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GENERALIZED CROSSED MODULES OVER GENERALIZED GROUP-GROUPOIDS

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Abstract. In this paper we define generalized double group-groupoids and crossed modules over generalized group-groupoids. Also we prove that these algebraic structures are categorically equivalent.

1. Introduction

A crossed module defined by Whitehead in [25,26] can be viewed as a 2-dimensional group [6] and is widely used in homotopy theory [10], the theory of identities among relations for group presentations [11], algebraic K-theory [16], and homological algebra [15,17]. See [10] for a discussion of the relation of crossed modules to crossed squares and so to homotopy 3-types.

In [8, Theorem 1] Brown and Spencer proved that there exists a categorical equivalence between the categories crossed modules and group-groupoids which are widely called in literature *2-groups* [4]. Following this equivalence normal and quotient objects in these two categories have been recently compared and associated objects in the category of group-groupoids have been characterized in [22]. In Porter [23], this categorical equivalence has been extended to internal groupoids in the category of groups with operations and in [2] this result is exemplified. Also monodromy groupoids for internal groupoids in the category of groups with operations are studied in [21]. A double groupoid introduced by Ehresmann in [12] is a groupoid object in the category of groupoids. In mathematical phyisics, double categories have been used as an application of categorical methods to deeper understanding of genuine features of some problems. In [24], a new categorical equivalence between crossed modules over group-groupoids and double group- groupoids is given.

The notion of generalized group is introduced by Molaei [19]. The interesting aspect of this definition is that while there is a unique identity element in a group, each element in the generalized group has a unique identity element. Many studies have been done on generalized groups in various fields of mathematics [1,3,18,20]. Combining this definition with groupoids was accomplished in [14] by giving the definitions of crossed modules over generalized groups and generalized group-groupoids. Also the categorical equivalence of these two notions is given in [14].

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In this paper, we generalize the structure given in [24] to generalized double group-groupoids. We first define crossed modules over generalized group-groupoids and generalized double group-groupoids. These definitions are reinforced with examples. Also we prove that these types of crossed modules and generalized double group-groupoids are equivalent, which enables us to have some varieties of examples for double groupoids.

2. Preliminaries

This section is formed to give basic concepts that will be used in later sections. In this section we give the definition of generalized group and some of its properties. Also we remind the notion of groupoid, generalized group-groupoid and generalized crossed module over generalized groups.

Definition 1. [19] A generalized group G is a nonempty set admitting an operation $G \times G, (x, y) \mapsto xy$ called multiplication which satisfies the following conditions:

(i) $x(yz) = (xy)z$, for all $x, y, z \in G$,
(ii) For each $x \in G$, there exists a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$,
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$.

The following lemma gives us some properties of the generalized groups.

Lemma 1. [19] *If G is a generalized group, then the followings are satisfied:*

(i) For each $x \in G$, there is a unique element $x^{-1} \in G$,
(ii) For each $x \in X$ we have $e(x) = e(x^{-1})$ and $(e(e(x)) = e(a)$,
(iii) For each $x \in G$, we have $(x^{-1})^{-1} = x$.

By the Definition 1, we can easily see that every group is a generalized group. But it is not true in general that every generalized group is a group.

Example 1. The set $G = \mathbf{R} \times (\mathbf{R} - \{0\})$ with the multiplication $(a, b)(c, d) = (bc, bd)$ is a generalized group. But G is not a group.

Lemma 2. [20] *Let G be a generalized group such that $xy = yx$ for all $x, y \in G$. Then G is a group.*

Example 2. [13] Let G be a generalized group with the multiplication m. Then $G \times G$ becomes a generalized group with the multiplication

$$m^*((x, y), (z, t)) = (m(x, y), m(z, t))$$

for $x, y, z, t \in G$. Here the identity element $e^*(x, y) = (e(x), e(y))$ and the inverse $(x, y)^{-1} = (x^{-1}, y^{-1})$ for $(x, y) \in G \times G$.

Definition 2. [18] A generalized group is called a *normal generalized group if $e(xy) = e(x)e(y)$ for all $x, y \in G$.*

Definition 3. [18] A generalized group homomorphism $f : G \to H$ is a map such that $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Definition 4. [18] Let G be a generalized group and H be a nonempty subset of G. Then H is is called *generalized subgroup if and only if for all $x, y \in H$, $xy^{-1} \in H$.*

Definition 5. [18] Let G be a generalized group and N be a generalized subgroup of G. If there exists a generalized group H and a homomorphism $f\colon G \to H$ such that for all $x \in G$, $N_x = \emptyset$ or $N_x = Kerf_x$, where $N_x = N \cap G_x$, $G_x = \{y \in G | e(x) = e(y)\}$ and $f_x = f|_{G_x}$, then N is called a *generalized normal subgroup.*

Theorem 1. [18] *Let $f\colon G \to H$ be a homomorphism of the distinct generalized groups G and H. Then the followings are obtained:*
(i) $f(e(x)) = e(f(x))$ is an identity element in H for all $x \in G$,
(ii) $f(x^{-1}) = (f(x))^{-1}$
(iii) If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H

A generalized action of generalized group G on generalized group H is given in [14] as follows:

Definition 6. Let G and H be two generalized groups. Then a generalized action of the generalized group G on H is a function

$$\cdot\colon G \times H \to H, (x, h) \mapsto x \cdot h$$

such that the following conditions are satisfied:
(i) $(x_1x_2) \cdot h = x_1 \cdot (x_2 \cdot h)$, for every $x_1, x_2 \in G$ and $h \in H$,
(ii) $x \cdot (h_1h_2) = (x \cdot h_1)(x \cdot h_2)$, for every $x \in G$ and $h_1, h_2 \in H$,
(iii) For every $h \in H$, there exists an element $e(x) \in G$ such that $e(x) \cdot h = h$,
(iv) $x \cdot e(h) = e(h)$ for every $x \in G$ and $h \in H$.

Proposition 1. [14] *The semidirect product of two generalized groups is also a generalized group.*

Let us give the definition of a groupoid.

A *groupoid* is a (small) category in which each morphism is an isomorphism [5, p.205]. Hence a groupoid G has the following:

- A set of arrows denoted by G,
- A set $\mathsf{Ob}(G)$ of **objects**,
- **Source** and **target** point maps $\partial_0, \partial_1\colon G \to G_0$
- **Object inclusion** map $\epsilon\colon G_0 \to G$ such that $s\epsilon = t\epsilon = 1_{G_0}$.
- There exists an associative partial composition defined by $G_t \times_s G \to G$, $(g, h) \mapsto g \circ h$, where $G_{\partial_1} \times_{\partial_0} G$ is the pullback of ∂_1 and ∂_0 and $\partial_0(g \circ h) = \partial_0(g)$, $\partial_1(g \circ h) = \partial_1(h)$,
- For $x \in G_0$ there is an identity denoted by 1_x,
- Each arrow g has an inverse g^{-1} such that $\partial_0(g^{-1}) = \partial_1(g)$, $t(g^{-1}) = s(g)$, $g \circ g^{-1} = \epsilon(\partial_0(g))$, $g^{-1} \circ g = \epsilon(\partial_1((g))$. The map $G \to G$, $g \mapsto g^{-1}$ is called the **inversion**.

In a groupoid G, the source and target points, the object inclusion, the inversion maps and the partial composition are called **structural maps**.

In a groupoid G for $x, y \in G_0$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *transitive* if for all $x, y \in \mathsf{Ob}(G)$, the set $G(x, y)$ is not empty. For $x \in G_0$ we denote the star $\{g \in G \mid \partial_0(g) = x\}$ of x by G_x.

A **group-groupoid** which is also knows as “2-group” is an internal category in the category of groups [7, 8]. In a group-groupoid G as convention, ba represents the product in group while $b \circ a$ the composite in groupoid for $a, b \in G$ with $\partial_0(b) = \partial_1(a)$; and a^{-1} is the inverse of a in group and $\bar{a}$ is that in groupoid.

We remind that for a group-groupoid G the interchange rule

$$(b \circ a)(d \circ c) = (bd) \circ (ac)$$

holds for $a, b, c, d \in G$ whenever the composites $b \circ a$ and $d \circ c$ are defined. Moreover $(b \circ a)^{-1} = b^{-1} \circ a^{-1}$ and $1_x{}^{-1} = 1_{x^{-1}}$ for $x \in G_0$.

In [14], the notion of generalized group-groupoid, which is a generalized group object in the category of groupoids, is defined as follows:

Definition 7. A generalized group-groupoid is a groupoid (G, G_0) such that the following conditions hold:

(i) (G, m, υ, n) and $(G_0, m_0, \upsilon_0, n_0)$ are generalized groups,
(ii) The maps $(m, m_0)\colon (G\times G, G_0\times G_0) \to (G, G_0)$, $\upsilon\colon \{\lambda\} \to G$ and $(n, n_0)\colon (G, G_0) \to (G, G_0)$ are groupoid homomorphisms.

Also the interchange law

$$(b \circ a)(d \circ c) = (bd) \circ (ac)$$

holds. Such a generalized group-groupoid is denoted by $(G, \partial_0, \partial_1, \circ, \varepsilon, \iota, m, G_0)$.

A generalized group-groupoid morphism $f\colon G \to H$ is a morphism of underlying groupoids preserving the generalized group structure.

In the following definition, a generalized crossed module which is a generalized of the crossed module over groups is given.

Definition 8. A generalized crossed module (M, P, α) consists of two generalized group M and P together with a generalized group homomorphisms $\alpha\colon M \to P$ and a generalized action of P and M,

$$P \times M \to M, (p, m) \mapsto p \cdot m$$

such that the followings are satisfied:

(1) $\alpha(p \cdot m) = p\alpha(m)p$, for all $m \in M$ and $p \in P$
(2) $\alpha(m)m_1 = mm_1m^{-1}$ for all $m, m_1 \in M$.

Let (M, P, α) and (M', P', α') be two generalized crossed modules over generalized group. A morphism $(f_1, f_2)\colon (M, P, \alpha) \to (M', P', \alpha')$ between generalized crossed modules is defined to be a pair of generalized group morphisms $f_1\colon M \to M'$ and $f_2\colon P \to P'$ such that (f_1, f_2) is a morphism of generalized crossed modules over generalized groups.Then we obtain the category $\mathsf{XMod}(GGp)$ of generalized crossed modules over generalized group groupoids and their morphisms. Let (M, P, α) be a generalized crossed module such that P is an abelian generalized group and A is a normal generalized group. Then let us denote the full subcategory as $\mathsf{XMod}(GGp)^*$. On the other hand, let the category of generalized group-groupoids whose object sets are abelian be $GGp/_{Ab}$.

Theorem 2. [14] *The categories* $\mathsf{XMod}(GGp)^*$ *and* $GGp/_{Ab}$ *are equivalent.*

3. Generalized Crossed Modules

In this section we define the concept of crossed module over generalized group-groupoids.

LEMMA 3. *Let M and P be two generalized group-groupoids. If the generalized group P acts on M, then there is a generalized action of P_0 on M_0 defined by*

$$y \cdot x = d_0(\varepsilon(y \cdot x)) = d_0(\varepsilon(y) \cdot \varepsilon(x)),$$

for all $x \in M_0$ and $y \in P_0$.

Proof. Let us prove that the conditions of the generalized action hold.

(i) For $y_1, y_2 \in P_0$ and $x \in M_0$,

$$\begin{aligned}(y_1y_2) \cdot x &= d_0(\varepsilon((y_1y_2) \cdot x) \\ &= d_0(\varepsilon((y_1y_2) \cdot \varepsilon(x)) \\ &= d_0(\varepsilon((y_1)\varepsilon(y_2)) \cdot \varepsilon(x) \\ &= d_0(\varepsilon(y_1)\varepsilon((y_2) \cdot \varepsilon(x)) \\ &= d_0(\varepsilon(y_1)\varepsilon((y_2 \cdot x)) \\ &= y_1 \cdot (y_2 \cdot x).\end{aligned}$$

(ii) For $y \in P_0$ and $x_1, x_2 \in M_0$,

$$\begin{aligned}y \cdot (x_1x_2) &= d_0(\varepsilon((y \cdot (x_1 \cdot x_2)) \\ &= d_0(\varepsilon(y) \cdot \varepsilon(x_1 \cdot x_2)) \\ &= d_0(\varepsilon(y) \cdot \varepsilon(x_1) \cdot \varepsilon(x_2)) \\ &= d_0(\varepsilon(y) \cdot \varepsilon(x_1)(\varepsilon(y) \cdot \varepsilon(x_2)) \\ &= d_0(\varepsilon(y \cdot (x_1))d_0(\varepsilon(y \cdot x_2)) \\ &= (y \cdot x_1)(y \cdot x_2).\end{aligned}$$

(iii) For $y \in P_0$ and $x \in M_0$,

$$\begin{aligned}e(y) \cdot x &= d_0(\varepsilon(e(y) \cdot x)) \\ &= d_0(\varepsilon(e(y)) \cdot \varepsilon(x)) \\ &= d_0(\varepsilon(x)) \\ &= x.\end{aligned}$$

(iii) For $y \in P_0$ and $x \in M_0$,

$$\begin{aligned}y \cdot e(x) &= d_0(\varepsilon(y \cdot e(x)) \\ &= d_0(\varepsilon(y) \cdot \varepsilon(e(x)) \\ &= d_0(\varepsilon(y)) \\ &= y.\end{aligned}$$

□

DEFINITION 9. Let M and P be two generalized group-groupoids. If there is a generalized action of P on M, then we say that P *generalized acts* on M.

We now give the definition of crossed module in the category of generalized group-groupoids.

Definition 10. Let $\alpha = (\alpha_1, \alpha_2)\colon M \to P$ be a morphism of generalized group-groupoids and P generalized act on M. If (M, P, α_1) is a generalized crossed module over generalized groups, then (M, P, α) is called a generalized crossed module over generalize group-groupoids.

It is also worth mentioning here that in a generalized crossed module (M, P, α) over generalize group-groupoids, $\alpha_0\colon M_0 \to P_0$ is also a generalized crossed module over generalized groups.

Lemma 4. *If (M, P, α) is a generalized crossed module over generalize group-groupoids, then P_0 generalized acts on M by*

$$y \cdot g = \varepsilon^P(y) \cdot g = 1_y \cdot g$$

for $y \in P_0$ and $g \in M$.

Proof. It is seen in the following equations that the conditions are satisfied.

(i) For $y_1, y_2 \in P_0$ and $m \in m$,

$$\begin{aligned}(y_1 y_2) \cdot m &= \varepsilon^P(y_1 y_2) \cdot m\\ &= 1_{y_1 y_2} \cdot m\\ &= (1_{y_1} 1_{y_2}) \cdot m\\ &= 1_{y_1} \cdot (1_{y_2}) \cdot m)\\ &= y_1 \cdot (y_2 \cdot m).\end{aligned}$$

(ii) For $y \in P_0$ and $m_1, m_2 \in M$,

$$\begin{aligned}y \cdot (m_1 m_2) &= \varepsilon^P(y) \cdot (m_1 m_2)\\ &= 1_y \cdot (m_1 m_2)\\ &= (1_y \cdot m_1)(1_y \cdot m_2)\\ &= (\varepsilon^P(y) \cdot m_1)(= \varepsilon^P(y) \cdot m_2)\\ &= (y \cdot m_1)(y \cdot m_2).\end{aligned}$$

(iii) For $y \in P_0$ and $m \in M$,

$$\begin{aligned}e(y) \cdot m &= \varepsilon^P(e(y)) \cdot m\\ &= 1_{e(y)} \cdot m\\ &= g.\end{aligned}$$

(iv) For $y \in P_0$ and $m \in M$,

$$\begin{aligned}y \cdot e(m) &= \varepsilon^M(y) \cdot m\\ &= 1_y \cdot e(m)\\ &= e(m).\end{aligned}$$

□

Lemma 5. *If (M, P, α) is a generalized crossed module over generalize group-groupoids, then P generalized acts on M_0 by*

$$p \cdot x = d_1^P(p) \cdot x = 1_y \cdot m$$

for $p \in P$ and $x \in M_0$.

Proof. The proof can be handled in a similar way in Lemma 4. □

Definition 11. Let G be a generalized group-groupoid and H a subgroupoid of G. If H_O is a generalized subgroup of G_O and the group of morphisms of H is a generalized subgroup of G, then H is called a generalized subgroup-groupoid of G.

Definition 12. Let G be a generalized group-groupoid and N a normal subgroupoid of G. If N_O is a generalized normal subgroup of G_O and the group of morphisms of N is a generalized normal subgroup of G, then N is called a generalized normal subgroup-groupoid of G.

Example 3. Let A be a generalized group and B a generalized normal subgroup of A. By [14, Example 3.1] $A \times A$ is a generalized group-groupoid with object set A. Then we can obtain that the direct product $N = B \times B$ becomes a normal subgroup-groupoid of $G = A \times A$. By the generalized group homomorphism $f\colon A \to C$, where C is a generzlized group, we have a morphism $f \times f\colon (A \times A) \to C \times C$. For all $(a,b) \in G$, $N_{(a,b)} = \emptyset$ or $N_{(a,b)} = ker(f \times f)_{(a,b)}$, where $N_{(a,b)} = N \times G_{(g,h)}$, $G_{(a,b)} = \{(a,b)|(e(a),e(b)) = ((e(g),e(h))$ and $(f \times f)_{(a,b)} = (f \times f)|_{G_{(a,b)}}$.

In the light of the above definitions, now we give the following example of generalized crossed module over generalized group-groupoids.

Example 4. Let G be a generalized group-groupoid and N be generalized normal subgroup-groupoid of G. Then we have a generalized crossed module over generalized group-groupoids with the inclusion functor

$$\alpha = inc\colon N \hookrightarrow G, n \mapsto n,$$

and the generalized action of G on N

$$G \times N \to N, (g,n) \mapsto g \cdot n = gng^{-1}.$$

In addition to this case, generalized crossed modules $(G, G, 1_G)$ and $(\mathbf{1}, G, 0)$ over generalized group-groupoids can be given as examples, where G is a generalized group-groupoids.

Example 5. Let (M, P, α) be a crossed module over generalized groups. Then we know that $M \rtimes M$ and $P \rtimes P$ are generalized group-groupoids on M and P, respectively. So we obtain a generalized crossed module $(M \rtimes M, P \rtimes P, \alpha \times \alpha)$ over generalized group groupoids.

Example 6. Let M be a generalized group. Then it is known by the detailed proof given in [14] that the direct product $M \times M$ is a generalized group groupoid on M. So a generalized crossed module over generalized groups (M, P, α) gives rise to generalized crossed module over generalized group-groupoids replacing M and P with the associated generalized group-groupoids.

Let (M, P, α) and (M', P', α') be two generalized crossed modules over generalized group-groupoids. A morphism $(f_1, f_2)\colon (M, P, \alpha) \to (M', P', \alpha')$ between generalized crossed modules is defined to be a pair of generalized group-groupoid morphisms $f_1\colon M \to M'$ and $f_2\colon P \to P'$ such that (f_1, f_2) is a morphism of generalized crossed modules over generalized groups.Then we obtain the category $\mathsf{XMod}(GGpGd)$ of generalized crossed modules over generalized group groupoids and their morphisms. If we assume that P is an abelian and M is a normal generalized group, then we have a new category denoted by $\mathsf{XMod}(GGpGd)^*$, which is a full subcategory of the category $\mathsf{XMod}(GGpGd)$.

4. Generalized Double Group-Groupoids

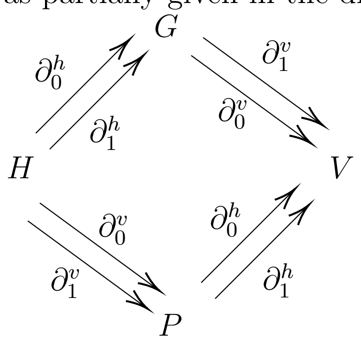
In this section, after reminding the notion of double groupoid, we introduce the notion of generalized double group-groupoids and give some properties to be used in equivalence of categories.

A double groupoid [9, 12] denoted by $\mathcal{G} = (G, H, V, P)$ is meant four related categories

$$(G, V, \partial_0^v, \partial_1^v, +, 0), \qquad (G, H, \partial_0^h, \partial_1^h, \circ, 1)$$

$$(V, P, \partial_0^h, \partial_1^h, \cdot, e), \qquad (H, P, \partial_0^v, \partial_1^v, \cdot, f)$$

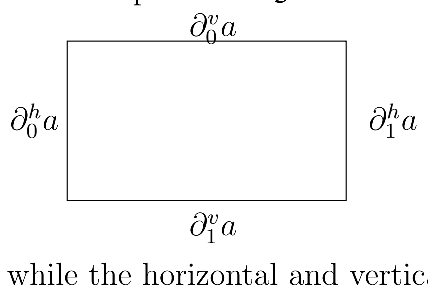
as partially given in the diagram



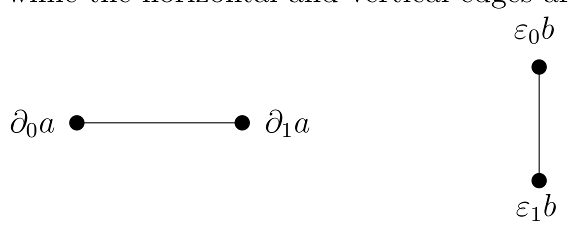
and satisfying the conditions of groupoid structures. In this new groupoid structure, the morphisms of G is called *squares*, H and V *horizontal and vertical edges* respectively, and of P points. There is a relations between the initial and final maps of structures as follows for $i, j = 0, 1$,

$$\partial_i^v \partial_j^h = \partial_j^h \partial_i^v.$$

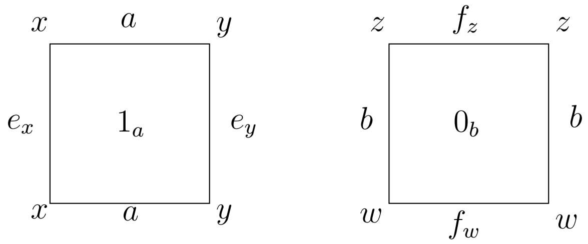
Thus a square a in $\mathcal{G}$ can be shown as



while the horizontal and vertical edges are shown as

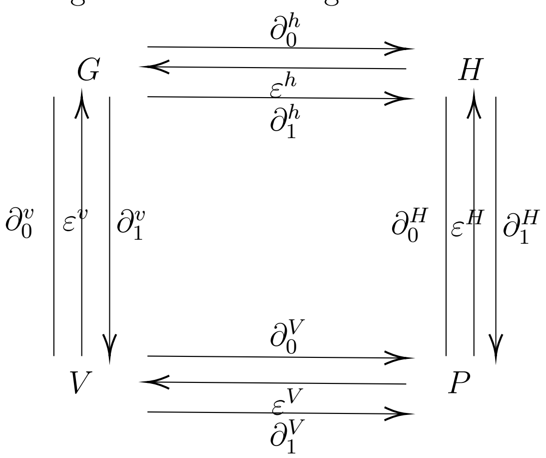


For squares the identities 1_a and 0_b can be pictured as:



Also we need the square $1_{1_x} = 0_{e_x}$, for $x \in P$. This square is denoted by $\odot$.

Now we assume that $\mathcal{G}$ is an internal groupoid in the category of generalized group-groupoids. Hence a *generalized double group-groupoid* is defined by four different but compatible generalized group groupoids (G, H), (G, V), (H, P) and (V, P) with the following commutative diagram:



Also the following interchange laws obtained by the horizontal and vertical compositions together with the group operations are satisfied, for the squares $\alpha, \beta, \gamma, \delta \in \mathcal{G}$:

$$\begin{aligned} (\alpha \circ_v \beta) \circ_h (\gamma \circ_v \delta) &= (\alpha \circ_h \gamma) \circ_v (\beta \circ_h \delta) \\ (\alpha \circ_v \beta) + (\gamma \circ_v \delta) &= (\alpha + \gamma) \circ_v (\beta + \delta) \\ (\alpha \circ_h \beta) + (\gamma \circ_h \delta) &= (\alpha + \gamma) \circ_h (\beta + \delta) \end{aligned} \tag{1}$$

whenever one side of the equations make sense.

Lemma 6. [24] *In a generalized double group-groupoid, as a consequence of interchange laws, the vertical and horizontal compositions of squares can be written in terms of the generalized group operations as:*

$$\gamma_1 \circ_h \gamma = \gamma_1 - \varepsilon^h \partial_1^h(\gamma) + \gamma = \gamma - \gamma_1 - \varepsilon^h \partial_1^h(\gamma) + \gamma_1 \tag{2}$$

$$\delta_1 \circ_v \delta = \delta_1 - \varepsilon^v \partial_1^h(\delta) + \delta = \delta - \delta_1 - \varepsilon^v \partial_1^v(\delta) + \delta_1 \tag{3}$$

for all $\gamma_1, \gamma, \delta_1, \delta \in G$ *such that* $\varepsilon^h \partial_1^h(\gamma) = \varepsilon^h \partial_0^h(\gamma_1)$ *and* $\varepsilon^v \partial_1^h(\delta) = \varepsilon^v \partial_0^h(\delta_1)$.

We can easily see that the horizontal inverse of $\gamma \in G$

$$\gamma^{-h} = \varepsilon^h \partial_0^h(\gamma) - \gamma + \varepsilon^h \partial_1^h(\gamma) = \varepsilon^h \partial_1^h(\gamma) - \gamma + \varepsilon^h \partial_0^h(\gamma)$$

and the vertical inverse of $\delta \in G$

$$\delta^{-v} = \varepsilon^v \partial_0^h(\delta) - \delta + \varepsilon^v \partial_1^v(\delta) = \varepsilon^v \partial_1(\delta) - \delta + \varepsilon^v \partial_0^v(\delta).$$

In particular, if $\gamma \in Ker\partial_0^h$ and $\delta \in Ker\partial_0^v$, then

$$\gamma^{-h} = -\gamma + \varepsilon^h \partial_1^h(\gamma) = \varepsilon^h \partial_1^h(\gamma) - \gamma$$

and

$$\delta^{-v} = -\delta + \varepsilon^v \partial_1^v(\delta) = \varepsilon^v \partial_1^v(\delta) - \delta.$$

Considering Lemma 6, if we take $\partial_1^h(\gamma) = 0$ and $\partial_1^v(\delta) = 0$, then we have the following result.

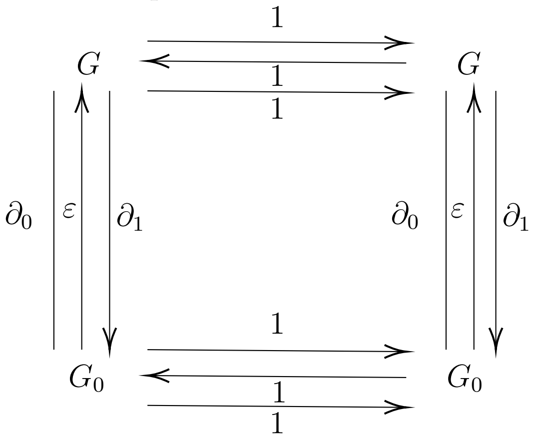
Corollary 1. *Let (G, H, V, P) be a generalized double group-groupoid and $\gamma, \gamma_1, \delta, \delta_1 \in G$ with $\partial_1^v(\delta) = 0 = \partial_0^v$ and $\partial_1^h(\gamma) = 0 = \partial_0^h$. Then*

$$\gamma_1 \circ_h \gamma = \gamma_1 + \gamma = \gamma + \gamma_1$$

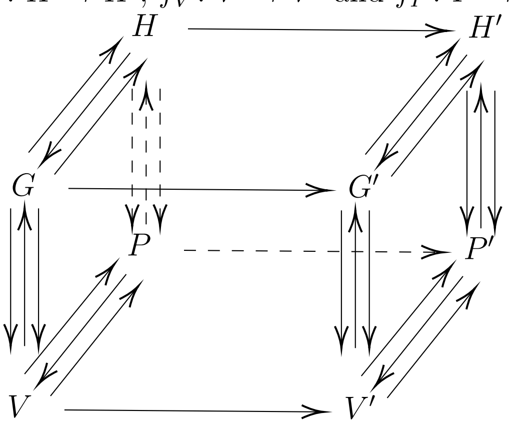
$$\delta_1 \circ_v \delta = \delta_1 + \delta = \delta + \delta_1$$

i.e. squares in $Ker\partial_0^h$(rep. $Ker\partial_0^v$) and $Ker\partial_1^h$(rep. $Ker\partial_1^v$) are commutative.

Example 7. Let G be a generalized group groupoid. Then (G, H, V, P) becomes a generalized double group-groupoid where $G = H$ and $V = P = G_O$ with the following structural maps:



Definition 13. Let (G, H, V, P) and (G', H', V', P') be two generalized double group-groupoids. A morphism of generalized double group-groupoids consists of four generalized group homomorphism $F = (f_G, f_H, f_V, F_P)$ such that $f_G\colon G \to G'$, $f_H\colon H \to H'$, $f_V\colon V \to V'$ and $f_P\colon P \to P'$. We denote such a morphism as follows:



Hence we obtain the category GDGpGd of generalized double group-groupoids and their morphisms.

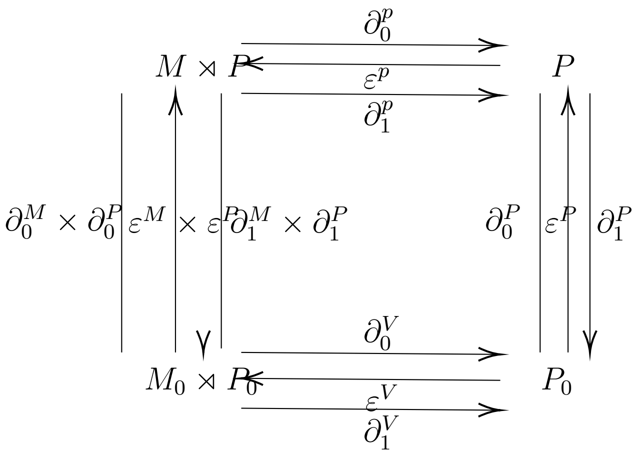
5. The equivalence of the categories

In this section, we prove that a generalized crossed module over generalized group-groupoids is equivalent to a generalized double group-groupoids (G, H, V, P) such that H and P are abelian generalized groups.

THEOREM 3. *Let (M, P, α) be a generalized crossed module in* $\mathsf{XMod}(GGpGd)^*$. *Then (M, P, α) induces a generalized double group-groupoid.*

Proof. We know from [14] that a generalized crossed module over generalized groups becomes a generalized group- groupoid such that the set of objects is the generalized group P and the set of morphisms is the semidirect product $M \rtimes P$. Here $M \rtimes P$ is a generalized group from [14, Prposition 2.11].

Now we show that how to get a generalized double group-groupoid from a generalized crossed module. If (M, P, α) is a generalized crossed module, then we have the following diagram:

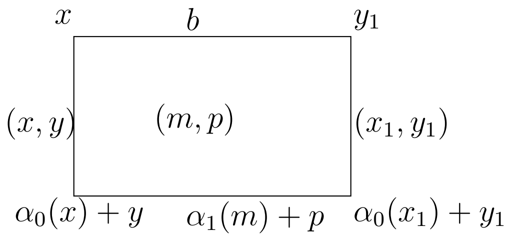


Here we have a generalized double group-groupoid $(M \rtimes P, M, M_0 \rtimes P_0, P_0)$. For the generalized group groupoid $M \rtimes P$, the groupoid structural maps are defined by $\partial_0^h(m, p) = m$, $\partial_1^h(m, p) = \alpha_1(m)+p$, $\varepsilon^h(p) = (0, p)$, $m^h((m_1, p_1), (m, p)) = (m_1+m, p)$ for $p_1 = \alpha_1(m) + p$.

Also we have a generalized group groupoid structure on $M_0 \rtimes P_0$ such that $\partial_0^V(x, y) = x$, $\partial_1^V(z, y) = \alpha_0(x) + y$, $\varepsilon^V(y) = (0, y)$, $m^V((x_1, y_1), (x, y)) = (x_1 + y, y)$ for $y_1 = \alpha_1(x) + y$.

It is easy to see that groupoid structural maps in $M \rtimes P$ and $M_0 \rtimes P_0$ are morphisms of generalized groups.

A square (m, p) in $M \rtimes P$ is denoted by



for $m \in M(x, x_1)$ and $p \in P(y, y_1)$. Here it is important to say that if (m, p),(m_1, p_1) and (m', p') are squares in $M \rtimes P$ where $m \in G(x, x_1)$, $m_1 \in (x_1, x_2)$, $(m' \in (x', x'_1)$,

$p \in G(y, y_1)$, $p_1 \in (y_1, y_2)$, $(p' \in (y', y_1')$ and $p' = \alpha_1(m) + p$, then,

$$(m_1, p_1) \circ_h (m, p) = (m_1 \circ m, p_1 \circ p), \tag{4}$$

$$(m', p') \circ_v (m, p) = (m' + m, p). \tag{5}$$

Finally we must show the interchange laws are satisfied.

For (m, p), (m', p'), (m_1, p_1) and (m_2, p_2) in $M \rtimes P$,

$$\begin{aligned} ((m_1, p_1) \circ_h (m, p)) \circ_v ((m_2, p_2) \circ_h (m', p')) &= (m_1 \circ m, p_1 \circ p) \circ_v (m_2 \circ m', p_1 \circ p) \\ &= ((m_1 \circ m) +_M (m_2 \circ m'), p_2 \circ p') \\ &= ((m_1 +_M m_2) \circ (m +_M m'), p_2 \circ p) \end{aligned} \tag{6}$$

and

$$\begin{aligned} ((m_1, p_1) \circ_v (m_2, p_2)) \circ_h ((m, p) \circ_h (m', p')) &= (m_1 + m_2, p_2) \circ_h (m + m', p') \\ &= ((m_1 \circ m) +_M (m_2 \circ m'), p_2 \circ p') \\ &= ((m_1 +^M m_2) \circ (m +^M m'), p_2 \circ p). \end{aligned} \tag{7}$$

Hence from 6 and 7, we have

$$((m_1, p_1) \circ_h (m, p)) \circ_v ((m_2, p_2) \circ_h (m', p')) = ((m_1, p_1) \circ_v (m_2, p_2)) \circ_v ((m, p) \circ_h (m', p')).$$

On the other hand

$$\begin{aligned} ((m_1, p_1) \circ_v (m, p)) + ((m_2, p_2) \circ_v (m', p')) &= (m_1 + m, p) + (m_2 + m', p') \\ &= ((m_1 + m) \cdot p' + (m_2 + m'), (p + p')) \end{aligned} \tag{8}$$

and

$$\begin{aligned} & ((m_1, p_1) + (m_2, p_2)) \circ_v ((m, p) + (m', p')) \\ = \; & ((m_1 + m) \cdot p_2) + (p_1 + p_2)) \circ_v ((m + m') \cdot p', p + p') \\ = \; & ((m_1 + m) \cdot p_2 + (m_2 + m')p', (p + p')). \end{aligned} \tag{9}$$

Here from the equations 8 and 9 we must get

$$((m_1 + m) \cdot p_2 + (m_2 + m')p' = ((m_1 + m) \cdot p' + (m_2 + m'). \tag{10}$$

Since $\partial_1^h(m_1, p_1) = \alpha(m_1) + p_1 = p = \partial_0^h(m, p)$ and the property $e(m) - n = -n + e(m)$ for $m, n \in M$ for the normal generalized group M, we obtain

$$((m_1, p_1) \circ_v (m, p)) + ((m_2, p_2) \circ_v (m', p'))) = ((m_1, p_1) + (m_2, p_2)) \circ_v ((m, p) + (m', p')). \tag{11}$$

Also since M is a normal generalized group, then from

$$\begin{aligned} & ((m_1, p_1) \circ_h (m, p)) + ((m_2, p_2) \circ_h (m', p')) \\ = \; & (m_1 \circ m, p_1 \circ p) + (m_2 \circ m', p_2 \circ p') \\ = \; & ((m_1 \circ m)(p_1 \circ p) \cdot (m_2 \circ m'), (p_1 \circ p) + (p_2 \circ p')) \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
& ((m_1,p_1)+(m_2,p_2))\circ_h((m,p)+(m',p')) \\
= \;& (m_1+(p_1\cdot m_2),(p_1+p_2)\circ_h(m+(p\cdot m'),p+p') \\
= \;& (m+(p_1\cdot m_2)\circ m+(p\cdot m'),(p_1\circ p)+(p_2\circ p')) \qquad (13)
\end{aligned}
$$

the interchange law

(14)
$$((m_1,p_1)\circ_h(m,p))+((m_2,p_2)\circ_h(m',p'))=((m_1,p_1)+(m_2,p_2))\circ_h((m,p)+(m',p'))$$

is satisfied.

It is easy to see that the interchange laws are satisfied for the generalized group groupoid $M_0\rtimes P_0$.

□

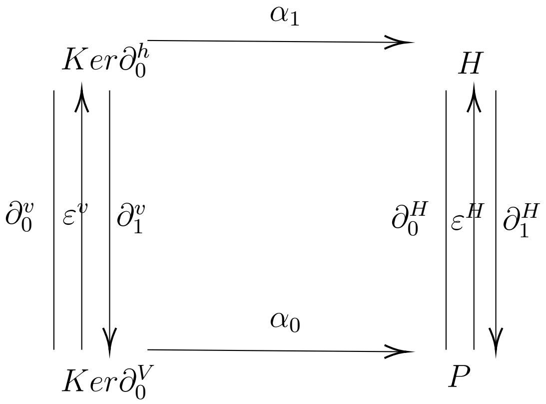
Theorem 4. *Let $\mathcal{G}=(G,H,V,P)$ be a generalized double group-groupoids such that the horizontal edges set H and points set P are abelian generalized groups. Then $\mathcal{G}$ becomes a generalized crossed module over generalized group-groupoids.*

Proof. A generalized crossed module (M,H,α) from the generalized double group-groupoid G is obtained by the following way:

The generalized group-groupoids M, H and the generalized group-groupoid morphism α are defined by

$$
\begin{aligned}
& M=(Ker\partial_0^h,Ker\partial_0^V,\partial_0^v,\partial_1^v,\varepsilon^v,m^v,\eta^v) \\
& H=(H,P,\partial_0^H,\partial_1,\varepsilon^H,m^H,\eta^H) \\
& \alpha=(\alpha_1=\partial_1^h,\alpha_0=\partial_0^h) \\
& \cdot: H\times Ker\partial_0^h\to Ker\partial_0^h,(h,m)=\varepsilon^h(h)+m-\varepsilon^h(h)
\end{aligned}
$$

Then we can show such a crossed module in the following diagram;



The action of H on $Ker\partial_0^h$ defined by $H\times Ker\partial_0^h\to Ker\partial_0^h$ is satisfied the conditions of generalized action.

□

Finally,we give the main theorem by using the Theorems 3 and 4. As we mentioned earlier, $\mathcal{G}=(G,H,V,P)$ will be considered as a generalized double group-groupoids such that the horizontal edges set H and points set P are abelian generalized groups.

THEOREM 5. *The categories* GDGpGd *and* $\mathsf{XMod}(GGpGd)^*$ *are equivalent.*

Proof. By the Theorem 3, we know that if (M, P, α) is a generalized crossed module over generalized group-groupoids, then we obtain a generalized double group-groupoid. Hence there exists a functor

$$\phi\colon \mathsf{XMod}(GGpGd)^* \to \mathsf{GDGpGd}$$

defined by $\phi(M, P, \alpha) = (M \rtimes P, P)$ and $\phi(f_1, f_2) = (f_1 \times f_2, f_2)$ where $(f_1, f_2)\colon (M, P, \alpha) \to (M', P', \alpha')$ is a morphism of generalized crossed modules.

Conversely, let $\mathcal{G} = (G, H, V, P)$ and $\mathcal{G}' = (G, H, V, P)$ be generalized double group-groupoids and $F = (f_G, f_H, f_V, F_P)$ be a morphism of generalized double group-groupoids such that $f_G\colon G \to G'$, $f_H\colon H \to H'$, $f_V\colon V \to V'$ and $f_P\colon P \to P'$. Then by the Theorem 4 we obtain a generalized crossed module over generalized group-groupoids and a generalized crossed module morphism $\alpha = (\alpha_1 = \partial_1^h, \alpha_0 = \partial_0^h)$ via the functor

$$\varphi\colon \mathsf{GDGpGd} \to \mathsf{XMod}(GGpGd)^*.$$

So as the last step of the proof we can see that $\varphi\phi \simeq 1_{\mathsf{XMod}(GGpGd)^*}$ and $\phi\varphi \simeq 1_{\mathsf{GDGpGd}}$.

□

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