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QUASILINEAR SCHRÖDINGER EQUATIONS FOR THE HEISENBERG FERROMAGNETIC SPIN CHAIN

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Abstract. In this paper, we consider a model problem arising from a classical planar Heisenberg ferromagnetic spin chain

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \lambda|u|^{p-2}u,\ x \in \mathbb{R}^N,$$

where $2 \leq p < 2^*, N \geq 3$. By the Ekeland variational principle, the cut off technique, the change of variables and the L^∞ estimate, we study the existence of positive solutions. Here, we construct the L^∞ estimate of the solution in an entirely different way. Particularly, all the constants in the expression of this estimate are so well known.

1. Introduction

This paper is concerned with the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + W(x)z - \rho(|z|^2)z - \kappa\Delta l(|z|^2)l'(|z|^2)z,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $W(x)$ is a given potential, κ is a real constant and ρ, l are real functions of essentially pure power forms. Quasilinear equations of the form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of l. For instance, the case of $l(s) = s$ is used for the superfluid film equation in plasma physics [5]. In the case $l(s) = (1+s)^{\frac{1}{2}}$, (1.1) models the self-channeling of a high-power ultra short laser in mater [7]. If $l(s) = (1-s)^{\frac{1}{2}}$, (1.1) also appears in the theory of the Heisenberg ferromagnetic spin chain. We refer to [2, 3, 12, 17] and their references for more details on this subject.

Here, our special interest is in the existence of standing wave solutions, that is, solutions of type $\phi(x,t) = \exp(iFt)u(x)$, where $F \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that ϕ satisfies (1.1) if and only if the function u

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solves the following equation of the elliptic type

$$-\Delta u + V(x)u - \kappa\Delta l(u^2)l'(u^2)u = \rho(u^2)u,\ x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x)+F$ is the new potential function. If we let $l(s) = (1-s)^{\frac{1}{2}}$, $\rho(s) = \varepsilon'(1-s)^{-\frac{1}{2}}$ and $V(x) = \lambda + \varepsilon'$, we get the equation

$$-\Delta u + \lambda u - \frac{\kappa u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \varepsilon'\frac{u}{\sqrt{1-u^2}} - \varepsilon' u,\ x \in \mathbb{R}^N, \tag{1.3}$$

which originally appears in the Heisenberg ferromagnetic spin chain. In the mathematical literature, few results are known on (1.3). In one dimensional space, Brüll et al. [4] studied the ground states u for (1.3) with $\lim_{|x|\to\infty} u(x) = 0$. For higher dimensional space, in [17], Takeno, Homma constructed the expression of the solution to boundary value problems for second order nonlinear ordinary differential equations.

More recently, Wang consider the following quasilinear Schrödinger equation:

$$-\Delta u + \lambda u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \varepsilon'\frac{u}{\sqrt{1-u^2}} - \varepsilon' u,\ x \in \mathbb{R}^3. \tag{1.4}$$

In [18], Wang generalized the result given in [4] to three dimensional space.

In the case $l(s) = s^\alpha$, $\rho(s) = \lambda s^{\frac{p-1}{2}}$, Liu and Wang in [8] first studied the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \alpha\kappa|u|^{2\alpha-2}u\Delta u^{2\alpha} = \lambda|u|^{p-1}u,\ x \in \mathbb{R}^N \tag{1.5}$$

by a minimization argument. In [8], the authors proved that (1.5) has a solution for a sequence of $\lambda_n \to \infty$ and a sequence of $\lambda_n \to 0$ if $\alpha > \frac{1}{2}$ and $4\alpha \le p+1 < 2\alpha 2^*$.

The main objective of the present paper is to study the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \lambda|u|^{p-2}u,\ x \in \mathbb{R}^N, \tag{1.6}$$

that is, the case $l(s) = (1-s)^{\frac{1}{2}}$, $\rho(s) = \lambda s^{\frac{p-2}{2}}$. To the best of our knowledge, up to now there are no results for (1.6) on $\mathbb{R}^N$ not only for the superlinear case, i.e., $p > 2$, but for the eigenvalue problem, that is, $p = 2$.

We observe that the minimizer of the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\left(1 - \frac{u^2}{1-u^2}\right)|\nabla u|^2 + V(x)u^2\right]\mathrm{d}x \tag{1.7}$$

constrained on the manifold

$$\Sigma = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}|u|^p\mathrm{d}x = 1\right\}$$

solves the Euler-Lagrange equation (1.6). From the variational point of view, there exist two difficulties to overcome for this functional (1.7). One is that the functional is not well defined in $H^1(\mathbb{R}^N)$. The other is how to guarantee the positiveness of the principle part. In order to overcome these two difficulties, we will focus the following functional

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[\left(1 - \frac{\kappa u^2}{1-\kappa u^2}\right)|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x, \tag{1.8}$$

where $\kappa > 0$ is a constant. Obviously, if

$$u_0 = \inf_{u\in\Sigma} I_0(u),$$

then u_0 solves the equation

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-\kappa u^2}}\Delta\sqrt{1-\kappa u^2} = \lambda'|u|^{p-2}u, \ x \in \mathbb{R}^N, \tag{1.9}$$

where $\lambda' = \lambda\kappa^{\frac{p-2}{2}}$. For the solution u_κ of (1.9), we rescale $u_0 = \kappa^{-\frac{1}{2}}u$. Then u satisfies (1.6). Furthermore, according to [14], (1.9) can be reformulated as the following problems of the form

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda'|u|^{p-2}u, \ x \in \mathbb{R}^N, \tag{1.10}$$

where $g(t) = \sqrt{1 - \frac{\kappa t^2}{1-\kappa t^2}}$. Now, by using the cut off technique introduced by Wang [18], we continuously extend the domain of the function $g(t)$ to all of $[0, +\infty)$. More precisely, we consider the function

$$g_\kappa(t) = \begin{cases} \sqrt{1 - \frac{\kappa t^2}{1-\kappa t^2}}, & \text{if } 0 \le t < \frac{1}{\sqrt{\theta\kappa}}; \\ \sqrt{\frac{2\theta}{(\theta-1)^2\sqrt{\theta\kappa}t} + \frac{\theta^2-5\theta+2}{(\theta-1)^2}}, & \text{if } t \ge \frac{1}{\sqrt{\theta\kappa}}, \end{cases} \tag{1.11}$$

where $\theta > \frac{5+\sqrt{17}}{2}$. Clearly, $g_\kappa(t) \in C^1([0, +\infty), [0, +\infty))$ and $g_\kappa(t)$ decreases in $[0, +\infty)$. Substituting this form for $g(t)$ in (1.10), we obtain the following Schrödinger equation:

$$-\mathrm{div}(g_\kappa^2(u)\nabla u) + g_\kappa(u)g_\kappa'(u)|\nabla u|^2 + V(x)u = \lambda'|u|^{p-2}u, \ x \in \mathbb{R}^N \tag{1.12}$$

and the minimizer of the functional

$$I_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N} [g_\kappa^2(u)|\nabla u|^2 + V(x)u^2]\mathrm{d}x \tag{1.13}$$

restricted to Σ satisfies the equation (1.12).

Here, the previously defined $g_\kappa(t)$ is obviously bounded. So we can discuss the mentioned constrained minimization problem in $H^1(\mathbb{R}^N)$ by the methods given by Shen, Yan in [15,16]. If we can prove the minimizer u_κ of the functional (1.13) constrained on Σ satisfies $\max_{x\in\mathbb{R}^N}|u_\kappa(x)| < \frac{1}{\sqrt{\theta\kappa}}$, then this minimizer u_κ is good for what we want since $g_\kappa(u_\kappa) = g(u_\kappa) = \sqrt{1 - \frac{\kappa u_\kappa^2}{1-\kappa u_\kappa^2}}$ under this situation. That is, in this case, the functional (1.13) is exactly the functional (1.8) and thus u_κ is a weak solution of equations (1.9) and (1.10). In [18], by using the Morse L^∞ estimate, the author proved that there exists some $\kappa_0 > 0$ such that for all $\kappa \in [0, \kappa_0)$ the solutions found verify the estimate $\max_{x\in\mathbb{R}^N}|u_\kappa(x)| < \frac{1}{\sqrt{\theta\kappa}}$. By the way, the same Morse L^∞ estimate was also used in [1]. But they did not detect the specific expression of κ_0 not only in [18] but in [1].

For the L^∞ estimate of the minimizer of the functional (1.13) constrained on the manifold Σ, we follow the ideas shown in [13, 14] and make the change of variables

$$v = G_\kappa(u) = \int_0^u g_\kappa(s)\mathrm{d}s, \quad u = G_\kappa^{-1}(v). \tag{1.14}$$

Thus the change of variable (1.14) transforms the quasilinear equations (1.12) into semilinear equations

$$-\Delta v + V(x)\frac{G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))} = \lambda' \frac{|G_\kappa^{-1}(v)|^{p-2}G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))}, \quad x \in \mathbb{R}^N.$$

By taking a suitable test function in the equality the weak solution $v_\kappa = G_\kappa(u_\kappa)$ satisfies, we achieve an integral inequality. Then, by using the method of converting integral inequalities into differential inequalities, which can be found in Lemma 5.1 on p. 71 in Ladyzhenskaya and Ural'tseva [6] and is used to study the L^∞ estimate of the nonlinear elliptic equations on bounded domains, we construct the L^∞ estimate of the solution v_κ. We must point out explicitly that all the constants in this estimate are so well known. At last, the desired expressions of κ_0 is based on the corresponding function $u_\kappa = G_\kappa^{-1}(v_\kappa)$ is the solution of (1.9) and the inequality $|u_\kappa|_\infty \leq 1/\sqrt{\theta\kappa}$. Consequently, as previously mentioned, $u = \sqrt{\kappa}u_\kappa$ is the solution of (1.6) if $\kappa < \kappa_0$.

In this paper, this L^∞ estimate is first used *on unbounded domain* $\mathbb{R}^N$ for the quasilinear Schrödinger equation.

Throughout this paper, we assume the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfies

(V_1) $V(x) \geq V_0 > 0$;
(V_2) $\max_{x\in\mathbb{R}^N} V(x) < +\infty$.

In this paper, we make use of the following notations: Let X be the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right]^{\frac{1}{2}}.$$

By (V_1) and (V_2), X is equivalent to $H^1(\mathbb{R}^N)$. The symbols $|u|_p$ and $|u|_\infty$ are used for the norm of the space $L^p(\mathbb{R}^N)$ with $2 \leq p < +\infty$ and $p = \infty$, respectively.

The corresponding results are as follows:

Theorem 1.1. *For all* $\theta > \frac{5+\sqrt{17}}{2}$, *let*

$$\kappa_0 := 2^{-2-\frac{1}{a}-\frac{1}{ap}}\theta^{-1}\theta_1^2\left(\lambda_{1,p}C_N\right)^{-\frac{1}{ap}}, \tag{1.15}$$

where $a = \frac{1}{p} - \frac{1}{2^*}$, $p \geq 2$, $\theta_1^2 = \frac{\theta^2 - 5\theta + 2}{(\theta-1)^2}$, $C_N = \frac{1}{\sqrt{n(n-2)}} \left(\frac{n\Gamma(n+1)}{\Gamma(\frac{n}{2})\Gamma(n+1-\frac{n}{2})\omega_n} \right)^{\frac{1}{n}}$ *is the best Sobolev constant and*

$$\lambda_{1,p} = \inf_{u \in \Sigma} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx$$

is the minimum eigenvalue of the self-adjoint operator $-\triangle + V(x)$ *with*

$$\left(1 - \frac{\theta}{(\theta-1)(\theta-2)}\right) \theta_1^2 \lambda_{1,p} \leq \lambda' \leq \lambda_{1,p}.$$

Then, for $\kappa \in (0, \kappa_0)$, *the quasilinear problem* (1.9) *admits a minimizer* u_κ *satisfying* $\max_{x \in \mathbb{R}^N} |u_\kappa(x)| < \sqrt{\frac{1}{\theta\kappa}}$ *under the situations of* (V_1) *and* (V_2).

Remark 1.2. Recalling that $\theta > \frac{5+\sqrt{17}}{2}$ and $\theta_1 = \frac{\sqrt{\theta^2 - 5\theta + 2}}{(\theta-1)}$, if we take $\theta = 5$, we achieve the expression

$$\kappa_0 = \frac{1}{5} \left(\frac{1}{2}\right)^{5 + \frac{1}{a} + \frac{4}{ap}} (\lambda_{1,p} C_N)^{-\frac{1}{ap}}.$$

Theorem 1.3. *Assume that* (V_1), (V_2) *and* κ_0 *defined by* (1.15). *Then, for any* $\kappa \in (0, \kappa_0)$, *the quasilinear problem* (1.6) *with*

$$\begin{aligned} \lambda &= \lambda' \kappa^{\frac{2-p}{2}} \\ &= \left[\int_{\mathbb{R}^N} (g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2 + g_\kappa'(u_\kappa) g_\kappa(u_\kappa) u_\kappa |\nabla u_\kappa|^2 + V(x) u_\kappa^2)\, dx \right] \kappa^{\frac{2-p}{2}} \end{aligned}$$

has a positive solution u *satisfying* $\max_{x \in \mathbb{R}^N} |u(x)| < \sqrt{\frac{1}{\theta}}$ *for any* $\theta > \frac{5+\sqrt{17}}{2}$.

In [9,10], the authors deal with the eigenvalue problems for the case $l(s) = s$ and the case $l(s) = (1+s)^{\frac{1}{2}}$, respectively, on bounded domains. However, to the best of our knowledge, there are no works studying the spectrum of the quasilinear Schrödinger operator on $\mathbb{R}^N$. The quasilinear Schrödinger equation (1.12) is an eigenvalue problem under the situation of $p = 2$ and can be abbreviated as

$$Au + V(x)u = \lambda' u, \ x \in \mathbb{R}^N,$$

where

$$Au = -\Delta u - \frac{u}{\sqrt{1-u^2}} \triangle \sqrt{1-u^2}.$$

We denote

$$\lambda_{1,\kappa} = \inf_{|u|_2 = 1} I_\kappa(u)$$

as the first eigenvalue of the operator $A + V(x)$, and the eigenfunction corresponding to $\lambda_{1,\kappa}$ is u_1. In the same way, we define

$$\lambda_{2,\kappa} = \inf_{u \in M_1} I_\kappa(u),$$

where

$$M_1 = \left\{ u \in H^1(\mathbb{R}^N) : |u|_2 = 1, \int_{\mathbb{R}^N} uu_1 \mathrm{d}x = 0 \right\},$$

as the second eigenvalue. To go step further, we can define multiple eigenvalues of the operator. If all the eigenvalues are expressed as $\sigma(A+V(x))$, namely the spectrum of the operator $A+V(x)$, then $\lambda_{1,2} = \inf \sigma(A+V(x))$ in Theorem 1.1 and Theorem 1.3.

We must point out that the operator $A+V(x)$ is not a self-adjoint operator since A is nonlinear. But if

$$\lambda_{1,-\Delta+V(x)} = \inf \sigma(-\Delta + V(x))$$

denotes the first eigenvalue of the Schrödinger equation

$$-\Delta u + V(x)u = \lambda' u, \ x \in \mathbb{R}^N$$

under the conditions (V_1) and (V_2), then it is easy to see that $\lambda_{1,-\Delta+V(x)} \in [V_0, |V(x)|_\infty]$. Let us collect the above results in the following result.

Proposition 1.4. *Assume that* (V_1), (V_2) *hold. Then the minimum eigenvalue* $\inf \sigma(A+V(x))$ *and the corresponding eigenfunction* φ_1 *of the operator* $A+V(x)$ *satisfy*

$$\left(1 - \frac{\theta}{(\theta-1)(\theta-2)}\right)\theta_1^2 V_0 \le \inf \sigma(A+V(x)) \le \lambda_{1,-\Delta+V(x)} \le |V(x)|_\infty$$

and $|\varphi_1|_\infty < \sqrt{\frac{1}{\theta}}$, *respectively.*

2. Existence of minimizers

In this section, we will identify that the functional $I_\kappa(u)$ restricted to Σ does indeed have a minimizer. At the beginning of this section, we need the following lemma to show important properties involving functions $g_\kappa(t)$ and $G_\kappa^{-1}(t)$.

Lemma 2.1. *For any* $\theta > \frac{5+\sqrt{17}}{2}$, *we have*

(1) $\theta_1 := \frac{\sqrt{\theta^2-5\theta+2}}{\theta-1} < g_\kappa(t) \le 1$ *for all* $t \ge 0$;
(2) $\lim\limits_{t\to 0} \frac{G_\kappa^{-1}(t)}{t} = 1$;
(3) $\lim\limits_{t\to\infty} \frac{G_\kappa^{-1}(t)}{t} = \frac{1}{\theta_1}$;
(4) $t \le G_\kappa^{-1}(t) \le \frac{1}{\theta_1} t$ *for all* $t \ge 0$;
(5) $-\frac{\theta}{(\theta-1)(\theta-2)} \le \frac{t}{g_\kappa(t)} g_\kappa'(t) \le 0$ *for all* $t \ge 0$.

Proof. This lemma is mainly from [18], the proof is provided to readers only as a convenience. By the definition of $g_\kappa(t)$ and L'Hospital's rule, properties

(1)-(3) are obvious. By (1), for $t > 0$, we have $\theta_1 t \le G_\kappa(t) \le g_\kappa(0)t$, which implies (4). Now, we prove the property (5). If $t < \frac{1}{\sqrt{\theta\kappa}}$, we have

$$\frac{tg_\kappa'(t)}{g_\kappa(t)} = \frac{t(g_\kappa^2(t))'}{2g_\kappa^2(t)} = \frac{-\kappa t^2}{(1-\kappa t^2)(1-2\kappa t^2)} \ge -\frac{\theta}{(\theta-1)(\theta-2)}$$

by direct computation. If $t \ge \frac{1}{\sqrt{\theta\kappa}}$, we also have

$$\frac{tg_\kappa'(t)}{g_\kappa(t)} \ge -\frac{\theta}{(\theta-1)(\theta-2)}.$$

□

Next, we establish that the functional $I_\kappa(u)$ has a minimizer among the functions in Σ.

Lemma 2.2. *The minimizer of the functional $I_\kappa(u)$ restricted to Σ is attained by some $u_\kappa \in \Sigma$, that is,*

$$I_\kappa(u_\kappa) = m := \inf_{u\in\Sigma} I_\kappa(u).$$

Moreover, for any $\psi \in H^1(\mathbb{R}^N)\bigcap L^\infty(\mathbb{R}^N)$, the function u_κ satisfies

$$\int_{\mathbb{R}^N} [g_\kappa^2(u_\kappa)\nabla u_\kappa \nabla\psi + g_\kappa'(u_\kappa)g_\kappa(u_\kappa)|\nabla u_\kappa|^2\psi + V(x)u_\kappa\psi - \lambda'|u_\kappa|^{p-2}u_\kappa\psi]dx = 0. \tag{2.1}$$

Proof. We will use the similar methods given in [15,16] to prove this lemma. By the Ekeland variational principle, we can select a minimizing sequence $\{u_n\} \in \Sigma$, such that

$$I_\kappa(u_n) < m + \frac{1}{n}, \tag{2.2}$$

$$I_\kappa(w) \ge I_\kappa(u_n) - \frac{\|w-u_n\|}{n} \quad \text{for } w \in \Sigma. \tag{2.3}$$

From (2.2), it is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, we may assume that $u_n \rightharpoonup u_\kappa$ weakly in $H^1(\mathbb{R}^N)$.

By taking $\psi \in H^1(\mathbb{R}^N)$ and $w = \frac{u_n+t\psi}{|u_n+t\psi|_p} \in \Sigma$, we have

$$\begin{aligned} I_\kappa(w) = {} & \frac{1}{2}\left(\frac{1}{|u_n+t\psi|_p^2}-1\right)\int_{\mathbb{R}^N} g_\kappa^2\left(\frac{u_n+t\psi}{|u_n+t\psi|_p^2}\right)|\nabla(u_n+t\psi)|^2\mathrm{d}x \\ & + \frac{1}{2}\int_{\mathbb{R}^N}\left(g_\kappa^2\left(\frac{u_n+t\psi}{|u_n+t\psi|_p^2}\right) - g_\kappa^2(u_n+t\psi)\right)|\nabla(u_n+t\psi)|^2\mathrm{d}x \\ & + \frac{1}{2}\int_{\mathbb{R}^N}(g_\kappa^2(u_n+t\psi)-g_\kappa^2(u_n))|\nabla(u_n+t\psi)|^2\mathrm{d}x \\ & + \frac{1}{2}\int_{\mathbb{R}^N} g_\kappa^2(u_n)|\nabla(u_n+t\psi)|^2\mathrm{d}x \\ & + \frac{1}{2}\frac{1}{|u_n+t\psi|_p^2}\int_{\mathbb{R}^N} V(x)(u^2+2tu_m\psi+t^2\psi^2)\mathrm{d}x. \end{aligned} \tag{2.4}$$

It is easy to show that

$$\lim_{t\to 0}\frac{1}{t}\left(\frac{1}{|u_m+t\psi|_p}-1\right)=-\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi\mathrm{d}x. \tag{2.5}$$

Combining (2.4), (2.5) and the Lebesgue dominated convergence theorem one get

$$\begin{aligned}&\lim_{t\to 0}\frac{I_\kappa(w)-I_\kappa(u_n)}{t}\\ &=\int_{\mathbb{R}^N}g_\kappa^2(u_n)\nabla u_n\nabla\psi\mathrm{d}x+\int_{\mathbb{R}^N}g_\kappa'(u_n)g_\kappa(u_n)|\nabla u_n|^2\psi\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}V(x)u_n\psi\mathrm{d}x-\lambda_n'\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi\mathrm{d}x\end{aligned} \tag{2.6}$$

with

$$\lambda_n'=\int_{\mathbb{R}^N}(g_\kappa^2(u_n)|\nabla u_n|^2+g_\kappa'(u_n)g_\kappa(u_n)u_n|\nabla u_n|^2+V(x)u_n^2)\mathrm{d}x. \tag{2.7}$$

On the other hand, noticing that

$$\|w-u_n\|\le|t|\|\psi\|+\left|\frac{1}{u_n-t\psi}-1\right|\|u_n+t\psi\|,$$

we achieve

$$\lim_{t\to 0}\frac{\|w-u_n\|}{t}\le C. \tag{2.8}$$

Thus, by (2.3), it follows that

$$\frac{I_\kappa(w)-I_\kappa(u_n)}{t}\begin{cases}\ge-\frac{1}{nt}\|w-u_n\|\ge-\frac{C}{n}, & \text{if } t>0;\\ \le\frac{1}{n|t|}\|w-u_n\|\le\frac{C}{n}, & \text{if } t<0.\end{cases} \tag{2.9}$$

Consequently, from (2.6) and (2.9), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}g_\kappa^2(u_n)\nabla u_n\nabla\psi\mathrm{d}x+\int_{\mathbb{R}^N}g_\kappa'(u_n)g_\kappa(u_n)|\nabla u_n|^2\psi\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}V(x)u_n\psi\mathrm{d}x\\ &=\lambda_n'\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi\mathrm{d}x+\mu_n,\end{aligned} \tag{2.10}$$

where $|\mu_n|\le\frac{C\|\psi\|}{n}$. Moreover, from Lemma 2.1(5) and (2.7), we have

$$\lambda_n'\le\int_{\mathbb{R}^N}[g_\kappa^2(u_n)|\nabla u_n|^2+V(x)u_n^2]\mathrm{d}x\le m+\frac{1}{n}.$$

Then the desired result will follow if we show that this weakly convergence $u_n\rightharpoonup u_\kappa$ is actually strong. Indeed, according to Lemma 1.1 of [11], it suffices to show that, given $\varepsilon > 0$, there exists $R > 0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}(|\nabla u_n|^2 + V(x)u_n^2)\mathrm{d}x < \varepsilon. \tag{2.11}$$

We consider a cutoff function η_R satisfying $\eta_R = 0$ on $B_{R/2}(0)$, $\eta_R = 1$ on $\mathbb{R}^N \setminus B_R(0)$, $0 \le \eta_R \le 1$ and $|\nabla\eta_R| \le C/R$ for some constant $C > 0$. By taking $\psi = u_n\eta_R$ in (2.10), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}(g_\kappa^2(u_n)|\nabla u_n|^2 + g_\kappa'(u_n)g_\kappa(u_n)u_n|\nabla u_n|^2)\eta_R\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N} g_\kappa^2(u_n)u_n\nabla u_n\nabla\eta_R\mathrm{d}x + \int_{\mathbb{R}^N} V(x)u_n^2\eta_R\mathrm{d}x\\ &= \lambda_n'\int_{\mathbb{R}^N}|u_n|^p\eta_R\mathrm{d}x + \mu_n.\end{aligned}$$

By Lemma 2.1(5), Hölder inequality and the property of η_R, we conclude

$$\begin{aligned}&\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\int_{\mathbb{R}^N\setminus B_R(0)}(g_\kappa^2(u_n)|\nabla u_n|^2 + V(x)u_n^2)\mathrm{d}x\\ &\le \frac{C}{R}|u_n|_2|\nabla u_n|_2 + o(1).\end{aligned} \tag{2.12}$$

Note that $|u_n|_2$, $|\nabla u_n|_2$ are bounded and the fact $g_\kappa^2(u_n) \ge \theta_1^2$, it follows from (2.12) that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}(|\nabla u_n|^2 + V(x)u_n^2)\mathrm{d}x < \frac{C}{R}$$

for R sufficiently large, which yields (2.11). That is to say, the convergence is indeed strong in $H^1(\mathbb{R}^N)$. Consequently, $\{u_n\}$ convergences strongly in $L^p(\mathbb{R}^N)$ for $p \in [2, 2^*)$. To go a step further, we conclude that $|u_\kappa|_p = 1$ under the situation of $|u_n|_p = 1$. Thus, combining the inequality

$$m \le I_\kappa(u_n) < m + \frac{1}{n}$$

and the boundedness of u_κ which can be inferred from Theorem 4.2 in [16], it follows that m is achieved at some $u_\kappa \in \Sigma$ which satisfies (2.1). □

Lemma 2.2 shows that the minimizer of the functional $I_\kappa(u)$ restricted to Σ is achieved by u_κ. Moreover, the Lagrange multiplier λ' has the expression of the form

$$\lambda' = \int_{\mathbb{R}^N}(g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2 + g_\kappa'(u_\kappa)g_\kappa(u_\kappa)u_\kappa|\nabla u_\kappa|^2 + V(x)u_\kappa^2)\mathrm{d}x.$$

The following lemma gives the range of the Lagrange multiplier λ'.

Lemma 2.3. *Let*

$$\lambda_{1,p} := \inf_{u\in\Sigma}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx.$$

Then, the Lagrange multiplier λ' satisfies

$$\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\theta_1^2\lambda_{1,p}\le\lambda'\le\lambda_{1,p}.$$

Proof. Applying Lemma 2.1(1), (5), we have

$$\begin{aligned}&\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\theta_1^2|\nabla u_\kappa|^2\\ &\le g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2+g_\kappa'(u_\kappa)g_\kappa(u_\kappa)u_\kappa|\nabla u_\kappa|^2\\ &\le|\nabla u_\kappa|^2.\end{aligned}\tag{2.13}$$

Thus, on the one hand,

$$\begin{aligned}\lambda'&=\int_{\mathbb{R}^N}(g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2+g_\kappa'(u_\kappa)g_\kappa(u_\kappa)u_\kappa|\nabla u_\kappa|^2+V(x)u_\kappa^2)\mathrm{d}x\\ &\le\int_{\mathbb{R}^N}(g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2+V(x)u_\kappa^2)\mathrm{d}x\\ &=\inf_{u\in\Sigma}\int_{\mathbb{R}^N}(g_\kappa^2(u)|\nabla u|^2+V(x)u^2)\mathrm{d}x\\ &\le\inf_{u\in\Sigma}\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\mathrm{d}x=\lambda_{1,p}.\end{aligned}\tag{2.14}$$

On the other hand,

$$\begin{aligned}\lambda'&=\int_{\mathbb{R}^N}(g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2+g_\kappa'(u_\kappa)g_\kappa(u_\kappa)u_\kappa|\nabla u_\kappa|^2+V(x)u_\kappa^2)\mathrm{d}x\\ &\ge\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\int_{\mathbb{R}^N}(g_\kappa^2(u_\kappa)|\nabla u_\kappa|^2+V(x)u_\kappa^2)\mathrm{d}x\\ &=\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\inf_{u\in\Sigma}\int_{\mathbb{R}^N}(g_\kappa^2(u)|\nabla u|^2+V(x)u^2)\mathrm{d}x\\ &\ge\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\theta_1^2\inf_{u\in\Sigma}\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\mathrm{d}x\\ &=\left(1-\frac{\theta}{(\theta-1)(\theta-2)}\right)\theta_1^2\lambda_{1,p}.\end{aligned}\tag{2.15}$$

Combining (2.14) and (2.15), we achieve the desired result. □

3. L^∞ estimate

This section is mainly to show the L^∞ estimate of the function $v_\kappa=G_\kappa(u_\kappa)$. To this aim, noticing that the equality (2.1) implies u_κ is the weak solution of (1.12), we need the following fact first.

Lemma 3.1. *For any $\phi\in H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, v_κ satisfies*

$$\int_{\mathbb{R}^N}\nabla v_\kappa\nabla\phi dx+\int_{\mathbb{R}^N}V(x)\frac{G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}dx=\lambda'\int_{\mathbb{R}^N}\frac{|G_\kappa^{-1}(v_\kappa)|^{p-2}G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}dx.$$

Proof. By (2.1), $\forall\ \psi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, u_κ satisfies

(3.1)
$$\int_{\mathbb{R}^N} [g_\kappa^2(u_\kappa)\nabla u_\kappa \nabla\psi + g_\kappa'(u_\kappa)g_\kappa(u_\kappa)|\nabla u_\kappa|^2\psi + V(x)u_\kappa\psi]\mathrm{d}x = \lambda'\int_{\mathbb{R}^N}|u_\kappa|^{p-2}u_\kappa\psi\mathrm{d}x.$$

For any $\phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, if we choose $\psi = \frac{1}{g_\kappa(u_\kappa)}\phi$, we conclude that

$$\nabla\psi = \frac{1}{g_\kappa(u_\kappa)}\nabla\phi - \frac{g_\kappa'(u_\kappa)}{g_\kappa^2(u_\kappa)}\phi\nabla u_\kappa.$$

In order to ensure that $\psi = \frac{1}{g_\kappa(u_\kappa)}\phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we need to show that $\frac{g_\kappa'(u_\kappa)}{g_\kappa^2(u_\kappa)}$ is bounded. In fact, if $u \le \frac{1}{\sqrt{\theta\kappa}}$, direct computation shows that

$$\left|\frac{g_\kappa'(u)}{g_\kappa(u)}\right| = \frac{2\kappa|u|}{(1-\kappa u^2)(1-2\kappa u^2)}.$$

Consequently, there exists some $u_0 > 0$ sufficiently small, such that $\left|\frac{g_\kappa'(u)}{g_\kappa(u)}\right| \le C$ for $0 < u < u_0$. On the other hand, by Lemma 2.1(5), we have

$$\left|\frac{ug_\kappa'(u)}{g_\kappa(u)}\right| \le \frac{\theta}{(\theta-1)(\theta-2)}.$$

So $\left|\frac{g_\kappa'(u)}{g_\kappa(u)}\right| \le C$ for $u > u_0$. Thus, combining the fact that $\theta_1 < g_\kappa(u) \le 1$, we conclude that $\frac{g_\kappa'(u)}{g_\kappa^2(u)}$ is bounded.

Substituting $\frac{1}{g_\kappa(u_\kappa)}\phi$ for ψ in (3.1), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[g_\kappa^2(u_\kappa)\nabla u_\kappa\left(\frac{1}{g_\kappa(u_\kappa)}\nabla\phi - \frac{g_\kappa'(u_\kappa)}{g_\kappa^2(u_\kappa)}\phi\nabla u_\kappa\right)\right]\mathrm{d}x\\ &+\int_{\mathbb{R}^N} g_\kappa'(u_\kappa)g_\kappa(u_\kappa)|\nabla u_\kappa|^2\frac{\phi}{g_\kappa(u_\kappa)}\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{u_\kappa\phi}{g_\kappa(u_\kappa)}\mathrm{d}x\\ =\ &\lambda'\int_{\mathbb{R}^N}\frac{|u_\kappa|^{p-2}u_\kappa\phi}{g_\kappa(u_\kappa)}\mathrm{d}x.\end{aligned}$$

That is, for any $\phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\nabla v_\kappa\nabla\phi\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\mathrm{d}x = \lambda'\int_{\mathbb{R}^N}\frac{|G_\kappa^{-1}(v_\kappa)|^{p-2}G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\mathrm{d}x.$$

□

The definition of the weak solution and Lemma 3.1 imply that v_κ is the solution of the equation

(3.2)
$$-\triangle v + V(x)\frac{G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))} = \lambda'\frac{|G_\kappa^{-1}(v)|^{p-2}G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))},\quad x \in \mathbb{R}^N.$$

Now, we construct the estimate of $|v_\kappa|_\infty$.

Lemma 3.2. *The solution* v_κ *of the semilinear equation* (3.2) *has the following estimate*:

$$|v_\kappa|_\infty \le 2^{1+\frac{1}{2a}}(2\lambda'\theta_1^{-2}C_N)^{\frac{1}{2ap}}.$$

Proof. For any $\phi \in H^1(\mathbb{R}^N)$, the solution v_κ of (3.2) satisfies

$$\int_{\mathbb{R}^N} \nabla v_\kappa \nabla \phi \mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\mathrm{d}x = \lambda' \int_{\mathbb{R}^N} \frac{|G_\kappa^{-1}(v_\kappa)|^{p-2}G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\mathrm{d}x. \tag{3.3}$$

By taking $\phi = (v_\kappa - l)^+ := \max\{v_\kappa - l, 0\}$ as a test function in (3.3) with $l > 0$, applying Lemma 2.1(1), we have

$$\begin{aligned} &\int_{A_l}\left[|\nabla v_\kappa|^2 + V(x)\frac{G_\kappa^{-1}(v_\kappa)(v_\kappa - l)^+}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\right]\mathrm{d}x \\ &= \lambda' \int_{A_l} \frac{|G_\kappa^{-1}(v_\kappa)|^{p-2}G_\kappa^{-1}(v_\kappa)(v_\kappa - l)^+}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\mathrm{d}x \\ &\le \frac{\lambda'}{\theta_1}\int_{A_l} |G_\kappa^{-1}(v_\kappa)|^{p-1}(v_\kappa - l)\mathrm{d}x, \end{aligned} \tag{3.4}$$

where $A_l = \{x \in \mathbb{R}^N : v(x) > l\}$ and $|A_l|$ denotes the Lebesgue measure of the set A_l. Furthermore, by Hölder inequality, Minkowski inequality and Lemma 2.1(4), (3.4) implies

$$\begin{aligned} &\int_{A_l}\left[|\nabla v_\kappa|^2 + V(x)\frac{G_\kappa^{-1}(v_\kappa)(v_\kappa - l)^+}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\right]\mathrm{d}x \\ &\le \frac{\lambda'}{\theta_1}\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{A_l}|G^{-1}(v_\kappa)|^p\mathrm{d}x\right)^{1-\frac{1}{p}} \\ &\le \frac{\lambda'}{\theta_1^2}\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{A_l}|v_\kappa|^p\mathrm{d}x\right)^{\frac{1}{p}} \\ &\le \frac{\lambda'}{\theta_1^2}\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{1}{p}} + l|A_l|^{\frac{1}{p}}\right) \\ &\le \frac{\lambda'}{\theta_1^2}\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{2}{p}} + \frac{\lambda'}{\theta_1^2}l|A_l|^{\frac{1}{p}}\left(\int_{A_l}|v_\kappa - l|^p\mathrm{d}x\right)^{\frac{1}{p}}. \end{aligned} \tag{3.5}$$

Moreover, using Hölder inequality again and Sobolev inequality $|v_\kappa - l|_{2^*}^2 \le C_N|\nabla v_\kappa|_2^2$, we achieve the following estimate

$$\begin{aligned} \int_{A_l}|v_\kappa - l|^p\mathrm{d}x &\le \left(\int_{A_l}|v_\kappa - l|^{2^*}\mathrm{d}x\right)^{\frac{p}{2^*}}|A_l|^{1-\frac{p}{2^*}} \\ &\le C_N^{\frac{p}{2}}\left(\int_{A_l}|\nabla v_\kappa|^2\mathrm{d}x\right)^{\frac{p}{2}}|A_l|^{1-\frac{p}{2^*}}. \end{aligned}$$

That is,

$$\left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{2}{p}} \le C_N |A_l|^{\left(1-\frac{p}{2^*}\right)\frac{2}{p}} \int_{A_l} |\nabla v_\kappa|^2 \mathrm{d}x \le C_N |A_l|^{2a} \int_{A_l} |\nabla v_\kappa|^2 \mathrm{d}x, \tag{3.6}$$

where $a := \frac{1}{p} - \frac{1}{2^*}$. Thus, combining (3.5) and (3.6), we have

$$\left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{2}{p}} \le C_N |A_l|^{2a} \frac{\lambda'}{\theta_1^2} \left[\left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{2}{p}} + l|A_l|^{\frac{1}{p}} \left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{1}{p}}\right]. \tag{3.7}$$

On the other hand, noticing that

$$l|A_l| \le \int_{A_l} v_\kappa \mathrm{d}x \le \left(\int_{A_l} |v_\kappa|^p \mathrm{d}x\right)^{\frac{1}{p}} |A_l|^{1-\frac{1}{p}} \le |A_l|^{1-\frac{1}{p}},$$

we obtain $l|A_l|^{\frac{1}{p}} \le 1$. To go a step further, we have

$$C_N |A_l|^{2a} \frac{\lambda'}{\theta_1^2} \le C_N \frac{\lambda'}{\theta_1^2} \left(\frac{1}{l}\right)^{2ap}. \tag{3.8}$$

Moreover, if we take $l_0 = (2\lambda' C_N \theta_1^{-2})^{\frac{1}{2ap}}$, we have

$$C_N \lambda' \theta_1^{-2} \left(\frac{1}{l_0}\right)^{2ap} = \frac{1}{2}. \tag{3.9}$$

Consequently, combining (3.7) and (3.9), we conclude, if $l > l_0$, that

$$\left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{1}{p}} \le 2\lambda' \theta_1^{-2} C_N |A_l|^{2a+\frac{1}{p}} l.$$

So

$$\int_{A_l} |v_\kappa - l| \mathrm{d}x \le \left(\int_{A_l} |v_\kappa - l|^p \mathrm{d}x\right)^{\frac{1}{p}} |A_l|^{1-\frac{1}{p}} \le 2\lambda' \theta_1^{-2} C_N l |A_l|^{1+2a}. \tag{3.10}$$

Inspired by Lemma 5.1, which is presented on p. 71 of [6], we consider the function

$$f(l) = \int_{A_l} |v_\kappa - l| \mathrm{d}x.$$

For this function, we have $-f'(l) = |A_l|$. Therefore, (3.10) can be rewritten as

$$f(l) \le 2\lambda' \theta_1^{-2} C_N l (-f'(l))^{2a+1}, \tag{3.11}$$

i.e.,

$$l^{-\frac{1}{1+2a}} \le \left(2\lambda' \theta_1^{-2} C_N\right)^{\frac{1}{1+2a}} f(l)^{-\frac{1}{1+2a}} (-f'(l)).$$

If we integrate this inequality with respect to l from l_0 to $l_{\max} := |v_\zeta|_\infty$, we obtain

$$\begin{aligned}
& l_{\max}^{1-\frac{1}{1+2a}} - l_0^{1-\frac{1}{1+2a}} \\
\le\ & \left(2\lambda'\theta_1^{-2}C_N\right)^{\frac{1}{1+2a}} \left((f(l_0))^{1-\frac{1}{1+2a}} - (f(l_{\max}))^{1-\frac{1}{1+2a}}\right) \\
\le\ & \left(2\lambda'\theta_1^{-2}C_N\right)^{\frac{1}{1+2a}} (f(l_0))^{1-\frac{1}{1+2a}}.
\end{aligned} \tag{3.12}$$

Moreover, jointly with (3.11), recalling that $l_0 = (2\lambda' C_N \theta_1^{-2})^{\frac{1}{2ap}}$, we infer that

$$\begin{aligned}
& (f(l_0))^{1-\frac{1}{1+2a}} \left(2\lambda'\theta_1^{-2}C_N\right)^{\frac{1}{1+2a}} \\
& \le (2\lambda'\theta_1^{-2}C_N l_0 |A_{l_0}|^{1+2a})^{\frac{2a}{1+2a}} (2\lambda'\theta_1^{-2}C_N)^{\frac{1}{1+2a}} \\
& \le 2\lambda'\theta_1^{-2}C_N |A_{l_0}|^{2a} l_0^{\frac{2a}{1+2a}} \\
& \le 2\lambda'\theta_1^{-2}C_N l_0^{-2ap} l_0^{\frac{2a}{1+2a}} \\
& = l_0^{\frac{2a}{1+2a}}.
\end{aligned} \tag{3.13}$$

Therefore, combining (3.12) and (3.13), we have

$$l_{\max}^{\frac{2a}{1+2a}} \le 2 l_0^{\frac{2a}{1+2a}} = 2(2\lambda'\theta_1^{-2}C_N)^{\frac{1}{p(1+2a)}},$$

which implies the desired inequality

$$|v_\kappa|_\infty = l_{\max} \le 2^{1+\frac{1}{2a}} (2\lambda'\theta_1^{-2}C_N)^{\frac{1}{2ap}}. \qquad \square$$

4. Proofs of two theorems

Proof of Theorem 1.1. By Lemma 2.2, u_κ is the minimizer of the functional $I_\kappa(u)$ restricted to Σ and the weak solution of (1.12). In addition, replacing u_κ by $|u_\kappa|$ if necessary, we can assume that $u_\kappa \ge 0$. Then, a direct consequence of Lemma 3.1, Lemma 2.3 and Lemma 3.2 is that $v_\kappa = G_\kappa(u_\kappa)$ solves (3.2) and has the estimate

$$\begin{aligned}
|v_\kappa|_\infty & \le 2^{1+\frac{1}{2a}} (2\lambda'\theta_1^{-2}C_N)^{\frac{1}{2ap}} \\
& \le 2^{1+\frac{1}{2a}+\frac{1}{2ap}} (\lambda_{1,p}C_N)^{\frac{1}{2ap}} \theta_1^{-\frac{1}{ap}}.
\end{aligned} \tag{4.1}$$

Jointly with Lemma 2.1(4) we infer that

$$|u_\kappa|_\infty \le \frac{1}{\theta_1}|v_\kappa|_\infty \le 2^{1+\frac{1}{2a}+\frac{1}{2ap}} \theta_1^{-1-\frac{1}{ap}} (\lambda_{1,p}C_N)^{\frac{1}{2ap}}. \tag{4.2}$$

Now, to ensure

$$|u_\kappa|_\infty < \frac{1}{\sqrt{\theta\kappa}}, \tag{4.3}$$

we select $\kappa_0 = 2^{-2-\frac{1}{a}-\frac{1}{ap}} \theta^{-1} \theta_1^{2+\frac{2}{ap}} (\lambda_{1,p}C_N)^{-\frac{1}{ap}}$. Thus inequality (4.3) can be satisfied if only $\kappa \in (0, \kappa_0)$. Obviously, the equation (1.12) is indeed the equation (1.9) under the situation of $|u_\kappa|_\infty < \frac{1}{\sqrt{\theta\kappa}}$. So u_κ solves (1.9) and we complete the proof. □

The proof of Theorem 1.3 is obvious since $u = \sqrt{\kappa}u_\kappa$ is the solution of the quasilinear equation (1.6).

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