_{\sharp}(A).$$

Thus we have an exact sequence

$$1 \longrightarrow \mathcal{E}^n_{\sharp}(\alpha; id_A) \xrightarrow{\mathbf{inc.}} \mathcal{E}^n_{\sharp}(\alpha) \xrightarrow{\pi_A} \mathcal{E}^n_{\sharp}(A)\ .$$

Theorem 3.7. *Let $X \diamond Y$ denote the product (resp. wedge) of spaces and $i\colon Y \to X \diamond Y$ be the inclusion map. Then, for a given ordered family $\mathcal{F}$ of covariant functors $F_k\colon HoTop_* \to Gr$ such that $F_k(X \diamond Y) = F_k(X) \times F_k(Y)$ for $k \le n$, there exists a split short exact sequence*

$$1 \longrightarrow \mathcal{E}^n_{\mathcal{F}}(i; id_Y) \xrightarrow{\mathbf{inc.}} \mathcal{E}^n_{\mathcal{F}}(i) \underset{s}{\overset{\pi_Y}{\rightleftarrows}} \mathcal{E}^n_{\mathcal{F}}(Y) \longrightarrow 1.$$

Proof. By Corollary 3.6, we have the following exact sequence:

$$1 \longrightarrow \mathcal{E}^n_{\mathcal{F}}(i; id_Y) \xrightarrow{\text{inc.}} \mathcal{E}^n_{\mathcal{F}}(i) \xrightarrow{\pi_Y} \mathcal{E}^n_{\mathcal{F}}(Y).$$

Define $s\colon \mathcal{E}^n_{\mathcal{F}}(Y) \to \mathcal{E}^n_{\mathcal{F}}(i)$ by $s([f]) = [f, id_X \diamond f]$. Then

$$\begin{aligned} F_k(id_X \diamond f) &= F_k(id_X) \times F_k(f) \\ &= F_k(id_X) \times F_k(id_Y) \\ &= F_k(id_X \diamond id_Y) \\ &= F_k(id_{X \diamond Y}) \end{aligned}$$

for $k \leq n$. Hence $s([f]) \in \mathcal{E}^n_{\mathcal{F}}(i)$. Since $\pi_Y \circ s([f]) = \pi_Y([f, id_X \diamond f]) = [f]$, $\pi_Y \circ s = id_{\mathcal{E}_F(Y)}$. ☐

Corollary 3.8. *Let $X \vee Y$ be the wedge product space of X and Y. Then we have a split short exact sequence*:

$$1 \longrightarrow \mathcal{E}^n_*(i; id_Y) \xrightarrow{\text{inc.}} \mathcal{E}^n_*(i) \underset{s}{\overset{\pi_Y}{\rightleftarrows}} \mathcal{E}^n_*(Y) \longrightarrow 1,$$

where $i\colon Y \to X \vee Y$ is the inclusion.

Proof. It follows immediately from Theorem 3.7. ☐

Corollary 3.9. *Let $X \times Y$ be the product of X and Y. Then we have a split short exact sequence*:

$$1 \longrightarrow \mathcal{E}^k_\sharp(i; id_Y) \longrightarrow \mathcal{E}^k_\sharp(i) \underset{s}{\overset{\pi_Y}{\rightleftarrows}} \mathcal{E}^k_\sharp(Y) \longrightarrow 1,$$

where $i\colon Y \to X \times Y$ is an inclusion.

Corollary 3.10. *Let X be $(r-1)$-connected and Y be $(\ell-1)$-connected for $r, \ell \geq 2$. Let $X \vee Y$ be the wedge product of X and Y. Then we have a split short exact sequence*:

$$1 \longrightarrow \mathcal{E}^k_\sharp(i; id_Y) \longrightarrow \mathcal{E}^k_\sharp(i) \underset{s}{\overset{\pi_Y}{\rightleftarrows}} \mathcal{E}^k_\sharp(Y) \longrightarrow 1,$$

where $i\colon Y \to X \vee Y$ is an inclusion.

Proof. It follows immediately from Theorem 3.7. ☐

This corollary has been proved in [7, Theorem 4.2].

Corollary 3.11. *Let X and Y be homotopy associative and inversive co-H-spaces such that the two sets $[X \wedge Y, X \flat Y]$ and $[X, Y]$ are trivial. Then there exists a split exact sequence*:

$$1 \longrightarrow \mathcal{E}^k_*(X) \longrightarrow \mathcal{E}^k_*(i) \rightleftarrows \mathcal{E}^k_*(Y) \longrightarrow 1,$$

where $i\colon Y \to X \vee Y$ is an inclusion and $X \flat Y = \Sigma(\Omega X \vee \Omega Y)$.

Proof. It can be proved similar to [7, Proposition 4.3]. ☐

4. Self-pair homotopy equivalences related to covariant functors of Moore spaces

In this section, we use the sequence to compute several subgroups of $\mathcal{E}(\alpha)$ described in Section 3. From now on, we consider a given ordered family $\mathcal{F}$ of covariant functors $F_k\colon HoTop_* \to Gr$ such that $F_k(X \diamond Y) = F_k(X) \times F_k(Y)$ for $k \leq n$. First, we prove the following proposition.

Proposition 4.1. *Let $X \diamond Y$ denote the product (resp. wedge) of spaces and $i_J\colon J \to X \diamond Y$ be the inclusion map for $J \in \{X, Y\}$. Define $\Phi^J\colon \mathcal{E}(i_J; id_J) \to \mathcal{E}(X \diamond Y)$ by $\Phi^J([id_J, f]) = [f]$. Then Φ^J is a monomorphism.*

Proof. Let $J = X$. If $(id_X, f) \simeq (id_X, g)$, then $f \simeq g$. Thus $\Phi^X([id_X, f]) = [f] = [g] = \Phi^X([id_X, g])$. Hence Φ^X is well-defined.

Now, we show that Φ^X is a monomorphism. We prove that if $[f] = [g]$, then $(id_X, f) \simeq (id_X, g)$. Since $id_X \simeq id_X$, there are two extensions $\tilde{f}$ and $\tilde{g}$ of f and g by [7, Proposition 5.1]. Thus

$$(id_X, f) \simeq (id_X, \tilde{f}) \text{ and } (id_X, g) \simeq (id_X, \tilde{g}).$$

Since $(X \diamond Y, X)$ is a homotopy extendable pair and both $\tilde{f}$ and $\tilde{g}$ are extensions of id_X, $(id_X, \tilde{f}) \simeq (id_X, \tilde{g})$ by [5, Proposition 3.2]. Therefore $(id_X, f) \simeq (id_X, g)$. So Φ^X is a monomorphism. Similarly, Φ^Y is a monomorphism. □

Corollary 4.2. *Let $X \diamond Y$ denote the product (resp. wedge) of spaces and $i_J\colon J \to X \diamond Y$ be the inclusion map for $J \in \{X, Y\}$. Define $\Phi^J_F\colon \mathcal{E}_F(i_J; id_J) \to \mathcal{E}_F(X \diamond Y)$ by $\Phi^J_F([id_J, f]) = [f]$ for any covariant functor F. Then Φ^J_F is a monomorphism.*

Moreover, $\Phi^J_{\mathcal{F}}\colon \mathcal{E}^n_{\mathcal{F}}(i_J; id_J) \to \mathcal{E}^n_{\mathcal{F}}(X \diamond Y)$ is given by $\Phi^J_{\mathcal{F}}([id_J, f]) = [f]$ for any ordered family of covariant functors. Then $\Phi^J_{\mathcal{F}}$ is a monomorphism.

Proof. If $[id_J, f] \in \mathcal{E}_F(i_J; id_J)$, then $F(f) = F(id_{X \diamond Y})$. Therefore, both Φ^J_F and $\Phi^J_{\mathcal{F}}$ are monomorphisms. □

Definition 4.3. For an ordered family $\mathcal{F}$ of covariant functors $F_k\colon HoTop_* \to Gr$, we define a subset $\mathcal{Z}^n_{\mathcal{F}}(X, Y)$ of $[X, Y]$ by

$$\mathcal{Z}^n_{\mathcal{F}}(X, Y) = \{[f] \in [X, Y] \mid F_k(f) = F_k(C_*) \text{ for } k \leq n\},$$

where $C_* : X \to Y$ is a constant map. In particular, if $\mathcal{F}$ is the ordered family of homotopy or homology functors, then we denote by

$$\mathcal{Z}^n_{\mathcal{F}}(X, Y) = \mathcal{Z}^n_{\sharp}(X, Y) \text{ and } \mathcal{Z}^n_{\mathcal{F}}(X, Y) = \mathcal{Z}^n_{*}(X, Y).$$

Corollary 4.4. *Let $X \diamond Y$ denote the product (resp. wedge) of spaces and $i_J\colon J \to X \diamond Y$ be the inclusion map for $J \in \{X, Y\}$. If $[id_J, f] \in \mathcal{E}_F(i_J; id_J)$, then $[f_{KJ}] \in \mathcal{Z}_F(J, K)$ and $[f_{KK}] \in \mathcal{E}_F(K)$ for any covariant functor F.*

Moreover, if $[id_J, f] \in \mathcal{E}^n_{\mathcal{F}}(i_J; id_J)$, then $[f_{KJ}] \in \mathcal{Z}^n_{\mathcal{F}}(J, K)$ and $[f_{KK}] \in \mathcal{E}^n_{\mathcal{F}}(K)$ for any ordered family functor $\mathcal{F}$.

Proof. Let $[id_J, f] \in \mathcal{E}_F(i_J; id_J)$. By Corollary 4.2, $[f] \in \mathcal{E}_F(X \diamond Y)$. Thus $F(f) = F(id_{X\diamond Y})$. Since $F(X \diamond Y) = F(X) \times F(Y)$, $F(f)$ can be represented by a matrix

$$F(f) \simeq \begin{pmatrix} F(f_{XX}) & F(f_{XY}) \\ F(f_{YX}) & F(f_{YY}) \end{pmatrix}$$

for all $f : X \diamond Y \to X \diamond Y$. For any $[id_J, f] \in \mathcal{E}_F(i_J; id_J)$, $F(f)$ can be represented by

$$F(f) \simeq \begin{pmatrix} F(id_X) & 0 \\ F(f_{YX}) & F(f_{YY}) \end{pmatrix} = \begin{pmatrix} F(id_X) & 0 \\ 0 & F(id_Y) \end{pmatrix} \text{ if } J = X,$$

$$F(f) \simeq \begin{pmatrix} F(f_{XX}) & F(f_{XY}) \\ 0 & F(id_Y) \end{pmatrix} = \begin{pmatrix} F(id_X) & 0 \\ 0 & F(id_Y) \end{pmatrix} \text{ if } J = Y.$$

Thus $f_{JJ} = id_J$ and $f_{JK} = C_*$, where C_* is a constant map. Therefore $F(f_{KJ}) = 0$ and $F(f_{KK}) = F(id_Y)$. □

In homotopy case, if $[f] \in \mathcal{E}^n_\sharp(X \times Y)$, then

$$\pi_k(f) \simeq \begin{pmatrix} \pi_k(f_{XX}) & \pi_k(f_{XY}) \\ \pi_k(f_{YX}) & \pi_k(f_{YY}) \end{pmatrix} = \begin{pmatrix} \pi_k(id_X) & 0 \\ 0 & \pi_k(id_Y) \end{pmatrix} \tag{1}$$

for all $k \leq n$.

Let X be $(r-1)$-connected and Y be $(\ell-1)$-connected for $r, \ell \geq 2$. By Proposition 2.3, $[P, X \vee Y] \equiv [P, X] \oplus [P, Y]$ for $\dim P < r + \ell - 1$. If $[f] \in \mathcal{E}^n_\sharp(X \vee Y)$ for $n < r + \ell - 1$, then

$$\pi_k(f) \simeq \begin{pmatrix} \pi_k(f_{XX}) & \pi_k(f_{XY}) \\ \pi_k(f_{YX}) & \pi_k(f_{YY}) \end{pmatrix} = \begin{pmatrix} \pi_k(id_X) & 0 \\ 0 & \pi_k(id_Y) \end{pmatrix} \tag{2}$$

for all $k \leq n$.

By Corollaries 4.2, 4.4 and equations (1) and (2), we have:

Lemma 4.5. *$\mathcal{E}^n_\sharp(i_J; id_J) \cong \mathcal{Z}^n_\sharp(J, K) \oplus \mathcal{E}^n_\sharp(K)$ for $J, K \in \{X, Y\}$ and $J \neq K$.*

Now, we recall some examples related to homotopy groups induced by identity.

Example 1 ([7, Example 1])**.** Let $X = S^m$ and $Y = S^n$ for $m \geq n \geq 1$. Let $i_J : J \to X \times Y$ be the inclusion map for $J \in \{X, Y\}$. By Corollary 3.9 and Lemma 4.5,

$$\mathcal{E}^k_\sharp(i_J) \cong \mathcal{Z}^n_\sharp(J, K) \oplus \mathcal{E}^n_\sharp(K) \oplus \mathcal{E}^k_\sharp(J)$$

for $J, K \in \{X, Y\}$ and $J \neq K$. If $m = 3, 7$, and $n = 1$, then $[Y, X] = 0$ and $[X, Y] = 0$. Then $\mathcal{Z}^k_\sharp(X, Y) = 0$ and $\mathcal{Z}^k_\sharp(Y, X) = 0$. Thus, $\mathcal{E}^k_\sharp(i_X; id_X) \cong \mathcal{E}^k_\sharp(Y)$ and $\mathcal{E}^k_\sharp(i_Y; id_Y) \cong \mathcal{E}^k_\sharp(X)$. Therefore,

$$\mathcal{E}^k_\sharp(i_X) \cong \mathcal{E}^k_\sharp(X) \oplus \mathcal{E}^k_\sharp(Y) \cong \mathcal{E}^k_\sharp(i_Y).$$

Hence,

	$k \geq m$	$m > k \geq 1$
$\mathcal{E}_\sharp^k(i_X) \cong \mathcal{E}_\sharp^k(i_Y)$	1	$\mathbb{Z}_2$

Example 2 ([7, Example 3])**.** Let $M_1 = M(\mathbb{Z}_2, n+1)$ and $M_2 = M(\mathbb{Z}_3, n)$ be Moore spaces for $n \geq 5$, and $X = M_1 \vee M_2$. By Corollary 3.10 and Lemma 4.5,

$$\mathcal{E}_\sharp^k(i_J) \cong \mathcal{Z}_\sharp^n(J,K) \oplus \mathcal{E}_\sharp^n(K) \oplus \mathcal{E}_\sharp^k(J)$$

for $J, K \in \{X, Y\}$ and $J \neq K$. By Theorem 2.6, $[M_1, M_2] = 0$ and $[M_2, M_1] = 0$. Thus $\mathcal{Z}_\sharp^{\dim X}(M_1, M_2) = \mathcal{Z}_\sharp^{\dim X}(M_2, M_1) = 0$. We have $\mathcal{E}_\sharp^{\dim X}(i_{M_1}; id_{M_1}) \cong \mathcal{E}_\sharp^{\dim X}(M_2)$ and $\mathcal{E}_\sharp^{\dim X}(i_{M_2}; id_{M_2}) \cong \mathcal{E}_\sharp^{\dim X}(M_1)$. Therefore,

$$\mathcal{E}_\sharp^{\dim X}(i_{M_1}) \cong \mathcal{E}_\sharp^{\dim X}(M_1) \oplus \mathcal{E}_\sharp^{\dim X}(M_2) \cong \mathcal{E}_\sharp^{\dim X}(i_{M_2}).$$

Since $\mathcal{E}_\sharp^{\dim X}(M_1) = \mathbb{Z}_2$ and $\mathcal{E}_\sharp^{\dim X}(M_2) = 1$,

$$\mathcal{E}_\sharp^{\dim X}(i_{M_1}) \cong \mathcal{E}_\sharp^{\dim X}(i_{M_2}) \cong \mathbb{Z}_2.$$

But $\mathcal{E}_\sharp^{\dim X}(i_{M_1}; id_{M_1})$ is not isomorphic to $\mathcal{E}_\sharp^{\dim X}(i_{M_2}; id_{M_2})$.

Let now G_1 and G_2 be abelian groups, and let $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$ be Moore spaces. Let $X = M_1 \vee M_2$ and $i_k \colon M_k \to X$ be the inclusion map and $p_j \colon X \to M_j$ be the projection for $j, k = 1, 2$. If $f \colon X \to X$ is a self-map, let $f_{jk} \colon M_k \to M_j$ be defined by $f_{jk} = p_j \circ f \circ i_k$ for $j, k = 1, 2$.

Let H be a finitely generated abelian group. Then H can be decomposed by $H = F \oplus T$, where F is a free abelian group and T is a finite abelian group. Then a Moore space $M(H, n)$ can be decomposed by $M(H, n) = M(F, n) \vee M(T, n)$ for $n \geq 3$. Moreover, we have

$$M(F, n) = \vee_{i=1}^r S_i^n \text{ and } M(T, n) = \vee_{j=1}^t S^n \cup_{q_j \iota} e^{n+1},$$

where S_i^n's are n-dimensional spheres and $q_j\iota$'s are attaching maps.

Theorem 4.6. *Let $H = F \oplus T$ be a finitely generated abelian group such that F is a free abelian group and T is a finite abelian group. If $M_1 = M(F, n)$, $M_2 = M(T, n)$, and $X = M_1 \vee M_2$, then we have*

$$\mathcal{E}_*^\infty(i_1) \cong \oplus^{(r+s)s}\mathbb{Z}_2 \text{ and } \mathcal{E}_*^\infty(i_2) \cong \oplus^{s^2}\mathbb{Z}_2,$$

where $i_k \colon M_k \to X$ is inclusion map for $k \in \{1, 2\}$, and s is the number of 2-torsion summands and r is the rank of H.

Proof. By Theorem 3.7, there is a split short exact sequence:

$$1 \longrightarrow \mathcal{E}_*^\infty(i_k; id_{M_k}) \longrightarrow \mathcal{E}_*^\infty(i_k) \rightleftarrows \mathcal{E}_*^\infty(M_k) \longrightarrow 1,$$

where $i_k \colon M_k \to X$ is an inclusion map and $k = 1, 2$. By [2, Theorem 3.2], we have

$$\mathcal{E}_*^\infty(X) \cong \oplus^{(r+s)s}\mathbb{Z}_2, \ \mathcal{E}_*^\infty(M_1) = 1, \text{ and } \mathcal{E}_*^\infty(M_2) = \oplus^{s^2}\mathbb{Z}_2.$$

Now, we determine that $\mathcal{E}_*^\infty(i_k; id_{M_k})$ for $k = 1, 2$.

Case 1. Since $\mathcal{E}_*^\infty(X) \cong [M_2, M_1] \oplus \mathcal{E}_*^\infty(M_2)$ by [2, Theorem 3.2], $f \circ i_1 = i_1$ for all $[f] \in \mathcal{E}_*^\infty(X)$. Thus $[id_{M_1}, g] \in \mathcal{E}_*^\infty(i_1; id_{M_1})$ for all $[g] \in \mathcal{E}_*^\infty(X)$. Therefore $\mathcal{E}_*^\infty(i_1; id_{M_1}) = \mathcal{E}_*^\infty(X)$. Consequently

$$\mathcal{E}_*^\infty(i_1) \cong \oplus^{(r+s)s}\mathbb{Z}_2.$$

Case 2. If $[id_{M_2}, f] \in \mathcal{E}_*^\infty(i_2; id_{M_2})$, then $f \circ i_2 = i_2$. Thus $f_{21} = 0$ and $f_{22} = id_{M_2}$. Hence $[f] \in \mathcal{E}_*^\infty(i_2; id_{M_2})$ if and only if $[f] = [id_X]$ by [2, Theorem 3.2]. Therefore $\mathcal{E}_*^\infty(i_2) \cong \mathcal{E}_*^\infty(M_2) \cong \oplus^{s^2}\mathbb{Z}_2$. Consequently

$$\mathcal{E}_*^\infty(i_2) \cong \oplus^{s^2}\mathbb{Z}_2. \qquad \square$$

Proposition 4.7. *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$ and let $X = M_1 \vee M_2$. Then we have*

$$\mathcal{E}_*^s(i_1) \cong \mathcal{E}_*^s(M_1) \oplus [M_1, M_2] \oplus \mathcal{E}_*^s(M_2)$$

and

$$\mathcal{E}_*^s(i_2) \cong \mathcal{E}_*^s(M_1) \oplus [M_2, M_1] \oplus \mathcal{E}_*^s(M_2).$$

Proof. By Theorem 3.7, there is a split short exact sequence:

$$1 \longrightarrow \mathcal{E}_*^s(i_k; id_{M_k}) \longrightarrow \mathcal{E}_*^s(i_k) \rightleftarrows \mathcal{E}_*^s(M_k) \longrightarrow 1,$$

where $i_k\colon M_k \to X$ is an inclusion and $k = 1, 2$. We have

$$\mathcal{E}_*^s(i_k) \cong \mathcal{E}_*^s(i_k; id_{M_k}) \oplus \mathcal{E}_*^s(M_k).$$

Now, we determine that $\mathcal{E}_*^s(i_k; id_{M_k})$ for $k = 1, 2$ and $s \geq n$.

Since $H_s(M_1) = \begin{cases} \mathbb{Z}_q & \text{if } s = n+1, \\ 0 & \text{others,} \end{cases}$ and $H_s(M_2) = \begin{cases} \mathbb{Z}_p & \text{if } s = n, \\ 0 & \text{others,} \end{cases}$

$$\mathcal{Z}_*^s(M_1, M_2) = [M_1, M_2] \text{ and } \mathcal{Z}_*^s(M_2, M_1) = [M_2, M_1]$$

for all s. From [3, Lemma 4.1], we have $\mathcal{E}(X) \cong \mathcal{E}(M_1) \oplus [M_1, M_2] \oplus [M_2, M_1] \oplus \mathcal{E}(M_2)$. Therefore

$$\mathcal{E}_*^s(X) \cong \mathcal{E}_*^s(M_1) \oplus [M_2, M_1] \oplus [M_1, M_2] \oplus \mathcal{E}_*^s(M_2)$$

for all s.

Case 1. If $[f] \in \mathcal{E}_*^s(X)$ such that $f_{11} = id_{M_1}$ and $f_{21} = C_*$, then $f \circ i_1 = i_1$. Thus $[id_{M_1}, f] \in \mathcal{E}_*^s(i_1; id_{M_1})$. Hence $[id_{M_1}, f] \in \mathcal{E}_*^s(i_1; id_{M_1})$ if and only if $[f_{21}] \in \mathcal{Z}_*^s(M_1, M_2) = [M_1, M_2]$ and $[f_{22}] \in \mathcal{E}_*^s(M_2)$ by Corollary 4.4. Therefore

$$\mathcal{E}_*^s(i_1; id_{M_1}) \cong [M_1, M_2] \oplus \mathcal{E}_*^s(M_2).$$

Consequently,

$$\mathcal{E}_*^s(i_1) \cong \mathcal{E}_*^s(M_1) \oplus [M_1, M_2] \oplus \mathcal{E}_*^s(M_2).$$

Case 2. If $[f] \in \mathcal{E}_*^s(X)$ such that $f_{22} = id_{M_2}$ and $f_{21} = C_*$, then $f \circ i_2 = i_2$. Similarly to Case 1, we have $[id_{M_2}, f] \in \mathcal{E}_*^s(i_2; id_{M_2})$ if and only if $[f_{11}] \in \mathcal{E}_*^s(M_1)$ and $[f_{12}] \in \mathcal{Z}_*^s(M_2, M_1)$. Therefore

$$\mathcal{E}_*^s(i_2; id_{M_2}) \cong \mathcal{E}_*^s(M_1) \oplus [M_2, M_1].$$

Consequently,

$$\mathcal{E}_*^s(i_2) \cong \mathcal{E}_*^s(M_1) \oplus [M_2, M_1] \oplus \mathcal{E}_*^s(M_2). \qquad \square$$

By Theorem 2.6,

$$[M_2, M_1] \cong \begin{cases} 0 & \text{if } (p,q) = 1, \\ \mathbb{Z}_d & \text{if } (p,q) = d \neq 1, \end{cases}$$

and

$$[M_1, M_2] \cong \begin{cases} 0 & \text{if } p \text{ or } q : \text{odd}, \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \text{if } p \equiv 2, q \equiv 0 \text{ or } p \equiv 0, q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 0 \pmod 4). \end{cases}$$

By [6, Thoerem 2.1],

$$\mathcal{E}(M_1) \cong \begin{cases} \mathbb{Z}_q^* & \text{if } q : \text{odd}, \\ \mathbb{Z}_2 \times \mathbb{Z}_q^* & \text{if } q : \text{even}, \end{cases} \text{ and}$$

$$\mathcal{E}(M_2) \cong \begin{cases} \mathbb{Z}_p^* & \text{if } p : \text{odd}, \\ \mathbb{Z}_2 \times \mathbb{Z}_p^* & \text{if } p : \text{even}, \end{cases}$$

where $\mathbb{Z}_p^*$ and $\mathbb{Z}_q^*$ are the automorphism groups of $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively.

Since

$$H_s(M_1) \cong \begin{cases} \mathbb{Z}_p & \text{if } s = n+1, \\ 0 & \text{others}, \end{cases} \text{ and}$$

$$H_s(M_2) \cong \begin{cases} \mathbb{Z}_q & \text{if } s = n, \\ 0 & \text{others}, \end{cases}$$

$$(3) \qquad \mathcal{E}_*^s(M_1) \cong \begin{cases} \mathcal{E}(M_1) & \text{if } s \leq n, \\ \mathcal{E}_*^\infty(M_1) & \text{if } s \geq n+1, \end{cases} \cong \begin{cases} \mathbb{Z}_q^* & \text{if } q : \text{odd and } s \leq n, \\ \mathbb{Z}_2 \times \mathbb{Z}_q^* & \text{if } q : \text{even and } s \leq n, \\ \oplus^{s_q^2} \mathbb{Z}_2 & \text{if } s \geq n+1, \end{cases}$$

where s_q is the number of 2-torsion summunds in $\mathbb{Z}_q$.

And

$$(4) \qquad \mathcal{E}_*^s(M_2) \cong \begin{cases} \mathcal{E}(M_2) & \text{if } s < n, \\ \mathcal{E}_*^\infty(M_2) & \text{if } s \geq n, \end{cases} \cong \begin{cases} \mathbb{Z}_p^* & \text{if } p : \text{odd and } s < n, \\ \mathbb{Z}_2 \times \mathbb{Z}_p^* & \text{if } p : \text{even and } s < n, \\ \oplus^{s_p^2} \mathbb{Z}_2 & \text{if } s \geq n, \end{cases}$$

where s_p is the number of 2-torsion summands in $\mathbb{Z}_p$.

Theorem 4.8. *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$ and let $X = M_1 \vee M_2$. Then*

$$\mathcal{E}_*^n(i_1) \cong \begin{cases} \mathbb{Z}_q^* \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } q : \text{odd},\\ (\mathbb{Z}_2 \times \mathbb{Z}_q^*) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p : \text{odd and } q : \text{even},\\ (\mathbb{Z}_2 \times \mathbb{Z}_q^*) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 2, q \equiv 0 \\ & \text{or } p \equiv 0, q \equiv 2 \pmod 4,\\ (\mathbb{Z}_2 \times \mathbb{Z}_q^*) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 2 \pmod 4,\\ (\mathbb{Z}_2 \times \mathbb{Z}_q^*) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 0 \pmod 4 \end{cases}$$

and

$$\mathcal{E}_*^s(i_1) \cong \begin{cases} \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \text{ or } q : \text{odd},\\ \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 2, q \equiv 0 \\ & \text{or } p \equiv 0, q \equiv 2 \pmod 4,\\ \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 2 \pmod 4,\\ \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 0 \pmod 4 \end{cases}$$

for $s \geq n+1$, where s_q is the number of 2-torsion summunds in $\mathbb{Z}_q$.

$$\mathcal{E}_*^n(i_2) \cong \begin{cases} \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus \mathbb{Z}_q^* & \text{if } (p,q) = 1 \text{ and } q : \text{odd},\\ \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus (\mathbb{Z}_2 \times \mathbb{Z}_q^*) & \text{if } (p,q) = 1 \text{ and } q : \text{even},\\ \mathbb{Z}_d \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus \mathbb{Z}_q^* & \text{if } (p,q) = d \neq 1 \text{ and } q : \text{odd},\\ \mathbb{Z}_d \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus (\mathbb{Z}_2 \times \mathbb{Z}_q^*) & \text{if } (p,q) = d \neq 1 \text{ and } q : \text{even} \end{cases}$$

and

$$\mathcal{E}_*^s(i_2) \cong \begin{cases} \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus \left(\oplus^{s_q^2}\mathbb{Z}_2\right) & \text{if } (p,q) = 1,\\ \mathbb{Z}_d \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) \oplus \left(\oplus^{s_q^2}\mathbb{Z}_2\right) & \text{if } (p,q) = d \neq 1 \end{cases}$$

for $s \geq n+1$, where s_p is the number of 2-torsion summands in $\mathbb{Z}_p$.

Proof. By Theorem 2.6 and Proposition 4.7 and (3), (4). □

Proposition 4.9. *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$ and let $X = M_1 \vee M_2$. Then*

$$\mathcal{E}_\sharp^s(i_1) \cong \mathcal{E}_\sharp^s(M_1) \oplus \mathcal{Z}_\sharp^n(M_1, M_2) \oplus \mathcal{E}_\sharp^s(M_2)$$

and

$$\mathcal{E}_\sharp^s(i_2) \cong \mathcal{E}_\sharp^s(M_1) \oplus \mathcal{Z}_\sharp^n(M_2, M_1) \oplus \mathcal{E}_\sharp^s(M_2)$$

for all s.

Proof. By Theorem 3.7, there is a split short exact sequence:

$$1 \longrightarrow \mathcal{E}^s_\sharp(i_k; id_{M_k}) \longrightarrow \mathcal{E}^s_\sharp(i_k) \rightleftarrows \mathcal{E}^s_\sharp(M_k) \longrightarrow 1,$$

where $i_k \colon M_k \to X$ is an inclusion and $k = 1, 2$. Then we have

$$\mathcal{E}^s_\sharp(i_1) \cong \mathcal{E}^s_\sharp(i_1; id_{M_1}) \oplus \mathcal{E}^s_\sharp(M_1) \text{ and } \mathcal{E}^s_\sharp(i_2) \cong \mathcal{E}^s_\sharp(i_2; id_{M_2}) \oplus \mathcal{E}^s_\sharp(M_2).$$

From [3, Lemma 4.1], we have $\mathcal{E}(X) \cong \mathcal{E}(M_1) \oplus [M_1, M_2] \oplus [M_2, M_1] \oplus \mathcal{E}(M_2)$. Therefore

$$\mathcal{E}^s_\sharp(X) \cong \mathcal{E}^s_\sharp(M_1) \oplus \mathcal{Z}^s_\sharp(M_2, M_1) \oplus \mathcal{Z}^s_\sharp(M_1, M_2) \oplus \mathcal{E}^s_\sharp(M_2)$$

for all s.

Now, we determine that $\mathcal{E}^s_\sharp(i_k; id_{M_k})$ for $k = 1, 2$ and all s.

Case 1. If $[f] \in \mathcal{E}^s_\sharp(X)$ such that $f_{11} = id_{M_1}$ and $f_{21} = C_*$, then $f \circ i_1 = i_1$. Thus $[id_{M_1}, f] \in \mathcal{E}^s_\sharp(i_1; id_{M_1})$. Hence $[id_{M_1}, f] \in \mathcal{E}^s_\sharp(i_1; id_{M_1})$ if and only if $[f_{21}] \in \mathcal{Z}^s_\sharp(M_1, M_2)$ and $[f_{22}] \in \mathcal{E}^s_\sharp(M_2)$ by Corollary 4.4 and (2). Therefore

$$\mathcal{E}^s_\sharp(i_1; id_{M_1}) \cong \mathcal{Z}^s_\sharp(M_1, M_2) \oplus \mathcal{E}^s_\sharp(M_2).$$

Consequently

$$\mathcal{E}^s_\sharp(i_1) \cong \mathcal{E}^s_\sharp(M_1) \oplus \mathcal{Z}^s_\sharp(M_1, M_2) \oplus \mathcal{E}^s_\sharp(M_2).$$

Case 2. If $[f] \in \mathcal{E}^s_\sharp(X)$ such that $f_{22} = id_{M_2}$ and $f_{12} = C_*$, then $f \circ i_2 = i_2$. Similarly to Case 1, we have $[id_{M_2}, f] \in \mathcal{E}^s_\sharp(i_2; id_{M_2})$ if and only if $[f_{11}] \in \mathcal{E}^s_\sharp(M_1)$ and $[f_{12}] \in \mathcal{Z}^s_\sharp(M_2, M_1)$. Therefore

$$\mathcal{E}^s_\sharp(i_2; id_{M_2}) \cong \mathcal{E}^s_\sharp(M_1) \oplus \mathcal{Z}^s_\sharp(M_2, M_1).$$

Consequently

$$\mathcal{E}^s_\sharp(i_2) \cong \mathcal{E}^s_\sharp(M_1) \oplus \mathcal{Z}^s_\sharp(M_2, M_1) \oplus \mathcal{E}^s_\sharp(M_2). \qquad \square$$

Lemma 4.10. *For $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$, we have*

$$\mathcal{Z}^m_\sharp(M_1, M_2) = [M_1, M_2]$$

for $m \leq n$ and

$$\mathcal{Z}^{n+1}_\sharp(M_1, M_2) \cong \begin{cases} 0 & \text{if } p \text{ or } q : \text{odd}, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \mathbb{Z}_4 & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 0 \pmod 4. \end{cases}$$

Proof. Since $\pi_m(M_1) = 0$ for $m \leq n$, $\mathcal{Z}^m_\sharp(M_1, M_2) = [M_1, M_2]$. And since $\pi_{n+1}(M_1) \cong \mathbb{Z}_q$ and $\pi_{n+1}(M_2) = \begin{cases} 0 & \text{if } p : \text{odd}, \\ \mathbb{Z}_2 & \text{if } p : \text{even}. \end{cases}$

Therefore

$$\mathcal{Z}_\sharp^{n+1}(M_1, M_2) \cong \begin{cases} 0 & \text{if } p \text{ or } q : \text{odd}, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \mathbb{Z}_4 & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 0 \pmod 4, \end{cases}$$

by Theorem 2.6. □

Lemma 4.11. *For $M(\mathbb{Z}_q, n+1)$ and $M(\mathbb{Z}_p, n)$, we have*

$$\mathcal{Z}_\sharp^m(M_2, M_1) = [M_2, M_1]$$

for $m \leq n$ and

$$\mathcal{Z}_\sharp^{n+1}(M_2, M_1) = \begin{cases} 0 & \textit{if } (p,q) = 1 \textit{ or } p : \textit{odd}, \\ \mathbb{Z}_d & \textit{if } (p,q) = d \neq 1 \textit{ and } p : \textit{even}. \end{cases}$$

Proof. Since $\pi_m(M_1) = 0$ for $m \leq n$, $\mathcal{Z}_\sharp^m(M_2, M_1) = [M_2, M_1]$. And since $\pi_{n+1}(M_1) \cong \mathbb{Z}_q$ and $\pi_{n+1}(M_2) = \begin{cases} 0 & \text{if } p : \text{odd}, \\ \mathbb{Z}_2 & \text{if } p : \text{even}. \end{cases}$

Therefore

$$\mathcal{Z}_\sharp^{n+1}(M_2, M_1) = \begin{cases} 0 & \text{if } (p,q) = 1 \text{ or } p : \text{odd}, \\ \mathbb{Z}_d & \text{if } (p,q) = d \neq 1 \text{ and } p : \text{even}, \end{cases}$$

by Theorem 2.6. □

Lemma 4.12.

$$\mathcal{E}_\sharp^s(M(\mathbb{Z}_q, n+1)) \cong \begin{cases} \mathbb{Z}_q & \textit{if } s < n+1, \\ \oplus^{s_q^2}\mathbb{Z}_2 & \textit{if } s = n+1, \\ 1 & \textit{if } s > n+1, \end{cases}$$

and

$$\mathcal{E}_\sharp^s(M(\mathbb{Z}_p, n)) \cong \begin{cases} \mathbb{Z}_p & \textit{if } s < n, \\ \oplus^{s_p^2}\mathbb{Z}_2 & \textit{if } s = n, \\ 1 & \textit{if } s > n, \end{cases}$$

where s_q and s_p are the number of 2-torsion summands in $\mathbb{Z}_q$ and $\mathbb{Z}_p$.

Proof. It follows from [2, Theorems 3.2 and 3.8]. □

Theorem 4.13. *Let $M_1 = M(\mathbb{Z}_q, n+1)$ and $M_2 = M(\mathbb{Z}_p, n)$ and let $X = M_1 \vee M_2$. Then*

$$\mathcal{E}_\sharp^s(i_1) \cong \begin{cases} \mathbb{Z}_q \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \text{ or } q: \text{odd}, \\ \mathbb{Z}_q \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \mathbb{Z}_q \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } p \equiv q \equiv 0 \pmod 4 \end{cases}$$

for $s \leq n$ and

$$\mathcal{E}_\sharp^{n+1}(i_1) \cong \begin{cases} \oplus^{s_q^2}\mathbb{Z}_2 & \text{if } p \text{ or } q: \text{odd}, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv 2, q \equiv 0 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_4 & \text{if } p \equiv 0, q \equiv 2 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 2 \pmod 4, \\ \oplus^{s_q^2}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } p \equiv q \equiv 0 \pmod 4, \end{cases}$$

where s_p is the number of 2-torsion summunds in $\mathbb{Z}_p$. And

$$\mathcal{E}_\sharp^s(i_2) \cong \begin{cases} \mathbb{Z}_q \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } (p,q) = 1, \\ \mathbb{Z}_q \oplus \mathbb{Z}_d \oplus \left(\oplus^{s_p^2}\mathbb{Z}_2\right) & \text{if } (p,q) = d \neq 1 \end{cases}$$

for $s \leq n$ and

$$\mathcal{E}_\sharp^{n+1}(i_2) \cong \begin{cases} \oplus^{s_q^2}\mathbb{Z}_2 & \text{if } (p,q) = 1 \text{ or } p: \text{odd}, \\ \left(\oplus^{s_q^2}\mathbb{Z}_2\right) \oplus \mathbb{Z}_d & \text{if } (p,q) = d \neq 1 \text{ and } p: \text{even}, \end{cases}$$

where s_q is the number of 2-torsion summunds in $\mathbb{Z}_q$.

Proof. By Proposition 4.9, we have

$$\mathcal{E}_\sharp^s(i_1) \cong \mathcal{E}_\sharp^s(M_1) \oplus \mathcal{Z}_\sharp^n(M_1, M_2) \oplus \mathcal{E}_\sharp^s(M_2)$$

and

$$\mathcal{E}_\sharp^s(i_2) \cong \mathcal{E}_\sharp^s(M_1) \oplus \mathcal{Z}_\sharp^n(M_2, M_1) \oplus \mathcal{E}_\sharp^s(M_2)$$

for all s.

By Lemmas 4.10 and 4.11, we calculate $\mathcal{Z}_\sharp^n(M_1, M_2)$ and $\mathcal{Z}_\sharp^n(M_2, M_1)$, respectively.

By Lemma 4.12, we determine $\mathcal{E}_\sharp^s(M_1)$ and $\mathcal{E}_\sharp^s(M_2)$. □

References

[1] S. Araki and H. Toda, *Multiplicative structures in* mod q *cohomology theories. I*, Osaka Math. J. **2** (1965), 71–115. http://projecteuclid.org/euclid.ojm/1200691225

[2] M. Arkowitz and K. Maruyama, *Self-homotopy equivalences which induce the identity on homology, cohomology or homotopy groups*, Topology Appl. **87** (1998), no. 2, 133–154. https://doi.org/10.1016/S0166-8641(97)00162-4

[3] H. W. Choi and K. Y. Lee, *Certain self-homotopy equivalences on wedge products of Moore spaces*, Pacific J. Math. **272** (2014), no. 1, 35–57. https://doi.org/10.2140/pjm.2014.272.35
[4] P. Hilton, *Duality in homotopy theory: a retrospective essay*, J. Pure Appl. Algebra **19** (1980), 159–169. https://doi.org/10.1016/0022-4049(80)90099-7
[5] K. Y. Lee, *The groups of self pair homotopy equivalences*, J. Korean Math. Soc. **43** (2006), no. 3, 491–506. https://doi.org/10.4134/JKMS.2006.43.3.491
[6] J. W. Rutter, *The group of homotopy self-equivalence classes of CW complexes*, Math. Proc. Cambridge Philos. Soc. **93** (1983), no. 2, 275–293. https://doi.org/10.1017/S0305004100060576
[7] H. S. Shin, K. Y. Lee, and H. W. Choi, *Certain subgroups of groups of self-pair homotopy equivalences*, Topology Appl. **264** (2019), 382–393. https://doi.org/10.1016/j.topol.2019.06.012
[8] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton Univ. Press, Princeton, NJ, 1962.

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