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CERTAIN ASPECTS OF ROUGH IDEAL STATISTICAL CONVERGENCE ON NEUTROSOPHIC NORMED SPACES

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Abstract. In this paper, we have presented rough ideal statistical convergence of sequence on neutrosophic normed spaces as a significant convergence criterion. As neutrosophication can handle partially dependent components, partially independent components and even independent components involved in real-world problems. By examining some properties related to rough ideal convergence in these spaces we have established some equivalent conditions on the set of ideal statistical limit points for rough ideal statistically convergent sequences.

1. Introduction

It has been seen that new quests are revealing in the real-life problems with time. So many methods are already existing, and researchers are still investigating new variants for existing and future problems. In modern logic, the three-way decision situations like from accepting/rejecting/pending, from yes/no/not-applicable, from sports win/lose/tie etc. the standard analysis is not sufficient, which leads to employ non-standard analysis. Smarandache [23] provided neutrosophic sets with regard to the powerful advancement of intuitionistic fuzzy sets, that established the concept for the classic sets, fuzzy sets, vague sets etc., for nonstandard analysis. Conception of neutrosophic set can manage indeterminate data for the problem whereas the impression of fuzzy set theory and intuitionistic fuzzy set theory are not able to provide solution when the relation is indeterminate. As every element of the neutrosophic set has a truth value, a false value and an indeterminacy value, respectively, which lies in the non-standard unit interval. Due to this nature neutrosophic is more adjustable and efficient tool because of its ability to handle, not only the free components of information but also partially independent and dependent information. In neutrosophic set elements may have inconsistent information (*i.e.* sum of the components > 1) or incomplete information (*i.e.* sum of the components < 1) or consistent information (*i.e.* sum of the components $= 1$), and other interval-valued components (*i.e.* without any restriction on the sum of superior or inferior components).

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Definition 1.1. [23] Let U be a subset of $\mathbb{X}$ (which is space of points) with $a \in \mathbb{X}$. Then set U with $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ in $\mathbb{X}$ is called neutrosophic set(NS) and expressed as

$$U = \{< a, \tau(a), \upsilon(a), \eta(a) > : \ a \in \mathbb{X}, \tau(a), \upsilon(a), \eta(a) \in I\}$$

where $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ denotes truth membership function, indeterminacy membership function and falsity membership function respectively such that $0^- \leq \tau(a) + \upsilon(a) + \eta(a) \leq 3^+$. Also $I =]0^-, 1^+[$ represents the non-standard unit interval.

However, Wang et al. [24] and Ye [25] have customized existing definition for neutrosophic sets by suggesting the structure of single-valued neutrosophic sets and simplified neutrosophic sets respectively using the interval $[0, 1]$, that can be utilized in engineering and scientific applications. Later, Mahapatra and Bera [7] studied the notion of neutrosophic soft linear spaces. Kirişci and Şimşek [14] has introduced neutrosophic metric spaces and established its basic characteristic properties like open ball, compactness, completeness and nowhere dense. Further, Kirişci and Şimşek [15] has given the thought of neutrosophic normed spaces as a notable consideration of neutrosophic metric spaces.

Definition 1.2. [15] A neutrosophic normed space(NNS) is 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ with vector space $\mathbb{X}$, normed space $\aleph = \{< \tau(a), \upsilon(a), \eta(a) > : a \in \mathbb{X}\}$ such that $\aleph : \mathbb{X} \times \mathbb{R}^+ \to [0, 1]$, continuous t-norm $\circledast$ and continuous t-conorm $\odot$ if for every $x, y \in \mathbb{X}$ and $s, t > 0$, we have

(i) $0 \leq \tau(x; t), \upsilon(x; t), \eta(x; t) \leq 1$,
(ii) $\tau(x; t) + \upsilon(x; t) + \eta(x; t) \leq 3$,
(iii) $\tau(x; t) = 1$, $\upsilon(x; t) = 0$ and $\eta(x; t) = 0$ for $t > 0$ iff $x = 0$,
(iv) $\tau(x; t) = 0$, $\upsilon(x; t) = 1$ and $\eta(x; t) = 1$ for $t \leq 0$,
(v) $\tau(\alpha x; t) = \tau\left(x; \frac{t}{|\alpha|}\right)$, $\upsilon(\alpha x; t) = \upsilon\left(x; \frac{t}{|\alpha|}\right)$ and $\eta(\alpha x; t) = \eta\left(x; \frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$,
(vi) $\tau(x; \circ)$ is continuous non-decreasing function,
(vii) $\tau(x; s) \circledast \tau(y; t) \leq \tau(x + y; s + t)$,
(viii) $\upsilon(x; \circ)$ is continuous non-increasing function,
(ix) $\upsilon(x; s) \odot \upsilon(y, t) \geq \upsilon(x + y; s + t)$,
(x) $\eta(x; \circ)$ is continuous non-increasing function,
(xi) $\eta(x; s) \odot \eta(y; t) \geq \eta(x + y; s + t)$,
(xii) $\lim_{t\to\infty} \tau(x; t) = 1$, $\lim_{t\to\infty} \upsilon(x; t) = 0$ and $\lim_{t\to\infty} \eta(x; t) = 0$.

Then (τ, υ, η) is known as neutrosophic norm.

Example 1.3. [15] Consider $(\mathbb{X}, \| . \|)$ as a normed space. For all $t > 0$ and $x \in \mathbb{X}$, if we take
(i) $\tau(x; t) = \frac{t}{t+\|x\|}$, $\upsilon(x; t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x; t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii) $\tau(x; t) = 0$, $\upsilon(x; t) = 1$ and $\eta(x; t) = 1$ when $t \leq \|x\|$.
Along with $\alpha_1 \circledast \alpha_2 = \alpha_1\alpha_2$ and $\alpha_1 \odot \alpha_2 = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$ for all $\alpha_1, \alpha_2 \in [0, 1]$. Then, a 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS which satisfies above mentioned conditions.

A generalized version of intuitionistic fuzzy norms is considered in neutrosophic normed spaces that help to explore the basic properties like convergence and completeness in such spaces using continuous and bounded linear operators. Further, Kirişci and Şimşek [15] have established the concept of convergence for sequences on neutrosophic normed spaces.

Also Kirişci and Şimşek [15] have established the statistical convergence for sequences in the neutrosophic normed spaces using natural density. A natural density for the set A, where $A \subseteq \mathbb{N}$, has been given by $d(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{a \leq n : a \in A\} \mid$, where $\mid . \mid$ used for the order of an enclosed set. Moreover, sequence $s = \{s_k\}$ statistically converges to the number s_0, provided $A(\epsilon) = \{k \in \mathbb{N} : |s_k - s_0| \geq \epsilon\}$ has zero natural density [11].

Definition 1.4. [15] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$ and $t > 0$, we have

$$d(\{k \in \mathbb{N} : \tau(s_k - s_0; t) \leq 1 - \epsilon \text{ or } \upsilon(s_k - s_0; t) \geq \epsilon,\ \eta(s_k - s_0; t) \geq \epsilon\}) = 0.$$

It is convenient to represent symbolically by $St_{(\tau,\upsilon,\eta)} - \lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{St_{(\tau,\upsilon,\eta)}} s_0$.

Some remarkable results on the theory of neutrosophic normed spaces and statistical convergence of sequences on neutrosophic normed spaces have been studied with different directions ($c.f.$ [12, 13, 16, 17, 19]) as this theory serves as key component in so many areas of mathematics, economics, science and technology. In this paper we associate this theory with the rough convergence of sequences along with the ideal convergence with statistical point of view.

Kostryko *et al.* [20] established the notion of ideal convergence (I-convergence) by generalizing the idea of statistical convergence [10].

Definition 1.5. [20] A family $I \subseteq P(\mathbb{X})$, where $\mathbb{X} \neq \emptyset$ and $P(\mathbb{X})$ is a power set of the set $\mathbb{X}$, has been called an ideal on $\mathbb{X}$ provided, (i) $\emptyset \in I$, (ii) $R, S \in I \implies R \cup S \in I$, (iii) For $R \in I, S \subset R \implies S \in I$.

If $I \neq P(\mathbb{X})$, then I termed as a non-trivial ideal and further any non-trivial ideal I is known as an admissible ideal on $\mathbb{X}$ provided all the possibles singleton sets are contained in I *i.e* if $I \supset \{\{x\} : x \in \mathbb{X}\}$.

Example 1.6. A family of finite subsets of $\mathbb{N}$, denoted by I_f, is an admissible ideal in $\mathbb{N}$.

Consider I as a non-trivial admissible ideal throughout this article.

Definition 1.7. [20] A non-empty class $\mathbb{F} \subset P(\mathbb{X})$, where $\mathbb{X} \neq \emptyset$, has been called filter on $\mathbb{X}$ provided, (i) $\emptyset \notin \mathbb{F}$, (ii) $R, S \in \mathbb{F} \implies R \cap S \in \mathbb{F}$, (iii) For each $R \in \mathbb{F}, R \subset S \implies S \in \mathbb{F}$.
Every ideal I is associated with a filter $\mathbb{F}(I)$ which is given as follows:

$$\mathbb{F}(I) = \{K \subseteq \mathbb{X} : K^c \in I\}.$$

Definition 1.8. [20] Any sequence $s = \{s_k\}$ from $\mathbb{X}$ is called statistically ideal convergent (I-St-convergent) to s_0 from $\mathbb{X}$ provided for every $\epsilon > 0$ and $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n} | \{k \leq n : |s_k - s_0| \geq \epsilon\} | \geq \delta\right\} \in I.$$

Here, s_0 is known as the I-St-limit of the sequence $x = \{s_k\}$.

Recently, Kişi [17] associated the ideal convergence of sequences with the neutrosophic normed spaces.

Definition 1.9. [17] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called I-convergent to $s_0 \in X$ corresponding to neutrosophic norm (τ, υ, η) if for every $\kappa \in (0, 1)$ and $t > 0$ we have

$$\{k \in \mathbb{N} : \tau(s_k - s_0; t) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; t) \geq \kappa,\ \eta(s_k - s_0; t) \geq \kappa\} \in I.$$

It is convenient to represent symbolically by $I_{(\tau,\upsilon,\eta)}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{I_{(\tau,\upsilon,\eta)}} s_0$.

In the last two decades, there has been growing interest in the study of rough convergence theory, which provides a rigorous framework for understanding rough convergence and its applications in different aspects (*c.f.* [1–3, 6, 8, 9, 18, 21]) which leads us to investigate and explore rough ideal statistical convergence with the theory of neutrosophic normed spaces. Rough convergence helps to affirm the correctness of approximate solutions for real-life situations obtained from the computer programs and to draw conclusions from scientific experiments. In some cases, rough convergence may be more appropriate than other types of convergence because it reflects the true nature of the system or phenomenon being studied.

Definition 1.10. [22] A sequence $s = \{s_k\}$ from normed linear space $(\mathbb{X}, \|\,.\,\|)$ is called rough convergent for some $r \geq 0$ to $s_0 \in \mathbb{X}$ if for every $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ provided $\|s_k - s_0\| < r + \epsilon$ for all $k \geq k_0$.

Aytar [4] utilized the notion of rough convergence to establish the statistical analogue about this concept as rough statistical convergence for the sequences, like usual convergence is refined to statistical convergence for sequences using natural density by Fast [10]. Further, Aytar [5] tried some criteria related to convexity and closeness associated with a collection of rough statistical limit points.

Definition 1.11. [4] A sequence $s = \{s_k\}$ from normed linear space $(\mathbb{X}, \|\,.\,\|)$ is called rough statistically convergent to $s_0 \in \mathbb{X}$ for some $r \geq 0$ if for every $\epsilon > 0$, we get

$$d(\{k \in \mathbb{N} : \|s_k - s_0\| \geq r + \epsilon\}) = 0,$$

s_0 is identified as r-St-limit of sequence $s = \{s_k\}$.

2. Main Results

We first mention the conception of I-statistical convergence and rough statistical convergence of sequences on neutrosophic normed spaces that will be helpful in studying the major results of our work.

Definition 2.1. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called I-statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$ and $\kappa \in (0, 1)$ provided

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \tau(s_k - s_0; \epsilon) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; \epsilon) \geq \kappa,\ \eta(s_k - s_0; \epsilon) \geq \kappa\}| \geq \delta\} \in I.$$

It is convenient to represent symbolically by $I\text{-}St_{(\tau,\upsilon,\eta)} - \lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{I\text{-}St_{(\tau,\upsilon,\eta)}} s_0$.

DEFINITION 2.2. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called rough convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$ and $\kappa \in (0, 1)$ we can find $k_0 \in \mathbb{N}$ provided

$$\tau(s_k - s_0; r + \epsilon) > 1 - \kappa,\ \upsilon(s_k - s_0; r + \epsilon) < \kappa \text{ and } \eta(s_k - s_0; r + \epsilon) < \kappa \text{ for all } k \geq k_0.$$

It is convenient to represent symbolically by $r_{(\tau,\upsilon,\eta)} - \lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{r_{(\tau,\upsilon,\eta)}} s_0$.

DEFINITION 2.3. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called rough statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$ and $\kappa \in (0, 1)$,

$$d(\{k \in \mathbb{N} : \tau(s_k - s_0; r + \epsilon) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; r + \epsilon) \geq \kappa,\ \eta(s_k - s_0; r + \epsilon) \geq \kappa\}) = 0.$$

It is convenient to represent symbolically by $r\text{-}St_{(\tau,\upsilon,\eta)}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{r\text{-}St_{(\tau,\upsilon,\eta)}} s_0$.

DEFINITION 2.4. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called rough I-convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$ and $\kappa \in (0, 1)$ we have

$$\{k \in \mathbb{N} : \tau(s_k - s_0; r + \epsilon) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; r + \epsilon) \geq \kappa,\ \eta(s_k - s_0; r + \epsilon) \geq \kappa\} \in I.$$

It is convenient to represent symbolically by $r\text{-}I_{(\tau,\upsilon,\eta)}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{r-I_{(\tau,\upsilon,\eta)}} s_0$.

DEFINITION 2.5. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called rough I-convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$ and $\kappa \in (0, 1)$ we have

$$\{k \in \mathbb{N} : \tau(s_k - s_0; r + \epsilon) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; r + \epsilon) \geq \kappa,\ \eta(s_k - s_0; r + \epsilon) \geq \kappa\} \in I.$$

It is convenient to represent symbolically by $r\text{-}I(\tau, \upsilon, \eta)\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{r-I_{(\tau,\upsilon,\eta)}} s_0$.

DEFINITION 2.6. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is called rough I-statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$, $\delta > 0$ and $\kappa \in (0, 1)$ we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \tau(s_k - s_0; r + \epsilon) \leq 1 - \kappa \text{ or } \upsilon(s_k - s_0; r + \epsilon) \geq \kappa,\ \eta(s_k - s_0; r + \epsilon) \geq \kappa\}| \geq \delta\} \in I.$$

It is convenient to represent symbolically by $r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}\text{-}\lim_{k\to\infty} s_k = s_0$ or $s_k \xrightarrow{r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}} s_0$.

REMARK 2.7. For $r = 0$, the impression of rough I-statistical convergence about neutrosophic norm (τ, υ, η) agrees with the impression of I-statistical convergence about neutrosophic norm (τ, υ, η) in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$.

The $r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}$-limit of a sequence may be not unique. Hence, we take into consideration $r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}$-limit set of $s = \{s_k\}$ as $I\text{-}St\text{-}LIM_s^{r_{(\tau,\upsilon,\eta)}} = \{s_0 : s_k \xrightarrow{r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}} s_0\}$. For this the next example is given.

Example 2.8. Consider a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ as mentioned in Example 1.3. Take a non-trivial ideal I which is an admissible ideal. Consider infinite set $K \in I$ with $d_I(K) = 0$ and $d(K)$ does not exist. Define a sequence

$$s_k = \begin{cases} (-1)^k & k \notin K \\ k & \text{otherwise.} \end{cases}$$

Then

$$I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} = \begin{cases} \emptyset & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

If $s' = \{s_{j_k}\}$ be a sub-sequence of $s = \{s_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ then $LIM_{s_k}^{r(\tau,\upsilon,\eta)} \subset LIM_{s_{j_k}}^{r(\tau,\upsilon,\eta)}$, where $LIM_{s_k}^{r(\tau,\upsilon,\eta)}$ and $LIM_{s_{j_k}}^{r(\tau,\upsilon,\eta)}$ are rough limit sets of sequence $\{s_k\}$ and $\{s_{j_k}\}$, respectively. But this result fails to be hold in case of rough I-statistical convergence in NNS, which can be justified with the following given example.

Example 2.9. Consider a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ as mentioned in Example 1.3 and I as an admissible ideal. Take an infinite set $K = \{j_1 < j_2 < j_3 < \dots\} \in I$ with $d_I(K) = 0$ and $d(I)$ does not exist. Define sequence

$$s_k = \begin{cases} k & k \in K \\ 0 & \text{otherwise} \end{cases}$$

which has $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} = [-r, r]$ for $r > 0$. But sub-sequence $s' = \{s_{j_k}\}$, where $j_k \in K$ have $I\text{-}St\text{-}LIM_{s'}^{r(\tau,\upsilon,\eta)} = \emptyset$.

Theorem 2.10. *$I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ of any sequence $s = \{s_k\}$ from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ is a closed set.*

Proof. For $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ nothing has to be proved.
Assume $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} \neq \emptyset$ for some $r > 0$ and take $y = \{y_k\}$ any convergent sequence from $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ corresponding to norm (τ, υ, η) to $y_0 \in \mathbb{X}$.
For $\lambda \in (0,1)$ take $\kappa \in (0,1)$ with $(1-\kappa) \circledast (1-\kappa) > 1-\lambda$ and $\kappa \odot \kappa < \lambda$. Then for $\epsilon > 0$ and $\kappa \in (0,1)$ we get $k_1 \in \mathbb{N}$ with

$$\tau\left(y_k - y_0; \frac{\epsilon}{2}\right) > 1-\kappa, \upsilon\left(y_k - y_0; \frac{\epsilon}{2}\right) < \kappa \text{ and } \eta\left(y_k - y_0; \frac{\epsilon}{2}\right) < \kappa \text{ for all } \ k \geq k_1.$$

Let us choose $\delta > 0$ and $y_m \in I\text{-}St_{(\tau,\upsilon,\eta)}\text{-}LIM_s^r$ with $m > k_1$ such that set $A \in I$, where $A = \left\{n \in \mathbb{N} : \frac{1}{n}|R| \geq \delta\right\}$ with $R = \{k \in \mathbb{N} : \tau\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1 - \kappa \text{ or } \upsilon\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \geq \kappa, \eta\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \geq \kappa\}$. Since I is an admissible ideal then $M = \mathbb{N} - A$ becomes a non-empty set. Take $n \in M$ which gives

$$\frac{1}{n}|\{k \in \mathbb{N} : \tau\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1-\kappa, \upsilon\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \geq \kappa.\text{and}$$
$$\eta\left(s_k - y_m; r + \frac{\epsilon}{2}\right) \geq \kappa\}| < \delta.$$

i.e

$$\frac{1}{n}|\{k \in \mathbb{N} : \tau\left(s_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\kappa, \upsilon\left(s_k - y_m; r + \frac{\epsilon}{2}\right) < \kappa \text{ and}$$
$$\eta\left(s_k - y_m; r + \frac{\epsilon}{2}\right) < \kappa\}| \geq 1-\delta. \tag{1}$$

Consider $B_n = \{k \in \mathbb{N} : \tau\left(s_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\kappa, \upsilon\left(s_k - y_m; r + \frac{\epsilon}{2}\right) < \kappa$ and $\eta\left(s_k - y_m; r + \frac{\epsilon}{2}\right) < \kappa\}$. Then, for $j \in B_n$ we have

$$\begin{aligned}\tau(s_j - y_0; r+\epsilon) &\geq \tau\left(s_j - y_m; r + \frac{\epsilon}{2}\right) \circledast \tau\left(y_m - y_0; \frac{\epsilon}{2}\right)\\ &> (1-\kappa) \circledast (1-\kappa)\\ &> 1-\lambda,\\ \upsilon(s_j - y_0; r+\epsilon) &\leq \upsilon\left(s_j - y_m; r + \frac{\epsilon}{2}\right) \odot \upsilon\left(y_m - y_0; \frac{\epsilon}{2}\right)\\ &< \kappa \odot \kappa\\ &< \lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(s_j - y_0; r+\epsilon) &\leq \eta\left(s_j - y_m; r + \frac{\epsilon}{2}\right) \odot \eta\left(y_m - y_0; \frac{\epsilon}{2}\right)\\ &< \kappa \odot \kappa\\ &< \lambda.\end{aligned}$$

Hence, $j \in C_n = \{k \in \mathbb{N} : \tau(s_k - y_0; r+\epsilon) > 1-t,\ \upsilon\left(s_k - y_0; r+\epsilon\right) < t$ and $\eta\left(s_k - y_0; r+\epsilon\right) < t\}$. Now we get $B_n \subseteq C_n$ and using (1) we get $1-\delta \leq \frac{|B_n|}{n} \leq \frac{|C_n|}{n}$. Therefore,

$$\frac{1}{n}\left|\{k \leq n : \tau(s_k - y_0; r+\epsilon) \leq 1-t,\ \upsilon\left(s_k - y_0; r+\epsilon\right) \geq t \text{ and } \eta\left(s_k - y_0; r+\epsilon\right) \geq t\}\right| \geq \delta,$$

$$\begin{aligned}\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \tau(s_k - y_0; r+\epsilon) \leq 1-\lambda,\ \upsilon\left(s_k - y_0; r+\epsilon\right) \geq \lambda \text{ and}\\ \eta\left(s_k - y_0; r+\epsilon\right) \geq t| \geq \delta\} \in I.\end{aligned}$$

Therefore, $y_0 \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$. □

In following result, we are going to provide the convexity for set $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$.

THEOREM 2.11. *Let $s = \{s_k\}$ be any sequence from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then, $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ corresponding to neutrosophic norm (τ, υ, η) is convex for some $r \geq 0$.*

Proof. Let $\xi_1, \xi_2 \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$. For existence of convexity for $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$, we have to show that $[(1-\beta)\xi_1 + \beta\xi_2] \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ for $\beta \in (0,1)$.
For $\lambda \in (0,1)$ take $\kappa \in (0,1)$ with $(1-\kappa) \circledast (1-\kappa) > 1-\lambda$ and $\kappa \odot \kappa < \lambda$. For every $\epsilon > 0$ and $\kappa \in (0,1)$, we can define

$$\begin{aligned}M_1 = \{k \in \mathbb{N} : \tau\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \leq 1-\kappa \text{ or } \upsilon\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \geq \kappa,\\ \eta\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \geq \kappa\},\end{aligned}$$

$$\begin{aligned}M_2 = \{k \in \mathbb{N} : \tau\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \leq 1-\kappa \text{ or } \upsilon\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \geq \kappa,\\ \eta\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \geq \kappa\}.\end{aligned}$$

Then,

$$\frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \leq \frac{1}{n}\left|\{k \leq n : k \in M_1\}\right| + \frac{1}{n}\left|\{k \leq n : k \in M_2\}\right|.$$

Using fact of $I-$ convergence, for $\delta > 0$ we get $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : k \in M_1 \cup M_2\}| \geq \delta\right\} \in I$. Take $0 < \delta_1 < 1$ with $0 < 1 - \delta_1 < \delta$.
Let $M = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \geq \delta_1\right\}$ such that $M \in I$. For $n \notin M$

$$\frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| < 1 - \delta_1 \implies \frac{1}{n}\left|\{k \leq n : k \notin M_1 \cup M_2\}\right| \geq \delta_1.$$

Therefore, $\{k \in \mathbb{N} : k \notin M_1 \cup M_2\} \neq \emptyset$. For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}
\tau(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) &\geq \tau((1-\beta)(s_k - \xi_1) + \beta(s_k - \xi_2); r + \epsilon) \\
&\geq \tau\left((1-\beta)(s_k - \xi_1); \frac{r+\epsilon}{2}\right) \circledast \tau\left(\beta(s_k - \xi_2); \frac{r+\epsilon}{2}\right) \\
&\geq \tau\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \circledast \tau\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \\
&> (1-\kappa) \circledast (1-\kappa) \\
&> 1 - \lambda,
\end{aligned}$$

$$\begin{aligned}
\upsilon(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) &\leq \upsilon((1-\beta)(s_k - \xi_1) + \beta(s_k - \xi_2); r + \epsilon) \\
&\leq \upsilon\left((1-\beta)(s_k - \xi_1); \frac{r+\epsilon}{2}\right) \odot \upsilon\left(\beta(s_k - \xi_2); \frac{r+\epsilon}{2}\right) \\
&\leq \upsilon\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \odot \upsilon\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \\
&< \kappa \odot \kappa \\
&< \lambda,
\end{aligned}$$

and

$$\begin{aligned}
\eta(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) &\leq \eta((1-\beta)(s_k - \xi_1) + \beta(s_k - \xi_2); r + \epsilon) \\
&\leq \eta\left((1-\beta)(s_k - \xi_1); \frac{r+\epsilon}{2}\right) \odot \eta\left(\beta(s_k - \xi_2); \frac{r+\epsilon}{2}\right) \\
&\leq \eta\left(s_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \odot \eta\left(s_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \\
&< \kappa \odot \kappa \\
&< \lambda.
\end{aligned}$$

Clearly $M_1^c \cap M_2^c \subseteq P^c$, where $P = \{k \in \mathbb{N} : \tau(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) \leq 1 - \lambda$ or $\upsilon(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) \geq \lambda$, $\eta(s_k - [(1-\beta)\xi_1 + \beta\xi_2]; r + \epsilon) \geq \lambda\}$. Thus, if $n \notin M$ then we have $\delta_1 \leq \frac{|M_1^c \cap M_2^c|}{n} \leq \frac{|P^c|}{n}$ *i.e* $\frac{1}{n}\left|\{k \leq n : k \in P\}\right| < 1 - \delta_1 < \delta$.
Thus, $M^c \subseteq \{k \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in P\}\right| < \delta\}$, since $M \in I$ then $\{k \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in P\}\right| < \delta\} \in F(I)$ so $\{k \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in P\}\right| \geq \delta\} \in I$.
Hence, $[(1-\beta)\xi_1 + \beta\xi_2] \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ *i.e* $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ becomes a convex set. □

THEOREM 2.12. *A sequence $s = \{s_k\}$ from NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ is rough I-statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if there exists a sequence $y = \{y_k\} \in \mathbb{X}$, which is I-statistically convergent to $s_0 \in \mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) and for $\kappa \in (0,1)$ and for $k \in \mathbb{N}$ satisfies $\tau(s_k - y_k; r) > 1 - \kappa$, $\upsilon(s_k - y_k; r) < \kappa$ and $\eta(s_k - y_k; r) < \kappa$.*

Proof. Consider $y_k \xrightarrow{I-St_{(\tau,\upsilon,\eta)}} s_0$ and $\tau(s_k - y_k; r) < \kappa$, $\upsilon(s_k - y_k; r) > 1 - \kappa$ and $\eta(s_k - y_k; r) > 1 - \kappa$ for all $k \in \mathbb{N}$. Let $\epsilon > 0$, and for given $\kappa \in (0,1)$ take $\lambda \in (0,1)$ with $(1-\lambda) \circledast (1-\lambda) > 1-\kappa$ and $\kappa \odot \kappa < \lambda$. Now we have $\tau(s_k - y_k; r) \leq 1-\lambda$ or $\upsilon(s_k - y_k; r) \geq \lambda$, $\eta(s_k - y_k; r) \geq \lambda$ for all $k \in \mathbb{N}$.
Define

$$A = \{k \in \mathbb{N} : \tau(y_k - s_0; \epsilon) \leq 1-\lambda \text{ or } \upsilon(y_k - s_0; \epsilon) \geq \lambda,\ \eta(y_k - s_0; \epsilon) \geq \lambda\}.$$

Clearly, for $\delta > 0$ we have $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : k \in A\}| \geq \delta\right\} \in I$. Also consider

$$B = \{k \in \mathbb{N} : \tau(s_k - y_k; r) \leq 1-\lambda \text{ or } \upsilon(s_k - y_k; r) \geq \lambda,\ \eta(s_k - y_k; r) \geq \lambda\}.$$

For $k \in B^c$,

$$\begin{aligned}\tau(s_k - s_0; r+\epsilon) &\geq \tau(s_k - y_k; r) \circledast \tau(y_k - s_0; \epsilon)\\ &> (1-\lambda) \circledast (1-\lambda)\\ &> 1-\kappa,\\ \upsilon(s_k - s_0; r+\epsilon) &\leq \upsilon(s_k - y_k; r) \odot \upsilon(y_k - s_0; \epsilon)\\ &< \lambda \odot \lambda\\ &< \kappa,\end{aligned}$$

and

$$\begin{aligned}\eta(s_k - s_0; r+\epsilon) &\leq \eta(s_k - y_k; r) \odot \eta(y_k - s_0; \epsilon)\\ &< \lambda \odot \lambda\\ &< \kappa.\end{aligned}$$

Therefore $\tau(s_k - s_0; r+\epsilon) > 1-\kappa$, $\upsilon(s_k - s_0; r+\epsilon) < \kappa$ and $\eta(s_k - s_0; r+\epsilon) < \kappa$ whenever $k \in B^c$.
Thus $\{k \in \mathbb{N} : \tau(s_k - s_0; r+\epsilon) \leq 1-\kappa \text{ or } \upsilon(s_k - s_0; r+\epsilon) \geq \kappa,\ \eta(s_k - s_0; r+\epsilon) \geq \kappa\} \subseteq B$.
Then,

$$\frac{1}{n}|\{k \leq n : \tau(s_k - s_0; r+\epsilon) \leq 1-\kappa \text{ or } \upsilon(s_k - s_0; r+\epsilon) \geq \kappa,\ \eta(s_k - s_0; r+\epsilon) \geq \kappa\}| \leq \frac{1}{n}|B|.$$

Further

$$\frac{1}{n}|\{k \leq n : \tau(s_k - s_0; r+\epsilon) \leq 1-\kappa \text{ or } \upsilon(s_k - s_0; r+\epsilon) \geq \kappa,\ \eta(s_k - s_0; r+\epsilon) \geq \kappa\}| \leq \delta.$$

Hence,

$$\{k \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \tau(s_k - s_0; r+\epsilon) \leq 1-\kappa \text{ or } \upsilon(s_k - s_0; r+\epsilon) \geq \kappa, \eta(s_k - s_0; r+\epsilon) \geq \kappa\}| \leq \delta\} \in I.$$

Therefore, $s_k \xrightarrow{r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}} s_0$. □

THEOREM 2.13. *Let $s = \{s_k\}$ be any sequence from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. There does not exist elements $a, b \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ for some $r > 0$ and every $\kappa \in (0,1)$ with $\tau(a-b; mr) \leq 1-\kappa$, $\upsilon(a-b; mr) \geq \kappa$ or $\eta(a-b; mr) \geq \kappa$ for $m > 2$.*

Proof. Let us assume the contrary. Suppose elements $a, b \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ exists with

$$\tau(a-b; mr) \leq 1-\kappa \text{ or } \upsilon(a-b; mr) \geq \kappa,\ \eta(a-b; mr) \geq \kappa \text{ for } m > 2. \quad (2)$$

For given $\kappa \in (0,1)$ take $\lambda \in (0,1)$ with $(1-\lambda) \circledast (1-\lambda) > 1-\kappa$ and $\kappa \odot \kappa < \lambda$. Then for every $\epsilon > 0$ and $\lambda \in (0,1)$. Consider $M_1 = \left\{k \in \mathbb{N} : \tau\left(s_k - a; r + \frac{\epsilon}{2}\right) \leq 1 - \right.$

λ or $\upsilon\left(s_k - a; r + \frac{\epsilon}{2}\right) \geq \lambda$, $\eta\left(s_k - a; r + \frac{\epsilon}{2}\right) \geq \lambda\}$ and $M_2 = \{k \in \mathbb{N} : \tau\left(s_k - b; r + \frac{\epsilon}{2}\right) \leq 1 - \lambda$ or $\upsilon\left(s_k - b; r + \frac{\epsilon}{2}\right) \geq \lambda, \eta\left(s_k - b; r + \frac{\epsilon}{2}\right) \geq \lambda\}$. Then

$$\frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \leq \frac{1}{n}\left|\{k \leq n : k \in M_1\}\right| + \frac{1}{n}\left|\{k \leq n : k \in M_2\}\right|.$$

Using characteristic of $I-$ convergence, for $\delta > 0$ we get

$$\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : k \in M_1 \cup M_2\}\right| \geq \delta\} \in I.$$

For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}\tau(a-b; 2r+\epsilon) &\geq \tau\left(s_k - b; r + \frac{\epsilon}{2}\right) \circledast \tau\left(s_k - a; r + \frac{\epsilon}{2}\right)\\ &> (1-\lambda) \circledast (1-\lambda)\\ &> 1-\kappa,\\ \upsilon(a-b; 2r+\epsilon) &\leq \upsilon\left(s_k - b; r + \frac{\epsilon}{2}\right) \odot \upsilon\left(s_k - a; r + \frac{\epsilon}{2}\right)\\ &< \lambda \odot \lambda\\ &< \kappa,\end{aligned}$$

and

$$\begin{aligned}\eta(a-b; 2r+\epsilon) &\leq \eta\left(s_k - z; r + \frac{\epsilon}{2}\right) \odot \eta\left(s_k - a; r + \frac{\epsilon}{2}\right)\\ &< \lambda \odot \lambda\\ &< \kappa.\end{aligned}$$

Hence,

$$\tau(a-b; 2r+\epsilon) > 1-\kappa,\ \upsilon(a-b; 2r+\epsilon) < \kappa \text{ and } \eta(a-b; 2r+\epsilon) < \kappa. \tag{3}$$

Thus, from (3)

$$\tau(a-b; mr) > 1-\kappa,\ \upsilon(a-b; mr) < \kappa \text{ and } \eta(a-b; mr) < \kappa \text{ for } m > 2.$$

which leads to contradiction to (2).
Thus, elements $a, b \notin I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$ such that $\tau(a-b; mr) \leq 1-\kappa$ or $\upsilon(a-b; mr) \geq \kappa$, $\eta(a-b; mr) \geq \kappa$ for $m > 2$. □

Further, we are mentioning I-statistically bounded sequence on NNS as follows:

Definition 2.14. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. A sequence $s = \{s_k\}$ from $\mathbb{X}$ is said to be I-statistically bounded with neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$, $\delta > 0$ and $\kappa \in (0,1)$ we can find a real number $M > 0$ satisfying

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \tau(s_k; M) \leq 1-\kappa \text{ or } \upsilon(s_k; M) \geq \kappa,\ \eta(s_k; M) \geq \kappa\}\right| \geq \delta\right\} \in I.$$

In view of the above definition, we got next result on rough I-statistical convergence on NNS.

Theorem 2.15. *Let $s = \{s_k\}$ be any sequence from NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then $s = \{s_k\}$ is I-statistically bounded with neutrosophic norm (τ, υ, η) if and only if $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} \neq \emptyset$ for some $r > 0$.*

Proof. Necessary part:
Consider sequence $s = \{s_k\}$ which is I-statistically bounded corresponding to neutrosophic norm (τ, υ, η) in NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then, for every $\epsilon > 0$, $\kappa \in (0,1)$ and some $r > 0$ we get a real number $M > 0$ satisfying

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \tau(s_k; M) \leq 1-\kappa \text{ or } \upsilon(s_k; M) \geq \kappa,\ \eta(s_k; M) \geq \kappa\}| \geq \delta\right\} \in I.$$

Assume $K = \{k \in \mathbb{N} : \tau(s_k; M) \leq 1-\kappa \text{ or } \upsilon(s_k; M) \geq \kappa,\ \eta(s_k, M) \geq \kappa\}$.
For any $k \in K^c$ we get $\tau(s_k; M) > 1-\kappa$, $\upsilon(s_k, M) < \kappa$ and $\eta(s_k, M) < \kappa$.
Also

$$\begin{aligned} \tau(s_k; r+M) &\geq \tau(0; r) \circledast \tau(s_k; M) \\ &> 1 \circledast (1-\kappa) \\ &= 1-\kappa, \end{aligned}$$

$$\begin{aligned} \upsilon(s_k; r+M) &< \upsilon(0; r) \odot \upsilon(s_k; M) \\ &< 0 \odot \kappa \\ &= \kappa, \end{aligned}$$

and

$$\begin{aligned} \eta(s_k; r+M) &< \eta(0; r) \odot \eta(s_k; M) \\ &< 0 \odot \kappa \\ &= \kappa. \end{aligned}$$

Hence, $0 \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$. Therefore, $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} \neq \emptyset$.

Sufficient Part:
Let $I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)} \neq \emptyset$ for some $r > 0$. There exists $s_0 \in \mathbb{X}$ such that $s_0 \in I\text{-}St\text{-}LIM_s^{r(\tau,\upsilon,\eta)}$. For every $\epsilon > 0$, δ and $\kappa \in (0,1)$ we have

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \tau(s_k - s_0; r+\epsilon) \leq 1-\kappa \,\text{or}\, \upsilon(s_k - s_0; r+\epsilon) \geq \kappa, \eta(s_k - s_0; r+\epsilon) \geq \kappa\}| \geq \delta\} \in I.$$

Consequently, most of the s_k's are part of some ball having center at s_0 which implies that $s = \{s_k\}$ is I-statistically bounded in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. □

Further, we define I-statistical cluster point for the sequence on NNS and establish some related results.

Definition 2.16. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η) and I be an admissible ideal. An element $\gamma \in \mathbb{X}$ is called rough I-statistical cluster point of the sequence $s = \{s_k\}$ in $\mathbb{X}$ corresponding to neutrosophic norm (τ, υ, η) for some $r \geq 0$ if for every $\epsilon > 0$, $\delta > 0$ and $\kappa \in (0,1)$, we have $d_I(\{k \in \mathbb{N} : \tau(s_k - \gamma; r+\epsilon) > 1-\kappa$ and $\upsilon(s_k - \gamma; r+\epsilon) < \kappa,\ \eta(s_k - \gamma; r+\epsilon) < \kappa\} \neq 0$, where $d_I(\{A\})$ denotes the I-asymptotic density of the set $\{A\}$ *i.e* $d_I(\{A\}) = I - \lim_{n\to\infty} \frac{1}{n} \mid A \mid$.
In this case, γ represents $r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}$-cluster point of a sequence $s = \{s_k\}$.

Let $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$ represents the set of all $r\text{-}I\text{-}St_{(\tau,\upsilon,\eta)}$-cluster points corresponding to neutrosophic norm (τ, υ, η) of sequence $s = \{s_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. If $r = 0$ then we get ordinary I-statistical cluster point corresponding to neutrosophic norm (τ, υ, η) from NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ *i.e* $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s) = \Gamma^{I}_{(\tau,\upsilon,\eta)}(s)$.

Theorem 2.17. *The set $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$ of all r-$St_{(\tau,\upsilon,\eta)}$-cluster points about neutrosophic norm (τ,υ,η) of any sequence $s=\{s_k\}$ form $\mathbb{X}$, is closed for some $r\geq 0$.*

Proof. (i) For $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)=\emptyset$, nothing has to be proved.

(ii) For $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)\neq\emptyset$ consider a sequence $y=\{y_k\}$ from $\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$ such that $y_k \xrightarrow{(\tau,\upsilon,\eta)} y_*$. Here, sufficient to establish $y_*\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$. Now for $\lambda\in(0,1)$ take $\kappa\in(0,1)$ with $(1-\kappa)\circledast(1-\kappa)>(1-\lambda)$ and $\kappa\odot\kappa<\lambda$.

As $y_k \xrightarrow{(\tau,\upsilon,\eta)} y_*$, then for every $\epsilon>0$ and $\kappa\in(0,1)$ we get $k_\epsilon\in\mathbb{N}$ with $\tau\left(y_k-y_*;\frac{\epsilon}{2}\right)>1-\kappa$, $\upsilon\left(y_k-y_*;\frac{\epsilon}{2}\right)<\kappa$ and $\eta\left(y_k-y_*;\frac{\epsilon}{2}\right)<\kappa$ for all $k\geq k_\epsilon$.

Now choose $k_0\in\mathbb{N}$ such that $k_0\geq k_\epsilon$. Then, we have $\tau\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)>1-\kappa$, $\upsilon\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)<\kappa$ and $\eta\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)<\kappa$. Again as $y=\{y_k\}\subseteq\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$, we have $y_{k_0}\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$. Then $R=\{k\in\mathbb{N}:\tau\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\kappa,\ \upsilon\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa$ and $\eta\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa\}$ with

$$d_I(R)>0. \tag{4}$$

Choose $j\in\{k\in\mathbb{N}:\tau\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\kappa,\ \upsilon\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa$ and $\eta(s_k-y_{k_0};r+\frac{\epsilon}{2})<\kappa\}$, then we have $\tau\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\kappa$, $\upsilon\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa$ and $\eta\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa$.

$$\begin{aligned}\tau(s_j-y_*;r+\epsilon)&\geq\tau\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)\circledast\tau\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)\\&>(1-\kappa)\circledast(1-\kappa)\\&>1-\lambda,\\ \upsilon(s_j-y_*;r+\epsilon)&\geq\upsilon\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)\odot\upsilon\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)\\&<\kappa\odot\kappa\\&<\lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(s_j-y_*;r+\epsilon)&\geq\eta\left(s_j-y_{k_0};r+\frac{\epsilon}{2}\right)\odot\eta\left(y_{k_0}-y_*;\frac{\epsilon}{2}\right)\\&<\kappa\odot\kappa\\&<\lambda.\end{aligned}$$

Thus, $j\in\{k\in\mathbb{N}:\tau(s_k-y_*;r+\epsilon)>1-t,\ \upsilon(s_k-y_*;r+\epsilon)<t$ and $\eta(s_k-y_*;r+\epsilon)<t\}$. Hence

$$\begin{aligned}&\{k\in\mathbb{N}:\tau\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\kappa,\ \upsilon\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa \text{ and } \eta\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa\}\\&\quad\subseteq\{k\in\mathbb{N}:\tau(s_k-y_*;r+\epsilon)>1-t\lambda,\ \upsilon(s_k-y_*;r+\epsilon)<\lambda \text{ and } \eta(s_k-y_*;r+\epsilon)<\lambda\}.\end{aligned}$$

Now,

(5)

$$\begin{aligned}&d_I(\{k\in\mathbb{N}:\tau\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)>1-\kappa,\ \upsilon\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa \text{ and } \eta\left(s_k-y_{k_0};r+\frac{\epsilon}{2}\right)<\kappa\})\\&\quad\leq d_I(\{k\in\mathbb{N}:\tau(s_k-y_*;r+\epsilon)>1-\lambda,\ \upsilon(s_k-y_*;r+\epsilon)<\lambda \text{ and } \eta(s_k-y_*;r+\epsilon)<\lambda\}).\end{aligned}$$

Then using equations (4) and (5), we get

$d_I(\{k\in\mathbb{N}:\tau(s_k-y_*;r+\epsilon)>1-\lambda,\ \upsilon(s_k-y_*;r+\epsilon)<\lambda$ and $\eta(s_k-y_*;r+\epsilon)<\lambda\})>0$.

Therefore, $y_*\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$. □

THEOREM 2.18. *Let $\Gamma^{I}_{(\tau,\upsilon,\eta)}(s)$ be a collection of all I-statistical cluster points corresponding to neutrosophic norm (τ,υ,η) of the sequence $s=\{s_k\}$ from $\mathbb{X}$ and r be some non-negative real number. Then, for an arbitrary $\gamma\in\Gamma_{(\tau,\upsilon,\eta)}(s)$ and $\kappa\in(0,1)$ we have $\tau(s_0-\gamma;r)>1-\kappa$, $\upsilon(s_0-\gamma;r)<\kappa$ and $\eta(s_0-\gamma;r)<\kappa$ for all $s_0\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$.*

Proof. For $\kappa\in(0,1)$ take $\lambda\in(0,1)$ with $(1-\lambda)\circledast(1-\lambda)>1-\kappa$ and $\lambda\odot\lambda<\kappa$. If $\gamma\in\Gamma^{I}_{(\tau,\upsilon,\eta)}(s)$ then for $\epsilon>0$ and $\lambda\in(0,1)$ we get

(6) $d_I(\{k\in\mathbb{N}:\tau(s_k-\gamma;\epsilon)>1-\lambda,\ \upsilon(s_k-\gamma;\epsilon)<\lambda \text{ and } \eta(s_k-\gamma;\epsilon)<\lambda\})>0.$

Next we will show that if $\xi\in\mathbb{X}$ have $\tau(\xi-\gamma;r)>1-\lambda$, $\upsilon(\xi-\gamma;r)<\lambda$ and $\eta(\xi-\gamma;r)<\lambda$ then $\xi\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$.
Take $j\in\{k\in\mathbb{N}:\tau(s_k-\gamma;\epsilon)>1-\lambda,\ \upsilon(s_k-\gamma;\epsilon)<\lambda \text{ and } \eta(s_k-\gamma;\epsilon)<\lambda\}$, then $\tau(s_j-\gamma;\epsilon)>1-\lambda$, $\upsilon(s_j-\gamma;\epsilon)<\lambda$ and $\eta(s_j-\gamma;\epsilon)<\lambda$. Now,

$$\begin{aligned}\tau(s_j-\xi;r+\epsilon)&\geq\tau(s_j-\gamma;\epsilon)\circledast\tau(\xi-\gamma;r)\\&>(1-\lambda)\circledast(1-\lambda)\\&>1-\kappa,\\ \upsilon(s_j-\xi;r+\epsilon)&\leq\upsilon(s_j-\gamma;\epsilon)\odot\upsilon(\xi-\gamma;r)\\&<\lambda\odot\lambda\\&<\kappa,\end{aligned}$$

and

$$\begin{aligned}\eta(s_j-\xi;r+\epsilon)&\leq\eta(s_j-\gamma;\epsilon)\odot\eta(\xi-\gamma;r)\\&<\lambda\odot\lambda\\&<\kappa.\end{aligned}$$

Thus $j\in\{k\in\mathbb{N}:\tau(s_k-\xi;r+\epsilon)>1-\kappa,\ \upsilon(s_k-\xi;\epsilon)<\kappa \text{ and } \eta(s_k-\xi;\epsilon)<\kappa\}$. Consequently,

$$\begin{aligned}&\{k\in\mathbb{N}:\tau(s_k-\gamma;\epsilon)>1-\lambda,\ \upsilon(s_k-\gamma;\epsilon)<\lambda \text{ and } \eta(s_k-\gamma;\epsilon)<\lambda\}\\ \subseteq&\{k\in\mathbb{N}:\tau(s_k-\xi;r+\epsilon)>1-\kappa,\ \upsilon(s_k-\xi;r+\epsilon)<\kappa \text{ and } \eta(s_k-\xi;r+\epsilon)<\kappa\}.\end{aligned}$$

Then

$$\begin{aligned}&d_I(\{k\in\mathbb{N}:\tau(s_k-\gamma;\epsilon)>1-\lambda,\ \upsilon(s_k-\gamma;\epsilon)<\lambda \text{ and } \eta(s_k-\gamma;\epsilon)<\lambda\})\\ \leq&\, d_I(\{k\in\mathbb{N}:\tau(s_k-\xi;r+\epsilon)>1-\kappa,\ \upsilon(s_k-\xi;\epsilon)<t \text{ and } \eta(s_k-s_0;r+\epsilon)<\kappa\}).\end{aligned}$$

By equation (6), $d_I(\{k\in\mathbb{N}:\tau(s_k-\xi;r+\epsilon)>1-\kappa,\ \upsilon(s_k-\xi;r+\epsilon)<\kappa$ and $\eta(s_k-\xi;r+\epsilon)<\kappa\})>0$. Therefore, $\xi\in\Gamma^{r\text{-}I}_{(\tau,\upsilon,\eta)}(s)$. □

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