

SIX DIMENSIONAL ALMOST COMPLEX TORUS MANIFOLDS WITH EULER NUMBER SIX

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ABSTRACT. An almost complex torus manifold is a $2n$ -dimensional compact connected almost complex manifold equipped with an effective action of a real n -dimensional torus $T^n \simeq (S^1)^n$ that has fixed points. For an almost complex torus manifold, there is a labeled directed graph which contains information on weights at the fixed points and isotropy spheres. Let M be a 6-dimensional almost complex torus manifold with Euler number 6. We show that two types of graphs occur for M , and for each type of graph we construct such a manifold M , proving the existence. Using the graphs, we determine the Chern numbers and the Hirzebruch χ_y -genus of M .

1. Introduction

The purpose of this paper is to study 6-dimensional almost complex torus manifolds with Euler number 6. Let M be a $2n$ -dimensional almost complex manifold and let a k -dimensional torus $T^k \simeq (S^1)^k$ act effectively on M . Let F be a fixed component and p a point in F . The normal space $N_p F$ of F at p decomposes into the sum of complex 1-dimensional vector spaces L_1, \dots, L_m , where on each L_i the torus T^k acts by multiplication by $g^{w_{p,i}}$ for all $g \in T^k$, for some non-zero element $w_{p,i}$ of \mathbb{Z}^k , $1 \leq i \leq m$. Here, $2m$ is the codimension of F in M . These elements $w_{p,1}, \dots, w_{p,m}$ are the same for all $p \in F$, and called the *weights* of F . Since the action is effective, these weights $w_{p,1}, \dots, w_{p,m}$ span \mathbb{Z}^k . In particular, if $k = n$, then $m = n$ and hence M has only isolated fixed points. Thus, any almost complex torus manifold has isolated fixed points only.

For a torus action on a compact manifold M , the Euler number of M is equal to the sum of Euler numbers of its fixed components (Theorem 5.4). The Euler number of a point is 1. Thus, for an almost complex torus manifold M ,

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the Euler number of M is equal to the number of fixed points since M has only isolated fixed points.

For an almost complex torus manifold, there is a (directed labeled) graph that contains information on weights at the fixed points and isotropy spheres; a vertex of the graph corresponds to a fixed point, an edge corresponds to an isotropy sphere, and the label of an edge gives a weight at a fixed point corresponding to its vertex; see Definitions 2.1 and 2.2 and Proposition 2.6.

Suppose that two almost complex torus manifolds have the same associated graph. Then they have the same weights at the fixed points, and thus the same Chern numbers (and hence have the same Hirzebruch χ_y -genus, signature, Todd genus, and Euler number). If in addition their actions are both equivariantly formal, they have the same equivariant cohomology, because the graph is a GKM (Goresky-Kottwitz-MacPherson) graph if we forget the direction of each edge [10, Corollary 2.4]; moreover they have the same Chern class.

In [8], the first author showed that a $2n$ -dimensional almost complex torus manifold with Euler number $n + 1$, which is minimal, has the same associated graph as one for some linear action on the complex projective space $\mathbb{C}P^n$. This implies that the aforementioned invariants agree for the two manifolds. Li studied an analogous problem for Hamiltonian S^1 -actions on symplectic manifolds with (almost) minimal fixed point sets; such a manifold shares some invariants with the complex projective spaces or Grassmannians [16, 17].

In this paper, we study 6-dimensional almost complex torus manifolds with Euler number 6 (6 fixed points), which is the next minimal Euler number in dimension 6; since any torus action on a $(4m + 2)$ -dimensional compact almost complex manifold cannot have an odd number of fixed points, 6 is the next possible number of fixed points.

For a 6-dimensional almost complex torus manifold with Euler number 6, we show that two types of graphs occur as a graph associated to the manifold.

Theorem 1.1. *Let M be a 6-dimensional almost complex torus manifold with Euler number 6. Then one of Figure 1 occurs as a graph describing M , for some a, b , and c in \mathbb{Z}^3 that span \mathbb{Z}^3 and for some integers k (and l for Figure 1b).*

Moreover, for each type of graph, we construct such a manifold M , proving the existence.

Theorem 1.2. *Each of graphs in Figure 1 describes some 6-dimensional almost complex torus manifold with Euler number 6; Figure 1a describes Example 1.3 and Figure 1b describes Example 1.4.*

Example 1.3 (More details are in Example 3.1). Fix an integer k . Let $M_1(k)$ be a (real) 6-dimensional compact complex manifold $M_1(k) = \{([z_0 : z_1 : z_2 : z_3], [w_0 : w_1 : w_2]) \in \mathbb{C}P^3 \times \mathbb{C}P^2 \mid z_0 w_1^k = z_1 w_0^k, z_1 w_2^k = z_2 w_1^k\}$. Let T^3 act on $M_1(k)$ by

$$g \cdot ([z_0 : z_1 : z_2 : z_3], [w_0 : w_1 : w_2])$$

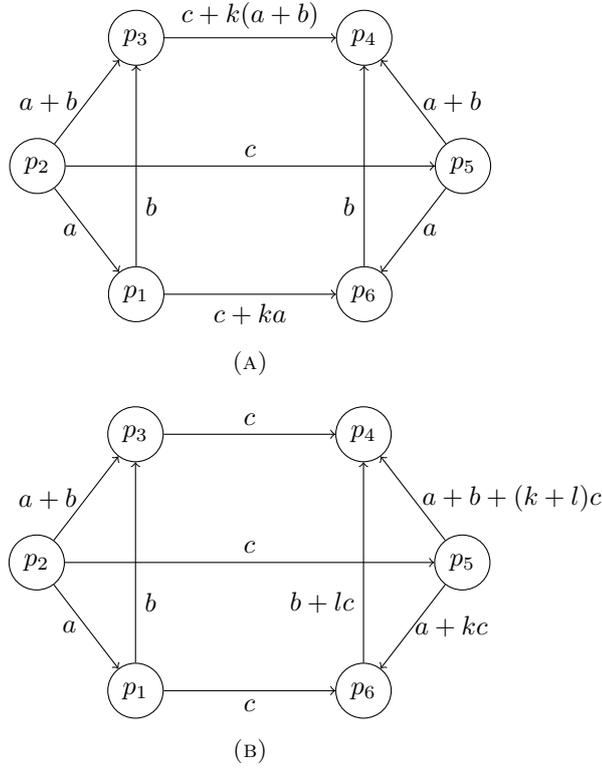


FIGURE 1. Graphs for 6-dimensional almost complex torus manifolds with Euler number 6

$$= ([z_0 : g^{ka} z_1 : g^{k(a+b)} z_2 : g^{-c} z_3], [w_0 : g^a w_1 : g^{a+b} w_2])$$

for all $g \in T^3 \subset \mathbb{C}^3$, where a, b , and c form a basis of \mathbb{Z}^3 . Figure 1a is a graph describing $M_1(k)$ with this action.

Example 1.4 (More details are in Example 3.2). Fix integers k and l . Let $M_2(k, l)$ be a (real) 6-dimensional compact complex manifold $M_2(k, l) = \{([z_0 : z_1 : z_2 : z_3 : z_4], [w_0 : w_1]) \in \mathbb{C}\mathbb{P}^4 \times \mathbb{C}\mathbb{P}^1 \mid z_0 w_1^k = z_3 w_0^k, z_2 w_1^{k+l} = z_4 w_0^{k+l}\}$. Let T^3 act on $M_2(k, l)$ by

$$g \cdot ([z_0 : z_1 : z_2 : z_3 : z_4], [w_0 : w_1]) = ([z_0 : g^{-a} z_1 : g^b z_2 : g^{kc} z_3 : g^{b+(k+l)c} z_4], [w_0 : g^c w_1])$$

for all $g \in T^3 \subset \mathbb{C}^3$, where a, b , and c form a basis of \mathbb{Z}^3 . Figure 1b is a graph describing $M_2(k, l)$ with this action.

As a consequence of Theorem 1.1, for a 6-dimensional almost complex torus manifold with Euler number 6, we determine its Chern numbers and the Hirzebruch χ_y -genus.

Theorem 1.5. *Let M be a 6-dimensional almost complex torus manifold with Euler number 6. Then the Hirzebruch χ_y -genus of M is $\chi_y(M) = 1 - 2y + 2y^2 - y^3$.*

- (1) *Suppose that Figure 1a describes M . Then the Chern numbers of M are $\int_M c_1 c_2 = 24$, $\int_M c_3 = 6$, and $\int_M c_1^3 = 2(k^2 + 27)$ for some integer k .*
- (2) *Suppose that Figure 1b describes M . Then the Chern numbers of M are $\int_M c_1 c_2 = 24$, $\int_M c_3 = 6$, and $\int_M c_1^3 = 54$.*

We note that for the manifold $M_1(k)$, if $|k|$ is large, then the Chern number $\int_M c_1^3 = 2(k^2 + 27)$ is large.

For a 4-dimensional almost complex torus manifold M , it need not hold that $a_0(M) \leq a_1(M)$, where $\chi_y(M) = a_0(M) - a_1(M)y + a_2(M)y^2$ is the Hirzebruch χ -genus of M ; see [9, 18].

Let M be a 6-dimensional almost complex torus manifold with Euler number 6. In theory, it is possible that the Hirzebruch χ_y -genus of M is $2 - y + y^2 - 2y^3$; see Proposition 5.5. On the other hand, Theorem 1.5 shows that this is not possible. It is plausible that for a 6-dimensional almost complex torus manifold M , if we let $\chi_y(M) = \sum_{i=0}^3 a_i(M) \cdot (-y)^i$ be the Hirzebruch χ_y -genus of M , then $a_0(M)$ is bounded above by some increasing function of $a_1(M)$.

Question 1.6. Let M be a 6-dimensional almost complex torus manifold. Then does $a_0(M) \leq f(a_1(M))$ hold for some increasing function $f(x)$? Here, $\chi_y(M) = \sum_{i=0}^3 a_i(M) \cdot (-y)^i$ is the Hirzebruch χ_y -genus of M .

In the language of the theory of multi-fans [18], we can rephrase this question as, if we can use $2(a_0(M) + a_1(M))$ fans to cover \mathbb{R}^3 $a_0(M)$ -times, where $a_0(M)$ is the Todd genus of M and $2(a_0(M) + a_1(M))$ is the total number of fixed points.

Let M be a 6-dimensional compact almost complex manifold. In Examples 1.3 and 1.4, we constructed examples of M equipped with T^3 -actions with 6 fixed points. On the other hand, if a smaller dimensional torus acts on M with 6 fixed points, then there are more examples. In [7], the first author showed that if the circle group S^1 acts on M with 4 fixed points, then 6 cases occur for the weights at the fixed points, and discussed the existence of such a manifold in each case. In particular, there is such a manifold M with Todd genus 0. If we blow up a fixed point of M , then we get another S^1 -manifold with 6 fixed points, whose Todd genus is 0. Such a manifold is distinct from the manifolds in Theorem 1.1; the latter manifolds have Todd genus 1, as shown in Theorem 1.5.

We discuss the ideas of the proof of Theorem 1.1. The first ingredient is an isotropy submanifold, which is the set of points in a given almost complex torus manifold that are fixed by an action of some subgroup of the torus. The

second is the Hirzebruch χ_y -genus and the Kosniowski formula (Theorem 5.1). The third is that a subgraph describes an almost complex torus submanifold (Proposition 2.4). The fourth is a relationship between weights at fixed points in an isotropy submanifold (Lemma 2.10).

The proof of Theorem 1.1 goes as follows. Let M be a 6-dimensional almost complex torus manifold with Euler number 6. The purpose of Theorem 1.1 is to determine graphs describing M .

First, in Lemma 5.6 we use the Hirzebruch χ_y -genera of isotropy submanifolds and the Kosniowski formula (Theorem 5.1) to obtain information on a 4-dimensional almost complex torus submanifold.

Second, in Proposition 6.1 we show that Figure 6a or Figure 6b describes M . Suppose that Figure 6b describes M . Using Proposition 2.4 and Lemma 2.10, in Proposition 6.2 we show that Figure 6b does not describe M . Therefore, Figure 6a describes M .

What remains is to determine the labels of Figure 6a. In Lemma 6.3, using Proposition 2.4 and Lemma 5.6, we find subgraphs describing 4-dimensional almost complex torus submanifolds. In Lemma 7.1, we determine a graph describing a 4-dimensional almost complex torus manifold with 3 or 4 fixed points. Then from Lemmas 6.3 and 7.1 we show in Lemma 7.2 that several cases occur for relationships between the labels of the subgraphs. Then our task is to determine the labels in each case that occurs, which we do so in the proof of Theorem 1.1 in Section 7.

The structure of this paper is as follows. In Section 2, we review background and preliminaries. In Section 3, we construct two 6-dimensional almost complex torus manifolds, one described by Figure 1a (Example 3.1) and another described by Figure 1b (Example 3.2). In Section 4, we discuss (equivariant) blow up of an almost complex torus manifold and show that an (equivariant) blow up of a linear T^3 -action on the complex projective space $\mathbb{C}P^3$ is the manifold $M_1(k)$ with $k = 1$ that we construct in Example 3.1. In Section 5, we study the Hirzebruch χ_y -genus of an almost complex torus manifold. Let M be a 6-dimensional almost complex torus manifold with Euler number 6. In Section 6, we show that Figure 6a describes M (Propositions 6.1 and 6.2), and we find subgraphs describing 4-dimensional almost complex torus submanifolds of M (Lemma 6.3). With all of these, in Section 7 we prove Theorem 1.1. In Section 8, we prove Theorem 1.5; we determine the Chern numbers and the Hirzebruch χ_y -genus of M .

We may use the ideas of this paper to study general $2n$ -dimensional almost complex torus manifolds with $n + 3$ fixed points, though it could be more complicated. To discuss, let M be such a manifold. As in Proposition 5.5, we may consider possible values of the Hirzebruch χ_y -genus of M . As in Lemma 5.6, we may conclude that almost complex torus submanifolds of M of codimension 2 have small numbers of fixed points. Together, we can use the ideas in

Propositions 6.1 and 6.2 to determine possible graphs for M . We may determine subgraphs of the almost complex torus submanifolds of codimension 2, as in Lemmas 6.3, 7.1, and 7.2. All of these could determine possible graphs of M . On the other hand, for higher dimensional M , all these steps will be more difficult because they will contain more cases to consider, and it will be harder to analyze those codimension 2 almost complex torus submanifolds.

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2. Background and preliminaries

Let M be a manifold. An *almost complex structure* on M is a bundle map $J : TM \rightarrow TM$ on the tangent bundle of M , which restricts to a linear complex structure on each tangent space. A pair (M, J) of a manifold M and an almost complex structure J on M is called an *almost complex manifold*. By definition, any almost complex manifold has even dimension. For an action of a group G on an almost complex manifold (M, J) , we say that the group action *preserves the almost complex structure* if $dg \circ J = J \circ dg$ for all elements g in the group G . Throughout this paper, we assume that any group action on an almost complex manifold preserves the almost complex structure.

For a torus action on a compact almost complex manifold with isolated fixed points, there exists a labeled directed multigraph that contains information on weights at the fixed points and isotropy submanifolds [10], which is an extension of the result for circle actions [3, 11]. The vertex set of the multigraph is the fixed point set of the manifold. If an edge e of the multigraph has label w , then a fixed point corresponding to the initial vertex of e has weight w , a fixed point corresponding to the terminal vertex of e has weight $-w$, and the two fixed points are in the same component of an isotropy submanifold.

Definition 2.1. A *labeled directed k -multigraph* Γ is a set V of vertices, a set E of edges, maps $i : E \rightarrow V$ and $t : E \rightarrow V$ giving the initial and terminal vertices of each edge, and a map $w : E \rightarrow \mathbb{Z}^k$ giving the label of each edge.

For an element $w = (w_1, \dots, w_k)$ in \mathbb{Z}^k , by $\ker w$ we mean the subgroup of T^k whose elements fix w . That is,

$$\ker w = \{g = (g_1, \dots, g_k) \in T^k \subset \mathbb{C}^k \mid g^w := g_1^{w_1} \cdots g_k^{w_k} = 1\}.$$

If a group G acts on a manifold M , we denote by M^G its fixed point set, that is,

$$M^G = \{m \in M \mid g \cdot m = m, \forall g \in G\}.$$

Definition 2.2. Let a k -dimensional torus T^k act on a compact almost complex manifold M with isolated fixed points. We say that a (labeled directed k -)multigraph $\Gamma = (V, E)$ *describes* M if the following hold:

- (i) The vertex set V is equal to the fixed point set M^{T^k} .

- (ii) The multiset of the weights at p is $\{w(e) \mid i(e) = p\} \cup \{-w(e) \mid t(e) = p\}$ for all $p \in M^{T^k}$.
- (iii) For each edge e , the two endpoints $i(e)$ and $t(e)$ are in the same component of the isotropy submanifold $M^{\ker w(e)}$.

Proposition 2.3 ([10]). *Let a k -dimensional torus T^k act on a compact almost complex manifold M with isolated fixed points. Then there exists a (labeled directed k -)multigraph Γ describing M that has no self-loops.*

Moreover, a sub-multigraph of the multigraph describes an isotropy submanifold of M , where the submanifold is given an action of the torus T^k .

Proposition 2.4. *Let a k -dimensional torus T^k act on a compact almost complex manifold M with isolated fixed points. Let Γ be a multigraph describing M that has no self-loops. Let S be a closed subgroup of T^k and let F be a component of the isotropy submanifold M^S . Let Γ_F be a sub-multigraph of Γ obtained as follows:*

- (1) *The vertex set of Γ_F is $F \cap M^{T^k}$.*
- (2) *For $p \in \Gamma_F$, edges e'_i of p in Γ_F are the edges e_i of p in Γ whose labels w_i satisfy $g^{w_i} = 1$ for all $g \in S$.*

Then Γ_F describes F , on which the T^k -action on M restricts to act. Moreover, $\dim F = 2m$ if and only if Γ_F is m -valent; each vertex of Γ_F has m edges.

Proof. The torus T^k acts on F as the restriction of the T^k -action on M to F . Thus, $F^{T^k} = F \cap M^{T^k}$ and hence Γ_F satisfies (i) of Definition 2.2 for the T^k -action on F .

Let e be an edge of Γ_F from p to q with label w . Because Γ_F is a sub-multigraph of Γ , p has weight w and q has weight $-w$. Because $g^w = 1$ for all $g \in S$ by (2), the weight w is a weight of the tangent space $T_p F$ to F at p . Conversely, for $p \in F^{T^k}$, if a weight w' in the tangent space $T_p M$ of M satisfies $g^{w'} = 1$ for all $g \in S$, then it is a weight of the tangent space $T_p F$ of F , and by (2) such a weight w' is the label of some edge of Γ_F at p . Thus, Γ_F satisfies (ii) of Definition 2.2 for the T^k -action on F .

Let e be an edge of Γ_F from p to q with label w . Because e is an edge of Γ , by (iii) of Definition 2.2, p and q are in the same component G of $M^{\ker w}$. By (2), $g^w = 1$ for all $g \in S$. Thus, $S \subset \ker w$ and hence $G \subset M^{\ker w} \subset M^S$. Because F is a connected component of M^S containing p and q and G is a connected component of $M^{\ker w} \subset M^S$ containing p and q , this implies that $G \subset F$. For any $x \in G$, every weight w' in $T_x G$ satisfies $g^{w'} = 1$ for $g \in \ker w$. Therefore, it follows that $G \subset F^{\ker w}$. Hence, Γ_F satisfies (iii) of Definition 2.2 for the T^k -action on F .

Since Γ_F describes F , $\dim F = 2m$ if and only if Γ_F is m -valent, by (ii) of Definition 2.2. □

The following example illustrates Proposition 2.4.

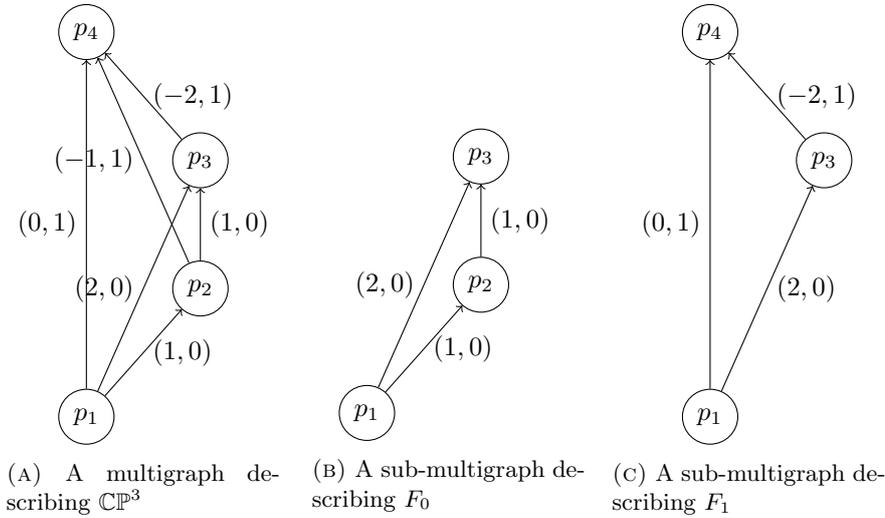


FIGURE 2

Example 2.5. Let a 2-dimensional torus T^2 act on $\mathbb{C}\mathbb{P}^3$ by

$$(g_1, g_2) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g_1 z_1 : g_1^2 z_2 : g_2 z_3]$$

for all $(g_1, g_2) \in T^2$ and for all $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathbb{P}^3$. The action has 4 fixed points, $p_1 = [1 : 0 : 0 : 0]$, $p_2 = [0 : 1 : 0 : 0]$, $p_3 = [0 : 0 : 1 : 0]$, and $p_4 = [0 : 0 : 0 : 1]$ that have weights $\{(1, 0), (2, 0), (0, 1)\}$, $\{(-1, 0), (1, 0), (-1, 1)\}$, $\{(-2, 0), (-1, 0), (-2, 1)\}$, and $\{(-2, 0), (1, -1), (2, -1)\}$, respectively. Figure 2a describes this action on $\mathbb{C}\mathbb{P}^3$.

Consider the subgroup $\{1\} \times S^1$ of T^2 . Its fixed point set is $\{[z_0 : z_1 : z_2 : 0]\} \cup [0 : 0 : 0 : 1]$. The 2-dimensional torus T^2 acts on the former fixed component $F_0 := \{[z_0 : z_1 : z_2 : 0]\}$ by

$$(g_1, g_2) \cdot [z_0 : z_1 : z_2 : 0] = [z_0 : g_1 z_1 : g_1^2 z_2 : 0]$$

for all $(g_1, g_2) \in T^2$, having fixed points $p_1 = [1 : 0 : 0 : 0]$, $p_2 = [0 : 1 : 0 : 0]$, and $p_3 = [0 : 0 : 1 : 0]$ whose weights are $\{(1, 0), (2, 0)\}$, $\{(-1, 0), (1, 0)\}$, and $\{(-2, 0), (-1, 0)\}$, respectively. Thus, Figure 2b describes the T^2 -action on F_0 .

Next, consider the subgroup $\mathbb{Z}_2 \times \{1\}$ of T^2 , where $\mathbb{Z}_2 = \{1, -1\}$. Its fixed point set is $\{[z_0 : 0 : z_2 : z_3]\} \cup [0 : 1 : 0 : 0]$. The 2-dimensional torus T^2 acts on the former fixed component $F_1 := \{[z_0 : 0 : z_2 : z_3]\}$ by

$$(g_1, g_2) \cdot [z_0 : 0 : z_2 : z_3] = [z_0 : 0 : g_1^2 z_2 : g_2 z_3]$$

for all $(g_1, g_2) \in T^2$, having fixed points $p_1 = [1 : 0 : 0 : 0]$, $p_3 = [0 : 0 : 1 : 0]$, and $p_4 = [0 : 0 : 0 : 1]$ whose weights are $\{(2, 0), (0, 1)\}$, $\{(-2, 0), (-2, 1)\}$, and $\{(0, -1), (2, -1)\}$, respectively. Thus, Figure 2c describes the T^2 -action on F_1 .

For an almost complex torus manifold, a multigraph describing it is a graph without any self-loops.

Proposition 2.6 ([10]). *Let M be an almost complex torus manifold. Then there is a multigraph describing M that has no multiple edges and no self-loops.*

Moreover, each edge corresponds to an isotropy sphere.

Proposition 2.7 ([10]). *Let M be a $2n$ -dimensional almost complex torus manifold. Then for each fixed point p , there are n 2-spheres on each of which the T^n -action on M restricts to act, sharing p as one fixed point in common with weight $w_{p,i}$, but no two of them share other fixed points. Here, $w_{p,1}, \dots, w_{p,n}$ are the weights at p .*

An almost complex manifold M has at least $\frac{1}{2} \dim M + 1$ fixed points [2, 18]. In other words, the Euler number of an almost complex torus manifold M is at least $\frac{1}{2} \dim M + 1$, because M has finitely many fixed points, the Euler number of M is equal to the sum of the Euler numbers of fixed points (Theorem 5.4), and the Euler number of a point is 1. Proposition 2.6 implies that a graph describing M has at least $\frac{1}{2} \dim M + 1$ vertices, proving the same result.

Proposition 2.8 ([2, 10, 18]). *Let M be an almost complex torus manifold. Then there are at least $\frac{1}{2} \dim M + 1$ fixed points. Alternatively, the Euler number of M is at least $\frac{1}{2} \dim M + 1$.*

Let a k -dimensional torus T^k act on a manifold M . The equivariant cohomology of M is

$$H_{T^k}^*(M) = H^*(M \times_{T^k} ET^k),$$

where ET^k is a contractible space on which T^k acts freely. Suppose that M is compact and oriented. Then the projection map $\pi : M \times_{T^k} ET^k \rightarrow BT^k$ induces a push-forward map

$$\pi_* : H_{T^k}^i(M; \mathbb{Z}) \rightarrow H^{i-\dim M}(BT^k; \mathbb{Z})$$

for all $i \in \mathbb{Z}$, where BT^k is the classifying space of T^k . The projection map π_* is also denoted by \int_M , and is given by integration over the fiber M .

Theorem 2.9 (The Atiyah-Bott-Berline-Vergne localization theorem, [1]). *Let a k -dimensional torus T^k act on a compact oriented manifold M . Let $\alpha \in H_{T^k}^*(M; \mathbb{Q})$. As an element of $\mathbb{Q}(t)$,*

$$\int_M \alpha = \sum_{F \subset M^{T^k}} \int_F \frac{\alpha|_F}{e_{T^k}(NF)},$$

where the sum is taken over all fixed components, and $e_{T^k}(NF)$ is the equivariant Euler class of the normal bundle to F in M .

A unitary manifold is a manifold with a complex structure on the bundle $TM \oplus \mathbb{R}^k$ for some $k \geq 0$, where \mathbb{R}^k denotes the k -dimensional trivial bundle over M . By definition, any almost complex manifold is unitary.

For a torus action on a unitary manifold, at each fixed component weights of the normal bundle are well-defined. Moreover, weights at fixed points in an isotropy submanifold have an intimate relation.

Lemma 2.10 ([10]). *Fix an element w in \mathbb{Z}^k . Let a k -dimensional torus T^k act on a compact unitary manifold M . Let p and p' be fixed points which lie in the same component of $M^{\ker w}$. Then the T^k -weights at p and at p' are equal modulo w .*

Since any almost complex manifold is unitary, Lemma 2.10 holds for almost complex manifolds.

Circle actions on almost complex manifolds with few fixed points are classified.

Theorem 2.11. *Let the circle act on a compact almost complex manifold M .*

- (1) *If there is exactly one fixed point p , then M is the point itself, that is, $M = \{p\}$.*
- (2) [6, 14] *If there are exactly two fixed points, then either M is the 2-sphere, or $\dim M = 6$ and weights at the fixed points are $\{-a - b, a, b\}$ and $\{-a, -b, a + b\}$ for some positive integers a and b .*
- (3) [6] *If there are exactly three fixed points, then $\dim M = 4$ and weights at the fixed points are $\{a + b, a\}$, $\{-a, b\}$, and $\{-b, -a - b\}$ for some positive integers a and b .*

The proof for the case of 3 fixed points in [6] is a careful modification of the proof for a symplectic action in [5].

3. Existence: examples

By Theorem 1.1, any 6-dimensional almost complex torus manifold with Euler number 6 is described by Figure 1a or Figure 1b. In this section, we construct examples of almost complex torus manifolds, one described by Figure 1a and the other described by Figure 1b.

We note that Example 3.1 with $k = 0$ is the same as Example 3.2 with $k = l = 0$. We also note that the manifold constructed in [10, Example 7.1] is a special case of Example 3.2 that we construct in this paper with $k = 0$. Moreover, Example 3.1 with $k = 1$ is also obtained as blowing up at a fixed point of a linear T^3 -action on $\mathbb{C}\mathbb{P}^3$; see Example 4.1 for details.

The first example is an almost complex torus manifold described by Figure 1a.

Example 3.1. Fix an integer k . By $M_1(k)$ we mean a compact complex manifold $M_1(k) = \{([z_0 : z_1 : z_2 : z_3], [w_0 : w_1 : w_2]) \in \mathbb{C}\mathbb{P}^3 \times \mathbb{C}\mathbb{P}^2 \mid z_0 w_1^k = z_1 w_0^k, z_1 w_2^k = z_2 w_1^k\}$. Note that $z_0 w_1^k = z_1 w_0^k$ and $z_1 w_2^k = z_2 w_1^k$ imply that $z_0 w_2^k = z_2 w_0^k$.

Let a, b , and c form a basis of \mathbb{Z}^3 . Let T^3 act on $M_1(k)$ by

$$g \cdot ([z_0 : z_1 : z_2 : z_3], [w_0 : w_1 : w_2])$$

$$= ([z_0 : g^{ka} z_1 : g^{k(a+b)} z_2 : g^{-c} z_3], [w_0 : g^a w_1 : g^{a+b} w_2])$$

for all $g \in T^3 \subset \mathbb{C}^3$. The action has 6 fixed points $p_1 = ([1 : 0 : 0 : 0], [1 : 0 : 0])$, $p_2 = ([0 : 1 : 0 : 0], [0 : 1 : 0])$, $p_3 = ([0 : 0 : 1 : 0], [0 : 0 : 1])$, $p_4 = ([0 : 0 : 0 : 1], [1 : 0 : 0])$, $p_5 = ([0 : 0 : 0 : 1], [0 : 1 : 0])$, and $p_6 = ([0 : 0 : 0 : 1], [0 : 0 : 1])$ that have weights $\{-c, a, a + b\}$, $\{-c - ka, -a, b\}$, $\{-c - k(a + b), -a - b, -b\}$, $\{c, a, a + b\}$, $\{c + ka, -a, b\}$, and $\{c + k(a + b), -a - b, -b\}$, respectively.

For instance, near p_1 , we have $z_0 \neq 0$, $w_0 \neq 0$, $\frac{z_1}{z_0} = (\frac{w_1}{w_0})^k$, and $\frac{z_2}{z_0} = (\frac{w_2}{w_0})^k$. Thus $(\frac{z_3}{z_0}, \frac{w_1}{w_0}, \frac{w_2}{w_0})$ are local coordinates at p_1 . Then T^3 acts on a neighborhood of p_1 by

$$\begin{aligned} g \cdot \left(\frac{z_3}{z_0}, \frac{w_1}{w_0}, \frac{w_2}{w_0} \right) &= \left(\frac{g^{-c} z_3}{z_0}, \frac{g^a w_1}{w_0}, \frac{g^{a+b} w_2}{w_0} \right) \\ &= \left(g^{-c} \frac{z_3}{z_0}, g^a \frac{w_1}{w_0}, g^{a+b} \frac{w_2}{w_0} \right). \end{aligned}$$

Therefore, the weights at p_1 are $\{-c, a, a + b\}$.

We have that p_1 has weight a , p_2 has weight $-a$, and p_1 and p_2 are in the same 2-sphere $F_{1,2} := \{([z_0 : z_1 : 0 : 0], [w_0 : w_1 : 0]) \mid z_0 w_1^k = z_1 w_0^k\}$ of $M^{\ker a}$. Thus, in a graph describing $M_1(k)$ with this action, there is an edge from p_1 to p_2 with label a .

By similar computations, one can see that Figure 1a describes $M_1(k)$ with this action.

The second example is an almost complex torus manifold described by Figure 1b.

Example 3.2. Fix integers k and l . By $M_2(k, l)$ we mean a compact complex manifold $M_2(k, l) = \{([z_0 : z_1 : z_2 : z_3 : z_4], [w_0 : w_1]) \in \mathbb{C}P^4 \times \mathbb{C}P^1 \mid z_0 w_1^k = z_3 w_0^k, z_2 w_1^{k+l} = z_4 w_0^{k+l}\}$. Let a, b , and c form a basis of \mathbb{Z}^3 . Let T^3 act on $M_2(k, l)$ by

$$\begin{aligned} g \cdot ([z_0 : z_1 : z_2 : z_3 : z_4], [w_0 : w_1]) \\ = ([z_0 : g^{-a} z_1 : g^b z_2 : g^{kc} z_3 : g^{b+(k+l)c} z_4], [w_0 : g^c w_1]) \end{aligned}$$

for all $g \in T^3 \subset \mathbb{C}^3$. The action has 6 fixed points $p_1 = ([1 : 0 : 0 : 0 : 0], [1 : 0])$, $p_2 = ([0 : 1 : 0 : 0 : 0], [1 : 0])$, $p_3 = ([0 : 1 : 0 : 0 : 0], [0 : 1])$, $p_4 = ([0 : 0 : 1 : 0 : 0], [1 : 0])$, $p_5 = ([0 : 0 : 0 : 1 : 0], [0 : 1])$, and $p_6 = ([0 : 0 : 0 : 0 : 1], [0 : 1])$ that have weights $\{-a, b, c\}$, $\{a, a + b, c\}$, $\{a + kc, a + b + (k + l)c, -c\}$, $\{-b, -a - b, c\}$, $\{-a - kc, b + lc, -c\}$, and $\{-a - b - (k + l)c, -b - lc, -c\}$, respectively.

For instance, near p_3 , we have $z_1 \neq 0$, $w_1 \neq 0$, $\frac{z_0}{z_1} = \frac{z_3}{z_1} (\frac{w_0}{w_1})^k$, and $\frac{z_2}{z_1} = \frac{z_4}{z_1} (\frac{w_0}{w_1})^{k+l}$. Thus, $(\frac{z_3}{z_1}, \frac{z_4}{z_1}, \frac{w_0}{w_1})$ are local coordinates at p_3 . Then T^3 acts on a neighborhood of p_3 by

$$\begin{aligned} g \cdot \left(\frac{z_3}{z_1}, \frac{z_4}{z_1}, \frac{w_0}{w_1} \right) &= \left(\frac{g^{kc} z_3}{g^{-a} z_1}, \frac{g^{b+(k+l)c} z_4}{g^{-a} z_1}, \frac{w_0}{g^c w_1} \right) \\ &= \left(g^{a+kc} \frac{z_3}{z_1}, g^{a+b+(k+l)c} \frac{z_4}{z_1}, g^{-c} \frac{w_0}{w_1} \right). \end{aligned}$$

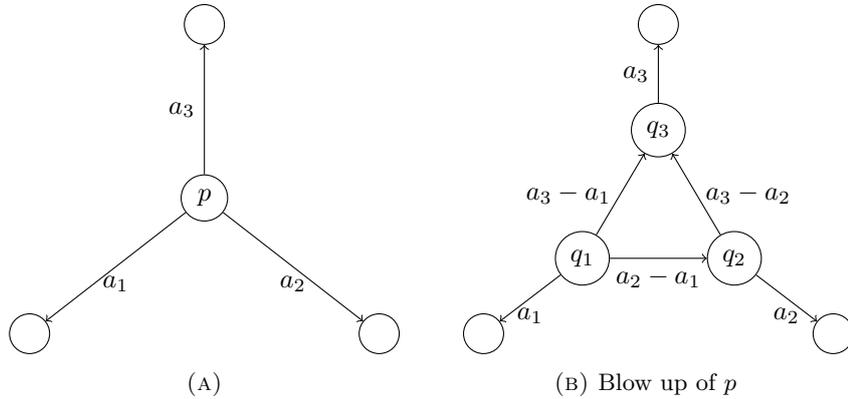


FIGURE 3. Blow up of M and its graph

Therefore, the weights at p_3 are $\{a + kc, a + b + (k + l)c, -c\}$.

We have that p_3 has weight $a + b + (k + l)c$, p_6 has weight $-a - b - (k + l)c$, and p_3 and p_6 are in the same 2-sphere $F'_{3,6} := \{([0 : z_1 : 0 : 0 : z_4], [0 : 1])\}$ of $M^{\ker a+b+(k+l)c}$. Thus, in a graph describing $M_2(k, l)$ with this action, there is an edge from p_3 to p_6 with label $a + b + (k + l)c$.

By similar computations, one can see that Figure 1b describes $M_2(k, l)$ with this action.

4. Blow up

In this section, we discuss blow up at a fixed point of an almost complex torus manifold and a graph describing it. We shall focus on dimension 6.

Let M be a 6-dimensional almost complex torus manifold. Let p be a fixed point and let a_1, a_2 , and a_3 be the weights at p . Then Figure 3a describes M near p . Now we shall blow up p . We can identify a neighborhood of p with a neighborhood of the origin in \mathbb{C}^3 , on which the torus T^3 acts by

$$g \cdot (z_1, z_2, z_3) = (g^{a_1} z_1, g^{a_2} z_2, g^{a_3} z_3)$$

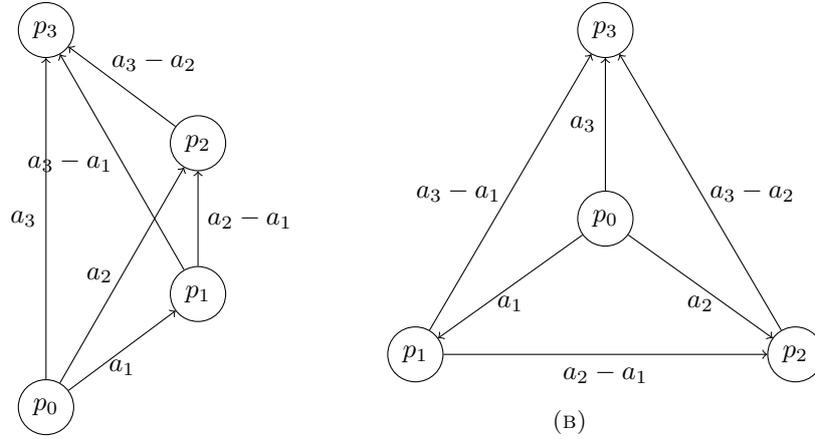
for all $g \in T^3$ and $(z_1, z_2, z_3) \in \mathbb{C}^3$. In this description, blowing up p corresponds to blowing up the origin in \mathbb{C}^3 , which is to replace the origin in \mathbb{C}^3 with all complex straight lines through it. Near p , the blown up space is described by

$$\{((z_1, z_2, z_3), [w_1, w_2, w_3]) \in \mathbb{C}^3 \times \mathbb{C}\mathbb{P}^2 \mid z_i w_j = z_j w_i, \forall i, j\}.$$

The T^3 -action near p naturally extends to act on the blown up space by

$$g \cdot ((z_1, z_2, z_3), [w_1, w_2, w_3]) = ((g^{a_1} z_1, g^{a_2} z_2, g^{a_3} z_3), [g^{a_1} w_1, g^{a_2} w_2, g^{a_3} w_3]),$$

which has 3 fixed points $q_1 = ((0, 0, 0), [1 : 0 : 0])$, $q_2 = ((0, 0, 0), [0 : 1 : 0])$, and $q_3 = ((0, 0, 0), [0 : 0 : 1])$ that have weights $\{a_1, a_2 - a_1, a_3 - a_1\}$,



(A)
FIGURE 4. Graph describing a standard T^3 -action on $\mathbb{C}\mathbb{P}^3$

$\{a_2, a_1 - a_2, a_3 - a_2\}$, and $\{a_3, a_1 - a_3, a_2 - a_3\}$, respectively. Let \widetilde{M} denote the blown up manifold. Then Figure 3b describes \widetilde{M} near the blow up of p .

Example 4.1. Let a_1, a_2 , and a_3 form a basis of \mathbb{Z}^3 . Let T^3 act on $\mathbb{C}\mathbb{P}^3$ by

$$g \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g^{a_1} z_1 : g^{a_2} z_2 : g^{a_3} z_3]$$

for all $g \in T^3 \subset \mathbb{C}^3$ and for all $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathbb{P}^3$. Then both Figure 4a and Figure 4b describe this linear standard T^3 -action on $\mathbb{C}\mathbb{P}^3$ as they are the same graph up to position of vertices. The fixed point $p_0 = [1 : 0 : 0 : 0]$ has weights $\{a_1, a_2, a_3\}$. Let $\widetilde{\mathbb{C}\mathbb{P}^3}$ denote blowing up of $\mathbb{C}\mathbb{P}^3$ at p_0 . As discussed above, in the blown up manifold $\widetilde{\mathbb{C}\mathbb{P}^3}$ the fixed point p_0 is replaced with three fixed points p_4, p_5 , and p_6 that have weights $\{a_1, a_2 - a_1, a_3 - a_1\}$, $\{a_2, a_1 - a_2, a_3 - a_2\}$, and $\{a_3, a_1 - a_3, a_2 - a_3\}$, respectively. Thus, Figure 5 describes $\widetilde{\mathbb{C}\mathbb{P}^3}$.

If we replace $a_2 - a_1, a_3 - a_2$, and a_1 with a, b , and c , respectively, then the T^3 -action on $\widetilde{\mathbb{C}\mathbb{P}^3}$ is the same as the action on $M_1(k)$ with $k = 1$ in Example 3.1, and Figure 5 is the same as Figure 1a with $k = 1$.

5. Hirzebruch genera

For a compact unitary manifold, the *Hirzebruch χ_y -genus*, the *Euler number*, the *Todd genus*, and the *L-genus* are the genera of power series $\frac{x(1+ye^{-x(1+y)})}{1-e^{-x(1+y)}}$, $1 + x, \frac{x}{1-e^{-x}}$, and $\frac{\sqrt{x}}{\tanh \sqrt{x}}$, respectively; the latter three power series are obtained from the first power series by evaluating at $y = -1, y = 0$, and $y = 1$. Therefore, the Hirzebruch χ_y -genus contains information on the Euler number, the Todd genus, and the *L-genus*.

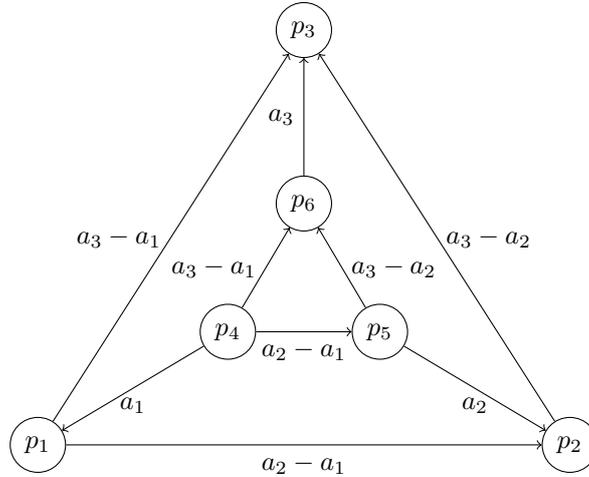


FIGURE 5

Let the circle group S^1 act on a unitary manifold. For a fixed component F , we denote by $d(-, F)$ ($d(+, F)$) the number of negative (positive) weights in the normal bundle of F .

Theorem 5.1 (Kosniowski formula, [4,13,15]). *Let the circle act on a compact unitary manifold M . Then the Hirzebruch χ_y -genus $\chi_y(M)$ of M satisfies*

$$\chi_y(M) = \sum_{F \subset M^{S^1}} (-y)^{d(-,F)} \cdot \chi_y(F) = \sum_{F \subset M^{S^1}} (-y)^{d(+,F)} \cdot \chi_y(F),$$

where the sum is taken over all fixed components.

Again, since any almost complex manifold is unitary, Theorem 5.1 holds for any compact almost complex S^1 -manifold. The below lemma is a simple application of the Kosniowski formula.

Lemma 5.2 ([10]). *Let a k -dimensional torus T^k act on a $2n$ -dimensional compact almost complex manifold M with isolated fixed points. Let $\chi_y(M) = \sum_{i=0}^n a_i(M) \cdot (-y)^i$ be the Hirzebruch χ_y -genus of M . Then the following hold:*

- (1) $a_i(M) \geq 0$ for $0 \leq i \leq n$.
- (2) The total number of fixed points is $\sum_{i=0}^n a_i(M)$.
- (3) For $0 \leq i \leq n$, $a_i(M) = a_{n-i}(M)$.

In [10], the first author used the Kosniowski formula to show that for an almost complex torus manifold, the coefficients of its Hirzebruch genus are non-zero.

Theorem 5.3 ([10]). *Let M be a $2n$ -dimensional almost complex torus manifold. Then $a_i(M) > 0$ for $0 \leq i \leq n$, where $\chi_y(M) = \sum_{i=0}^n a_i(M) \cdot (-y)^i$ is the Hirzebruch χ_y -genus of M . In particular, the Todd genus of M is positive.*

For a compact manifold equipped with a torus action, its Euler number is equal to the Euler number of its fixed point set.

Theorem 5.4 ([12]). *Let a torus T^k act on a compact manifold M . Then the Euler number $\chi(M)$ of M is equal to the sum of Euler numbers of its fixed components. That is,*

$$\chi(M) = \sum_{F \subset M^{T^k}} \chi(F).$$

To prove Theorem 1.1, we first determine the Hirzebruch χ_y -genus of a 6-dimensional almost complex torus manifold with Euler number 6.

Proposition 5.5. *Let M be a 6-dimensional almost complex torus manifold with Euler number 6. Then M has exactly 6 fixed points, and the Hirzebruch χ_y -genus $\chi_y(M)$ of M is either $\chi_y(M) = 1 - 2y + 2y^2 - y^3$ or $\chi_y(M) = 2 - y + y^2 - 2y^3$.*

Proof. Since there are only finitely many fixed points and the Euler number of a point is 1, by Theorem 5.4, M has exactly 6 fixed points.

Let $\chi_y(M) = \sum_{i=0}^3 a_i(M) \cdot (-y)^i$. By Lemma 5.2 and Theorem 5.3, it follows that $a_0(M) + \dots + a_3(M) = 6$, $a_0(M) = a_3(M) > 0$, and $a_1(M) = a_2(M) > 0$. These imply that either $a_0(M) = a_3(M) = 1$ and $a_1(M) = a_2(M) = 2$, or $a_0(M) = a_3(M) = 2$ and $a_1(M) = a_2(M) = 1$. □

Later, in Section 8 we shall show that in fact $\chi_y(M) = 1 - 2y + 2y^2 - y^3$. With the above, one important step for proving Theorem 1.1 is that for a 6-dimensional almost complex torus manifold with Euler number 6, a 4-dimensional isotropy submanifold contains at most 4 fixed points.

Lemma 5.6. *Let M be a 6-dimensional almost complex torus manifold with Euler number 6. Let p be a fixed point and let a, b, c be the weights at p . Let $S = \ker a \cap \ker b$ and let F_0 be a component of M^S containing p . Then F_0 is a 4-dimensional almost complex torus submanifold and has at most 4 fixed points.*

Proof. By Proposition 5.5, M has exactly 6 fixed points. Since the weights at p form a basis of \mathbb{Z}^3 , S is a 1-dimensional subtorus of T^3 , that is, S is a subcircle of T^3 . Since S only fixes a and b among the weights at p , it follows that F_0 is 4-dimensional. The 2-dimensional torus T^3/S acts effectively on F_0 , having p as a fixed point. Thus, F_0 is an almost complex torus submanifold¹.

Suppose on the contrary that F_0 has at least 5 fixed points. The T^3 -action on M restricts to act on F_0 . Since the T^3 -action on F_0 has at least 5 fixed

¹By Theorem 2.11, F_0 has at least 3 fixed points.

points, applying Lemma 5.2 to the T^3 -action on F_0 , we have that $a_0(F_0), a_1(F_0), a_2(F_0) \geq 0, a_0(F_0) + a_1(F_0) + a_2(F_0) \geq 5$, and $a_0(F_0) = a_2(F_0)$, where $\chi_y(F_0) = a_0(F_0) - a_1(F_0) \cdot y + a_2(F_0) \cdot y^2$. In particular, one of the following holds:

- (1) $a_0(F_0) = a_2(F_0)$ is bigger than or equal to 2.
- (2) $a_1(F_0)$ is bigger than or equal to 3.

Let F be any fixed component of the set M^S . The T^3 -action on F has isolated fixed points because its fixed point set is $F \cap M^{T^3}$. Applying Lemma 5.2 to the T^3 -action on F , we have that $a_i(F) \geq 0$ for $0 \leq i \leq \frac{1}{2} \dim F$, where $\chi_y(F) = \sum_{i=0}^{\frac{1}{2} \dim F} a_i(F) \cdot (-y)^i$.

Applying Theorem 5.1 to the S -action on M , we have

$$(5.1) \quad \chi_y(M) = \sum_{F \subset M^S} (-y)^{d(-, F)} \cdot \chi_y(F),$$

where for each fixed component $F \subset M^S$, $d(-, F)$ is the number of negative S -weights in the normal bundle of F .

Because F_0 is a 4-dimensional submanifold in a 6-dimensional manifold M , its normal bundle NF_0 has exactly one weight. Thus, either (a) $d(-, F_0) = 0$ or (b) $d(-, F_0) = 1$.

Suppose that Case (1) holds, that is, $a_0(F_0) = a_2(F_0) \geq 2$. Assume that (a) $d(-, F_0) = 0$. By Proposition 5.5, the Hirzebruch χ_y -genus of M is either (i) $\chi_y(M) = 1 - 2y + 2y^2 - y^3$ or (ii) $\chi_y(M) = 2 - y + y^2 - 2y^3$. The case (i) cannot hold because $a_0(F_0) \geq 2$ and $d(-, F_0) = 0$ imply $a_0(M) \geq 2$ by Equation (5.1); other fixed components F contribute the coefficients of the constant term of $\chi_y(M)$ by non-negative integers. Similarly, the case (ii) cannot hold because $a_2(F_0) \geq 2$ and $d(-, F_0) = 0$ imply $a_2(M) \geq 2$ by Equation (5.1).

Next, assume that (b) $d(-, F_0) = 1$. We cannot have (i) $\chi_y(M) = 1 - 2y + 2y^2 - y^3$ because $a_2(F_0) \geq 2$ and $d(-, F_0) = 1$ imply that $a_3(M) \geq 2$, and we cannot have (ii) $\chi_y(M) = 2 - y + y^2 - 2y^3$ because $a_0(F_0) \geq 2$ and $d(-, F_0) = 1$ imply that $a_1(M) \geq 2$.

Finally, Case (2) cannot hold; by Equation (5.1), $a_1(F_0) \geq 3$ implies that $a_1(M)$ or $a_2(M)$ must be at least 3, but $\chi_y(M) = 1 - 2y + 2y^2 - y^3$ or $\chi_y(M) = 2 - y + y^2 - 2y^3$. □

6. Graphs - dimension 6 and Euler number 6

We determine possible graphs describing a 6-dimensional almost complex torus manifold with Euler number 6; two possible types of graphs occur.

Proposition 6.1. *Let M be a 6-dimensional almost complex torus manifold with Euler number 6. Then one of graphs in Figure 6 describes M .*

Proof. By Proposition 5.5, M has exactly 6 fixed points. We label the fixed points by q_1, \dots, q_6 . By Proposition 2.6, there is a multigraph Γ describing M that has no multiple edges and no self-loops. To each fixed point q_i we assign

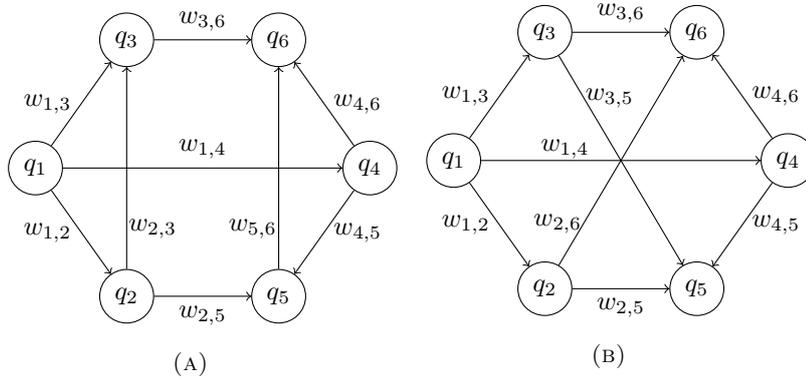


FIGURE 6. Possible graphs

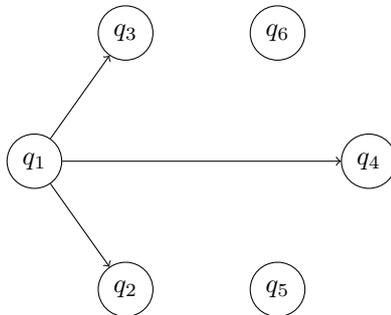


FIGURE 7

a vertex, also denoted by q_i . Without loss of generality, by relabeling vertices, we may assume that for each $i \in \{2, 3, 4\}$, there is an edge between q_1 and q_i , as in Figure 7.

Suppose that there is an edge e from q_i to q_j with label w . Let Γ' be a multigraph obtained from Γ by reversing the direction of the edge e and replacing the label w of the edge e with its negative $-w$. Then Γ' also describes M . Therefore, by reversing the direction of an edge and replacing the label of the edge with its negative, we may assume that if there is an edge from q_i to q_j , then $i < j$ (index increasing). If there is an edge $e_{i,j}$ from q_i to q_j , let $w_{i,j}$ be the label of the edge $e_{i,j}$.

We have following two cases.

- (A) There is an edge between q_2, q_3 , and q_4 .
- (B) There is no edge between q_2, q_3 , and q_4 .

Assume that Case (A) holds. Without loss of generality, by relabeling q_2, q_3 , and q_4 , we may assume that there is an edge from q_2 to q_3 . Then we show that

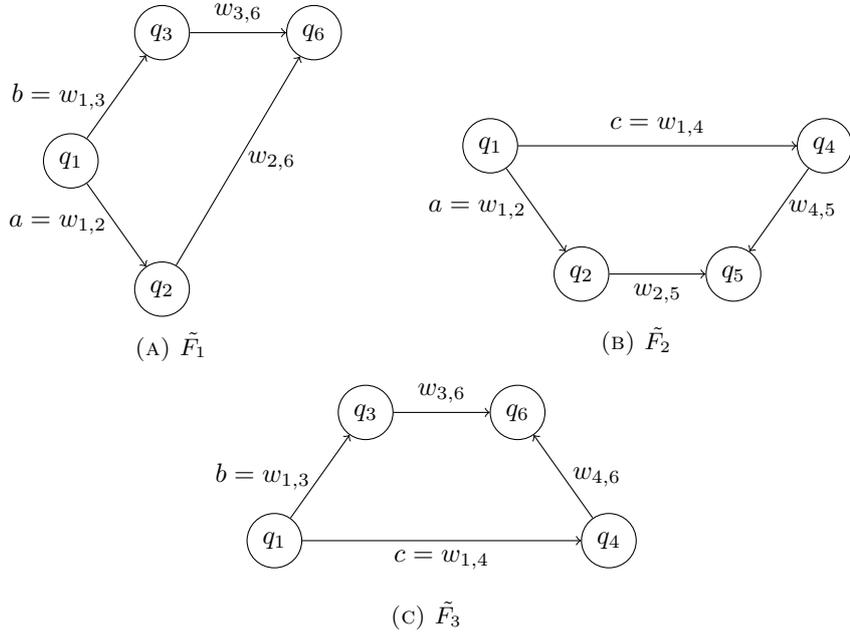


FIGURE 8

there does not exist an edge between q_2 and q_4 . To show this, suppose on the contrary that there is an edge between q_2 and q_4 . Since Γ has no self-loops and no multiple edges, the three edges at q_5 must be connected to q_3 , q_4 , and q_6 , respectively. But then q_6 must have a self-loop, which leads to a contradiction.

Similarly, by symmetry of Γ there is no edge between q_3 and q_4 . Then there must be an edge between p_4 and p_5 and an edge between p_4 and p_6 . Thus, the remaining edge of q_2 is connected to either q_5 or q_6 . Without loss of generality, assume that there is an edge between q_2 and q_5 . If we complete drawing the multigraph with the required properties, the multigraph Γ we get is Figure 6a.

Assume that Case (B) holds. Then for each $i \in \{2, 3, 4\}$ and for each $j \in \{5, 6\}$, there must be an edge between q_i and q_j . The multigraph Γ we get is Figure 6b. \square

We show that Figure 6b does not occur as a multigraph describing an almost complex torus manifold.

Proposition 6.2. *Figure 6b cannot describe an almost complex torus manifold.*

Proof. Assume on the contrary that Figure 6b describes an almost complex torus manifold M ; then $\dim M = 6$ and there are 6 fixed points.

Let $w_{1,2} = a$, $w_{1,3} = b$, and $w_{1,4} = c$. Let $\tilde{S}_1 := \ker a \cap \ker b$ and consider the isotropy submanifold \tilde{F}_1 of $M^{\tilde{S}_1}$ that contains q_1 . By Lemma 5.6, \tilde{F}_1 is a 4-dimensional almost complex torus submanifold of M .

By Proposition 2.4, some 2-valent sub-multigraph $\Gamma_{\tilde{F}_1}$ of Figure 6b describes the T^3 -action on \tilde{F}_1 . Because there is an edge from q_1 to q_2 (from q_1 to q_3) with label a (b) that satisfies $g^a = 1$ ($g^b = 1$) for all $g \in \tilde{S}_1$, Proposition 2.4 (2) implies that $q_2 \in \tilde{F}_1$ ($q_3 \in \tilde{F}_1$). Moreover, since $\Gamma_{\tilde{F}_1}$ is 2-valent, two edges of q_2 (q_3) must be in $\Gamma_{\tilde{F}_1}$. On the other hand, by Lemma 5.6, $\Gamma_{\tilde{F}_1}$ contains at most 4 vertices. Thus, $\Gamma_{\tilde{F}_1}$ cannot contain q_4 , and must contain exactly one of q_5 and q_6 .

Without loss of generality, by vertical symmetry of Figure 6b, we may assume that \tilde{F}_1 contains q_6 . Then Figure 8a describes the T^3 -action on \tilde{F}_1 . Applying Lemma 2.10 to the T^3 -action on \tilde{F}_1 with $w = a$ ($w = b$), since q_1 and q_2 (q_1 and q_3) are in the same component of $M^{\ker a}$ ($M^{\ker b}$), the T^3 -weights $\{a, b\}$ in $T_{q_1}\tilde{F}_1$ and $\{-a, w_{2,6}\}$ in $T_{q_2}\tilde{F}_1$ ($\{a, b\}$ in $T_{q_1}\tilde{F}_1$ and $\{-b, w_{3,6}\}$ in $T_{q_3}\tilde{F}_1$) are equal modulo a (modulo b). Thus $w_{2,6} = b + k_1a$ for some integer k_1 ($w_{3,6} = a + k_2b$ for some integer k_2).

Second, let \tilde{F}_2 be the 4-dimensional component of $M^{\tilde{S}_2}$ that contains q_1 , where $\tilde{S}_2 = \ker a \cap \ker c$. By Proposition 2.4 and Lemma 5.6, some 2-valent sub-multigraph $\Gamma_{\tilde{F}_2}$ of Figure 6b describes the T^3 -action on \tilde{F}_2 and contains at most 4 vertices. Thus $\Gamma_{\tilde{F}_2}$ contains exactly one of q_5 and q_6 . Suppose that $\Gamma_{\tilde{F}_2}$ contains q_6 . Then the weights $\{-w_{4,6}, -b - k_1a\}$ in $T_{q_6}\tilde{F}_2$ are linear combinations of a and c , which leads to a contradiction since a, b, c form a basis of \mathbb{Z}^3 . Thus \tilde{F}_2 contains q_1, q_2, q_4 , and q_5 ; see Figure 8b. Applying Lemma 2.10 to the T^3 -action on \tilde{F}_2 with $w = c$, since q_1 and q_4 are in the same component of $M^{\ker c}$, the T^3 -weights $\{a, c\}$ in $T_{q_1}\tilde{F}_2$ and $\{-c, w_{4,5}\}$ in $T_{q_4}\tilde{F}_2$ are equal modulo c . Thus $w_{4,5} = a + k_3c$ for some integer k_3 .

Third, let \tilde{F}_3 be the 4-dimensional component of $M^{\tilde{S}_3}$ that contains q_1 , where $\tilde{S}_3 = \ker b \cap \ker c$. By Proposition 2.4 and Lemma 5.6, some 2-valent sub-multigraph $\Gamma_{\tilde{F}_3}$ of Figure 6b describes the T^3 -action on \tilde{F}_3 and contains at most 4 vertices. Thus $\Gamma_{\tilde{F}_3}$ contains exactly one of q_5 and q_6 . Suppose that \tilde{F}_3 contains q_5 . Then the weights $\{-w_{3,5}, -a - k_3c\}$ in $T_{q_5}\tilde{F}_3$ are linear combinations of b and c , which leads to a contradiction. Therefore, \tilde{F}_3 contains q_6 ; see Figure 8c. Then the weights $\{-b, a + k_2b\}$ in $T_{q_3}\tilde{F}_3$ are linear combinations of b and c , which leads to a contradiction. □

Next, for a 6-dimensional almost complex torus manifold with Euler number 6, we study 4-dimensional almost complex torus submanifolds and their sub-multigraphs.

Lemma 6.3. *Let M be an almost complex torus manifold described by Figure 6a. Then there exist 4-dimensional almost complex torus submanifolds F_1, \dots, F_5 described by Figure 9a, ..., 9e, respectively.*

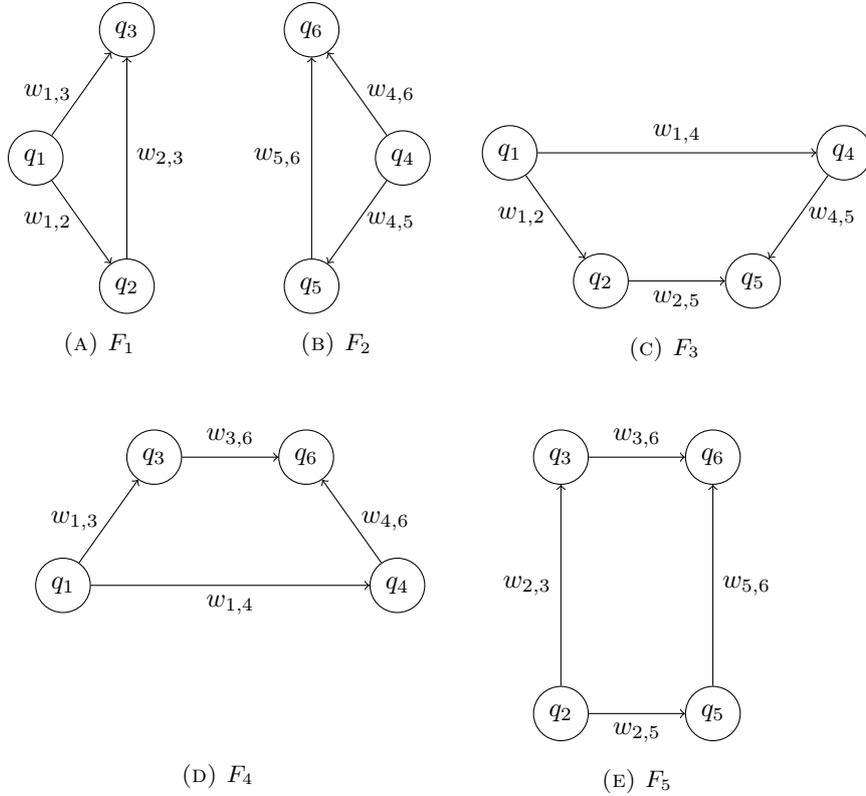


FIGURE 9

Proof. The idea of proof is similar to the one of Proposition 6.2.

Let F_1 be the 4-dimensional component of $M^{\ker w_{1,2} \cap \ker w_{1,3}}$ that contains q_1 . By Proposition 2.4, some 2-valent sub-multigraph Γ_{F_1} of Figure 6a describes the T^3 -action on F_1 . Since there is an edge from q_1 to q_2 with label $w_{1,2}$, $q_2 \in F_1$ and similarly $q_3 \in F_1$. By Lemma 5.6, Γ_{F_1} contains at most 4 vertices. If Γ_{F_1} contains one of q_4, q_5 , and q_6 , then Γ_{F_1} must contain at least 5 vertices. Thus, Γ_{F_1} is Figure 9a.

All the other cases are analogous. If we let F_2 (F_3, F_4 , and F_5) be the 4-dimensional component of $M^{\ker w_{4,5} \cap \ker w_{4,6}}$ ($M^{\ker w_{1,2} \cap \ker w_{1,4}}$, $M^{\ker w_{1,3} \cap \ker w_{1,4}}$, and $M^{\ker w_{2,3} \cap \ker w_{2,5}}$) that contains q_4 (q_1, q_1 , and q_2), then by Proposition 2.4 and Lemma 5.6, Figure 9b (Figure 9c, Figure 9d, and Figure 9e) describes the T^3 -action on F_2 (F_3, F_4 , and F_5 , respectively).

For each $1 \leq i \leq 5$, quotienting out by the subgroup of T^3 that acts trivially on F_i , it follows that F_i is a 4-dimensional almost complex torus manifold. \square

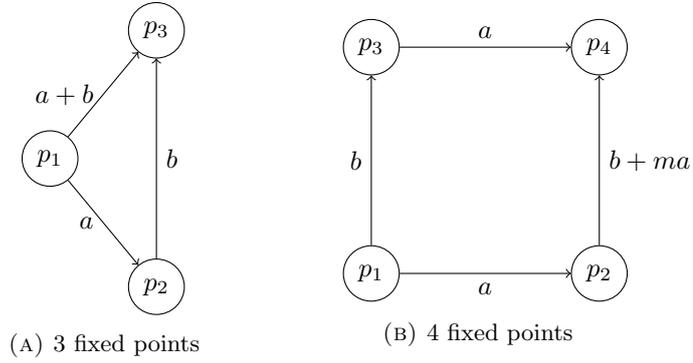


FIGURE 10

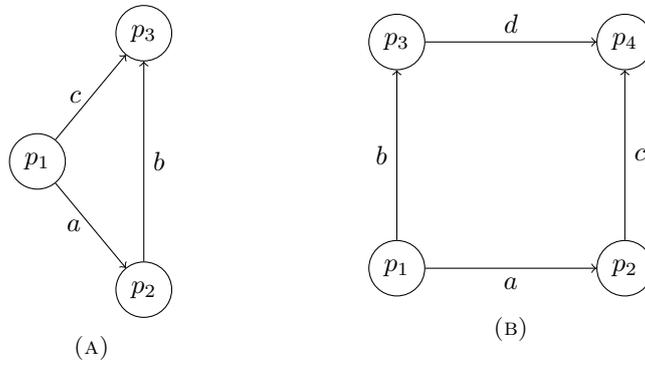


FIGURE 11

7. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For this, we first determine weights at the fixed points, for a 4-dimensional almost complex torus manifold with at most 4 fixed points; by Theorem 2.11, such a manifold has at least 3 fixed points.

Lemma 7.1. *Let M be a 4-dimensional almost complex torus manifold.*

- (1) *If there are 3 fixed points, Figure 10a describes M for some a and b that span \mathbb{Z}^2 .*
- (2) *If there are 4 fixed points, Figure 10b describes M for some a and b that span \mathbb{Z}^2 and for some integer m .*

Proof. Suppose that M has 3 fixed points $p_1, p_2,$ and p_3 . By Proposition 2.6, Figure 11a describes M for some non-zero $a, b, c \in \mathbb{Z}^2$. In particular, $p_1, p_2,$ and p_3 have weights $\{a, c\}, \{-a, b\},$ and $\{-b - c\},$ respectively.

Since \int_M is a map from $H_{T^2}^i(M; \mathbb{Z})$ to $H^{i-4}(BT^2; \mathbb{Z})$ for all $i,$ the image of the equivariant cohomology class 1 under the map \int_M is zero, that is,

$$\int_M 1 = 0.$$

By Theorem 2.9,

$$\int_M 1 = \sum_{p \in M^{T^2}} \frac{1}{e_{T^2}(N_p M)} = \sum_{p \in M^{T^2}} \frac{1}{e_{T^2}(T_p M)} = \frac{1}{ac} + \frac{1}{-ab} + \frac{1}{(-b)(-c)}.$$

Therefore, $c = a + b$.

Next, suppose that M has 4 fixed points $p_1, p_2, p_3,$ and p_4 . By Proposition 2.6, Figure 11b describes M for some non-zero $a, b, c, d \in \mathbb{Z}^2$. In particular, $p_1, p_2, p_3,$ and p_4 have weights $\{a, b\}, \{-a, c\}, \{-b, d\},$ and $\{-c, -d\},$ respectively.

Applying Theorem 2.9 to the equivariant cohomology class 1,

$$\begin{aligned} 0 = \int_M 1 &= \sum_{p \in M^{T^2}} \frac{1}{e_{T^2}(N_p M)} = \sum_{p \in M^{T^2}} \frac{1}{e_{T^2}(T_p M)} \\ &= \frac{1}{ab} + \frac{1}{-ac} + \frac{1}{-bd} + \frac{1}{(-c)(-d)} \\ &= \left(\frac{1}{a} - \frac{1}{d}\right) \left(\frac{1}{b} - \frac{1}{c}\right). \end{aligned}$$

Therefore, $a = d$ or $b = c$.

Suppose that $a = d$. Because there is an edge from p_1 to p_2 with label $a,$ p_1 and p_2 are in the same component F of $M^{\ker a}$. Applying Lemma 2.10 to the T^2 -action on $F,$ taking $w = a,$ the weights $\{a, b\}$ at p_1 and the weights $\{-a, c\}$ at p_2 are equal modulo a . Thus, $c = b + ma$ for some integer m .

Similarly, if $b = c,$ it follows from Lemma 2.10 that $a = b + mb$ for some integer $m;$ in this case, by relabeling fixed points and weights, this lemma follows. \square

From Lemma 7.1, we obtain the following lemma.

Lemma 7.2. *Suppose that Figure 6a describes an almost complex torus manifold. Then the followings hold.*

- (1) $w_{1,2} + w_{2,3} = w_{1,3}.$
- (2) $w_{4,5} + w_{5,6} = w_{4,6}.$
- (3) $w_{1,2} = w_{4,5}$ and $w_{1,4} \equiv w_{2,5} \pmod{w_{1,2}},$ or $w_{1,4} = w_{2,5}$ and $w_{1,2} \equiv w_{4,5} \pmod{w_{1,4}}.$
- (4) $w_{1,3} = w_{4,6}$ and $w_{1,4} \equiv w_{3,6} \pmod{w_{1,3}},$ or $w_{1,4} = w_{3,6}$ and $w_{1,3} \equiv w_{4,6} \pmod{w_{1,4}}.$

- (5) $w_{2,3} = w_{5,6}$ and $w_{2,5} \equiv w_{3,6} \pmod{w_{2,3}}$, or $w_{2,5} = w_{3,6}$ and $w_{2,3} \equiv w_{5,6} \pmod{w_{2,5}}$.

Proof. By Lemma 6.3, Figure 9a describes the 4-dimensional almost complex torus manifold F_1 . Thus, by Lemma 7.1, $w_{1,2} + w_{2,3} = w_{1,3}$.

Similarly, since Figures 9b, . . . , 9e describe the 4-dimensional almost complex torus manifolds F_2, \dots, F_5 , respectively, all the other cases follow from Lemma 7.1. \square

With all of the above, we are ready to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. By Proposition 6.1 and Proposition 6.2, Figure 6a describes M . By Lemma 6.3, there exist 4-dimensional almost complex torus submanifolds F_1, \dots, F_5 described by Figures 9a, . . . , 9e, respectively. For convenience, let $a := w_{1,2}$, $b := w_{2,3}$, and $c := w_{1,4}$. By Lemma 7.2(1), $w_{1,3} = a + b$. The fixed point q_1 has weights $\{a, a + b, c\}$ that span \mathbb{Z}^3 , and hence a, b, c span \mathbb{Z}^3 .

By Lemma 7.2(3) we have the following two cases.

- (A) $w_{4,5} = a$ and $w_{2,5} = c + ka$ for some integer k .
- (B) $w_{2,5} = c$ and $w_{4,5} = a + kc$ for some integer k .

Suppose that Case (A) holds. By Lemma 7.2(4) we have the following two cases.

- (i) $w_{4,6} = a + b$ and $w_{3,6} = c + l(a + b)$ for some integer l .
- (ii) $w_{3,6} = c$ and $w_{4,6} = a + b + lc$ for some integer l .

Suppose that Case (A-i) holds. By Lemma 7.2(2) it follows that $w_{5,6} = b$. Since $w_{2,3} = w_{5,6} = b$, by Lemma 7.2(5) it follows that $c + ka = w_{2,5} \equiv w_{3,6} = c + l(a + b) \pmod{w_{2,3}} = b$. Since a, b, c form a basis of \mathbb{Z}^3 , we have $l = k$ and Figure 1a describes M .

Suppose that Case (A-ii) holds. By Lemma 7.2(2) it follows that $w_{5,6} = b + lc$. By Lemma 7.2(5), (A-ii-1) $b = w_{2,3} = w_{5,6} = b + lc$ and $c + ka = w_{2,5} \equiv w_{3,6} = c \pmod{w_{2,3}} = b$, or (A-ii-2) $c + ka = w_{2,5} = w_{3,6} = c$ and $b = w_{2,3} \equiv w_{5,6} = b + lc \pmod{w_{2,5}} = c + ka$. Since a, b, c form a basis of \mathbb{Z}^3 , in the former case (A-ii-1) we have $l = 0$ and $k = 0$ and both Figure 1a with $k = 0$ and Figure 1b with $k = l = 0$ describe M , and in the latter case (A-ii-2) we have $k = 0$ and Figure 1b with $k = 0$ describes M .

Suppose that Case (B) holds. By Lemma 7.2(5) we have the following two cases.

- (i) $w_{5,6} = b$ and $w_{3,6} = c + lb$ for some integer l .
- (ii) $w_{3,6} = c$ and $w_{5,6} = b + lc$ for some integer l .

Suppose that Case (B-i) holds. By Lemma 7.2(2) it follows that $w_{4,6} = a + b + kc$. By Lemma 7.2(4), (B-i-1) $a + b = w_{1,3} = w_{4,6} = a + b + kc$ and $c = w_{1,4} \equiv w_{3,6} = c + lb \pmod{w_{1,3}} = a + b$, or (B-i-2) $c = w_{1,4} = w_{3,6} = c + lb$ and $a + b = w_{1,3} \equiv w_{4,6} = a + b + kc \pmod{w_{1,4}} = c$. Because a, b, c span \mathbb{Z}^3 , in the former case (B-i-1) we have $k = l = 0$ and both Figure 1a with $k = 0$ and

Figure 1b with $k = l = 0$ describe M , and in the latter case (B-i-2) we have $l = 0$ and Figure 1b with $l = 0$ describes M .

Suppose that Case (B-ii) holds. By Lemma 7.2(2) it follows that $w_{4,6} = a + b + (k + l)c$. In this case, Figure 1b describes M . \square

8. Invariants

In this section, we prove Theorem 1.5; we determine invariants of 6-dimensional almost complex torus manifolds with Euler number 6.

Proof of Theorem 1.5. By Proposition 5.5, M has 6 fixed points. Because $\dim M = 6$, for a fixed point p , the third equivariant Chern class $c_3(M)$ at p is the same as the equivariant Euler class of the normal space at p , which is the product of the weights at p . That is, $c_3(M)(p) = e_{T^3}(Np) = \prod_{i=1}^3 w_{p,i}$. Therefore, by Theorem 2.9, the Chern number $\int_M c_3$ is

$$\int_M c_3 = \sum_{p \in M^{T^3}} \frac{c_3(M)(p)}{e_{T^3}(Np)} = \sum_{p \in M^{T^3}} \frac{\prod_{i=1}^3 w_{p,i}}{\prod_{i=1}^3 w_{p,i}} = \sum_{p \in M^{T^3}} 1 = 6.$$

First, suppose that Figure 1a describes M . The weights at the fixed points p_1, \dots, p_6 are $\{-a, b, c + ka\}$, $\{a, a + b, c\}$, $\{-b, -a - b, c + k(a + b)\}$, $\{-c - k(a + b), -a - b, -b\}$, $\{-c, a + b, a\}$, $\{-c - ka, b, -a\}$, respectively.

For a non-negative integer i and real numbers x_1, x_2, \dots, x_n , let $\sigma_i(x_1, x_2, \dots, x_n)$ denote the i -th elementary symmetric polynomial in x_j . For instance, $\sigma_0(x_1, \dots, x_n) = 1$, $\sigma_1(x_1, \dots, x_n) = \sum_{i=1}^n x_i$, and $\sigma_2(x_1, \dots, x_n) = \sum_{i < j} x_i x_j$.

By Theorem 2.9,

$$\begin{aligned} \int_M c_1 c_2 &= \sum_{p \in M^{T^3}} \frac{\sigma_1(w_{p,1}, w_{p,2}, w_{p,3}) \sigma_2(w_{p,1}, w_{p,2}, w_{p,3})}{\prod_{i=1}^3 w_{p,i}} \\ &= - \frac{\{(k-1)a + b + c\} \{-ka^2 + (k-1)ab - ac + bc\}}{ab(c + ka)} \\ &\quad + \frac{(2a + b + c)(a^2 + ab + 2ac + bc)}{a(a + b)c} \\ &\quad - \frac{\{(k-1)a + (k-2)b + c\} \{ka^2 + (3k-1)ab + ac + (2k-1)b^2 + 2bc\}}{b(a + b)\{c + k(a + b)\}} \\ &\quad + \frac{\{(k+1)a + (k+2)b + c\} \{ka^2 + (3k+1)ab + ac + (2k+1)b^2 + 2bc\}}{\{c + k(a + b)\}(a + b)b} \\ &\quad - \frac{(2a + b - c)(a^2 + ab - 2ac - bc)}{c(a + b)a} \\ &\quad + \frac{\{(-k-1)a + b - c\} \{ka^2 + (-k-1)ab + ac - bc\}}{(c + ka)ba}. \end{aligned}$$

Combining the terms that have the same denominator, this is equal to

$$\begin{aligned} & \frac{2(-a^2+3ab-b^2)(ka+c)}{ab(c+ka)} + \frac{2(5a^2+5ab+b^2)c}{a(a+b)c} + \frac{2(a^2+5ab+5b^2)(ka+kb+c)}{(a+b)b\{c+k(a+b)\}} \\ &= \frac{2}{ab(a+b)} \{(-a^2+3ab-b^2) \times (a+b) + (5a^2+5ab+b^2) \times b + (a^2+5ab+5b^2) \times a\} \\ &= \frac{2}{ab(a+b)} \times 12ab(a+b) = 24. \end{aligned}$$

Therefore, $\int_M c_1 c_2 = 24$.

Since $\int_M T_0^3 = \int_M -T_3^3 = \int_M \frac{c_1 c_2}{24} = 1$ and $\int_M T_1^3 = \int_M -T_2^3 = \int_M \frac{c_1 c_2 - 12c_3}{24} = -2$, the Hirzebruch χ_y -genus of M is

$$\chi_y(M) = \sum_{i=0}^3 \left(\int_M T_i^3 \right) y^i = 1 - 2y + 2y^2 - y^3.$$

Next we compute $\int_M c_1^3$. By Theorem 2.9,

$$\begin{aligned} \int_M c_1^3 &= \sum_{p \in M^{T^3}} \frac{\sigma_1(w_{p,1}, w_{p,2}, w_{p,3})^3}{\prod_{i=1}^3 w_{p,i}} \\ &= -\frac{\{(k-1)a+b+c\}^3}{ab(c+ka)} + \frac{(2a+b+c)^3}{a(a+b)c} \\ &\quad + \frac{\{(k-1)a+(k-2)b+c\}^3}{b(a+b)\{c+k(a+b)\}} - \frac{\{(-k-1)a+(-k-2)b-c\}^3}{\{c+k(a+b)\}(a+b)b} \\ &\quad - \frac{(2a+b-c)^3}{c(a+b)a} + \frac{\{(-k-1)a+b-c\}^3}{(c+ka)ba}. \end{aligned}$$

Combining the terms that have the same denominator, this is equal to

$$\begin{aligned} & \frac{-2\{(k^2+3)a^2-6ab+2kac+3b^2+c^2\}}{ab} + \frac{2(12a^2+12ab+3b^2+c^2)}{a(a+b)} \\ &+ \frac{2\{(k^2+3)a^2+(2k^2+12)ab+2kac+(k^2+12)b^2+2kbc+c^2\}}{b(a+b)} \\ &= \frac{2}{ab(a+b)} \times [-\{(k^2+3)a^2-6ab+2kac+3b^2+c^2\} \times (a+b) \\ &\quad + (12a^2+12ab+3b^2+c^2) \times b \\ &\quad + \{(k^2+3)a^2+(2k^2+12)ab+2kac+(k^2+12)b^2+2kbc+c^2\} \times a] \\ &= \frac{2}{ab(a+b)} \times (k^2+27)ab(a+b) = 2(k^2+27). \end{aligned}$$

Second, suppose that Figure 1b describes M . The weights at the fixed points p_1, \dots, p_6 are $\{-a, b, c\}$, $\{a, a+b, c\}$, $\{-b, -a-b, c\}$, $\{-c, -a-b-(k+l)c, -b-lc\}$, $\{-c, a+b+(k+l)c, a+kc\}$, $\{-c, b+lc, -a-kc\}$, respectively. We shall replace $a+kc$, $b+lc$, and $-c$ by A , B , and C , respectively. Then the weights at

the latter three fixed points are $\{C, -A - B, -B\}$, $\{C, A + B, A\}$, $\{C, B, -A\}$, which are of the same form as the former three fixed points. By Theorem 2.9,

$$\begin{aligned} \int_M c_1 c_2 &= \sum_{p \in M^{T^3}} \frac{\sigma_1(w_{p,1}, w_{p,2}, w_{p,3}) \sigma_2(w_{p,1}, w_{p,2}, w_{p,3})}{\prod_{i=1}^3 w_{p,i}} \\ &= \frac{(-a + b + c)(-ab - ac + bc)}{-abc} + \frac{(2a + b + c)(a^2 + ab + ac + bc)}{a(a + b)c} \\ &\quad + \frac{(-a - 2b + c)(ab - ac + b^2 - 2bc)}{b(a + b)c} \\ &\quad + \frac{(-A - 2B + C)(AB - AC + B^2 - 2BC)}{B(A + B)C} \\ &\quad + \frac{(2A + B + C)(A^2 + AB + AC + BC)}{A(A + B)C} \\ &\quad + \frac{(-A + B + C)(-AB - AC + BC)}{-ABC}. \end{aligned}$$

Computing the first three terms, we get

$$\begin{aligned} &\frac{1}{ab(a + b)c} \times \{ -(-a + b + c)(-ab - ac + bc) \times (a + b) \\ &\quad + (2a + b + c)(a^2 + ab + ac + bc) \times b \\ &\quad + (-a - 2b + c)(ab - ac + b^2 - 2bc) \times a \} \\ &= \frac{1}{ab(a + b)c} \times 12abc(a + b) = 12. \end{aligned}$$

Hence the sum of the latter three terms is also 12. Therefore, $\int_M c_1 c_2 = 24$. As in the first case, $\int_M T_0^3 = \int_M -T_3^3 = \int_M \frac{c_1 c_2}{24} = 1$ and $\int_M T_1^3 = \int_M -T_2^3 = \int_M \frac{c_1 c_2 - 12c_3}{24} = -2$ and so the Hirzebruch χ_y -genus of M is

$$\chi_y(M) = \sum_{i=0}^3 \left(\int_M T_i^3 \right) y^i = 1 - 2y + 2y^2 - y^3.$$

We compute $\int_M c_1^3$. By Theorem 2.9,

$$\begin{aligned} \int_M c_1^3 &= \sum_{p \in M^{T^3}} \frac{\sigma_1(w_{p,1}, w_{p,2}, w_{p,3})^3}{\prod_{i=1}^3 w_{p,i}} \\ &= \frac{(-a + b + c)^3}{-abc} + \frac{(2a + b + c)^3}{a(a + b)c} + \frac{(-a - 2b + c)^3}{b(a + b)c} \\ &\quad + \frac{(-A - 2B + C)^3}{B(A + B)C} + \frac{(2A + B + C)^3}{A(A + B)C} + \frac{(-A + B + C)^3}{-ABC}. \end{aligned}$$

Computing the first three terms, we get

$$\begin{aligned} & \frac{1}{abc(a+b)} \times \{ -(-a+b+c)^3 \times (a+b) + (2a+b+c)^3 \times b + (-a-2b+c)^3 \times a \} \\ &= \frac{1}{abc(a+b)} \times 27abc(a+b) = 27. \end{aligned}$$

Thus, the sum of the latter three terms is also 27. Therefore, $\int_M c_1^3 = 54$. \square

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