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# THE BOUNDEDNESS OF BILINEAR PSEUDODIFFERENTIAL OPERATORS IN TRIEBEL-LIZORKIN AND BESOV SPACES WITH VARIABLE EXPONENTS

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Abstract. In this paper, using the Fourier transform, inverse Fourier transform and Littlewood-Paley decomposition technique, we prove the boundedness of bilinear pseudodifferential operators with symbols in the bilinear Hörmander class $BS^m_{1,1}$ in variable Triebel-Lizorkin spaces and variable Besov spaces.

## 1. Introduction

In recent years, variable exponent function spaces have received more and more attention in many fields, such as image processing, partial differential equations, fluid dynamics, harmonic analysis and variational calculus, see [1,4, 11,17,18,25,26], etc. Accordingly, many classical constant exponent function spaces have been generalized to variable exponent setting, such as variable exponent Lebesgue spaces [8,10], variable exponent Sobolev spaces [12], variable exponent Hardy spaces [20,29], variable exponent Bessel potential spaces [13], variable exponent Herz spaces [15], variable exponent Morrey spaces [2,3], variable exponent local Hardy spaces [22], variable exponent weak Hardy spaces [28], variable exponent Triebel-Lizorkin spaces and variable exponent Besov spaces [24,27], etc.

For the theory of bilinear pseudodifferential operators in function spaces, different from their linear counterparts $\mathcal{S}^0_{\rho,\delta}$, $0 \leqslant \delta \leqslant \rho < 1$, whose corresponding pseudodifferential operators are bounded on $L^2(\mathbb{R}^n)$, the classes

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$B\mathcal{S}^0_{\rho,\delta}$ contain symbols for which the corresponding bilinear pseudodifferential operators do not map any product $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ into any $L^q(\mathbb{R}^n)$ with $1/q = 1/q_1 + 1/q_2$, see [5]. Moreover, $B\mathcal{S}^0_{1,1}$ contains symbols for which the corresponding bilinear operators are unbounded from any $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ into any $L^q(\mathbb{R}^n)$ with $1/q = 1/q_1 + 1/q_2$. However, in [7], the operators with symbols in $B\mathcal{S}^0_{1,1}$ are proved to be bounded on products of Sobolev spaces with positive smoothness by Bényi and Torres. In [6], Bényi et al. gave the properties of symbols, and boundedness of bilinear pseudodifferential operators in Lebesgue spaces. In [5], for pseudodifferential operators with symbols in the bilinear Hörmander classes of sufficiently negative order, their boundedness in Sobolev spaces, weak-type spaces, BMO and Lebesgue spaces were obtained. In [14], Herbert and Naibo showed that bilinear pseudodifferential operators with symbols in Besov spaces are bounded on products of Lebesgue spaces. In [19], Naibo established boundedness properties on the scales of inhomogeneous Triebel-Lizorkin and Besov spaces of positive smoothness for pseudodifferential operators with symbols in certain bilinear Hörmander classes. In [27], Xu and Zhu gave the Leibniz-type estimates of bilinear pseudodifferential operators associated to bilinear Hörmander classes $BS^0_{1,1}$ in Besov space with variable exponents $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Triebel-Lizorkin space with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. In [16], Koezuka and Tomita discussed the bilinear pseudo-differential operators with symbols in the bilinear Hörmander symbol class $BS^m_{1,1}$ $(m \in \mathbb{R})$ in Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$.

In this paper, we will prove the boundedness of bilinear pseudodifferential operators with symbols in the bilinear Hörmander class $BS^m_{1,1}$ $(m \in \mathbb{R})$ in Triebel-Lizorkin space with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Besov space with variable exponents $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Local Hardy spaces with variable exponents $h^{p(\cdot)}(\mathbb{R}^n)$. The detailed definitions of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$, $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $h^{p(\cdot)}$ can be found in Section 2.

For the purpose of this article, the Fourier transform and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ will be denoted by $\mathcal{F}(f)$ or $\widehat{f}$ and $\mathcal{F}^{-1}$ or $\check{f}$; in particular, we use the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,$$

and

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi.$$

For a function $h \in L^\infty(\mathbb{R}^n)$, we denote $h(D)$ as the linear Fourier multiplier operator given by

$$h(D)f(x) = \mathcal{F}^{-1}[h\widehat{f}](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} h(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi,\ f \in \mathcal{S}(\mathbb{R}^n).$$

For $1 < p < \infty$ and $s \in \mathbb{R}$, the Sobolev space $L^p_s(\mathbb{R}^n)$ consists of all $g \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|g\|_{L^p_s} = \|(I - \triangle)^{s/2} g\|_{L^p} < \infty,$$

where $(I - \triangle)^{s/2} g = \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \widehat{g}]$.

Let $0 \leqslant \delta, \rho \leqslant 1$ and $m \in \mathbb{R}$. A function $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for all multi-indices $\gamma, \alpha, \beta \in \mathbb{N}^n_0$ there exist some positive constants $C_{\gamma,\alpha,\beta}$ such that

$$|\partial^\gamma_x \partial^\alpha_\xi \partial^\beta_\eta \sigma(x, \xi, \eta)| \leqslant C_{\gamma,\alpha,\beta}(1 + |\xi| + |\eta|)^{m+\delta|\gamma|-\rho(|\alpha|+|\beta|)}$$

for all $x, \xi, \eta \in \mathbb{R}^n$, where $|\gamma|, |\alpha|, |\beta|$ denote the sum of their components. The bilinear pseudodifferential operators associated to $\sigma$ is defined by

$$T_\sigma(f, g)(x) := \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix\cdot(\xi+\eta)} d\xi d\eta$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$.

Given $\sigma \in B\mathcal{S}^m_{\rho,\delta}$. Define

$$\|\sigma\|_{BS^m_{\rho,\delta,N}} := \max_{|\alpha|,|\beta|,|\gamma|\leqslant N} \sup_{x,\xi,\eta\in\mathbb{R}^n} |\partial^\gamma_x \partial^\alpha_\xi \partial^\beta_\eta \sigma(x, \xi, \eta)|(1 + |\xi| + |\eta|)^{-m-\delta|\gamma|+\rho(|\alpha|+|\beta|)}.$$

In this paper, we focus on the case $\rho = \delta = 1$.

Our main results are as follows:

**Theorem 1.1.** *Let $m \in \mathbb{R}$, $p(\cdot), p_1(\cdot), p_2(\cdot), q(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ such that $p^+, q^+ < \infty$ and $\frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} = \frac{1}{p(\cdot)}$. Let $s(\cdot) \in C^{\log}_{\rm loc}(\mathbb{R}^n)$. Then there exists a positive integer $N$ such that*

(1)

$$\|T_\sigma(f, g)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|\sigma\|_{BS^m_{1,1,N}} \left( \|f\|_{F^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|g\|_{F^{m+s(\cdot)}_{p_2(\cdot),q(\cdot)}} \right)$$

*for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\sigma \in B\mathcal{S}^m_{1,1}$.*

**Theorem 1.2.** *Let $m \in \mathbb{R}$, $p(\cdot), p_1(\cdot), p_2(\cdot), q(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ such that $p^+, q^+ < \infty$ and $\frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} = \frac{1}{p(\cdot)}$. Let $s(\cdot) \in C^{\log}_{\rm loc}(\mathbb{R}^n)$. Then there exists a positive integer $N$ such that*

(2)

$$\|T_\sigma(f, g)\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|\sigma\|_{BS^m_{1,1,N}} \left( \|f\|_{B^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|g\|_{B^{m+s(\cdot)}_{p_2(\cdot),q(\cdot)}} \right)$$

*for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\sigma \in B\mathcal{S}^m_{1,1}$.*

In the end of this section, we give two notations. If $a \leqslant cb$ and $b \leqslant ca$, we say that $a \approx b$. The symbol $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leqslant CB$.

## 2. Preliminaries

In this section, we introduce some necessary definitions and notations, see [19, 21, 23].

A measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called a variable exponent. For a measurable subset $A \subset \mathbb{R}^n$, we write

$$p^+(A) \equiv \operatorname*{ess\,sup}_{x \in A} p(x), \quad p^-(A) \equiv \operatorname*{ess\,inf}_{x \in A} p(x).$$

When $A = \mathbb{R}^n$, we write simply $p^+ := p^+(\mathbb{R}^n)$ and $p^- := p^-(\mathbb{R}^n)$. The collection of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ is denote by $\mathcal{P}(\mathbb{R}^n)$, and the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ satisfying $p^- > 0$ is denote by $\mathcal{P}_0(\mathbb{R}^n)$.

We say that $p(\cdot)$ have $LH$ condition if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \lesssim \frac{1}{-\log(|x - y|)} \quad \text{for} \quad |x - y| \leqslant \frac{1}{2}$$

and

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e + |x|)} \quad \text{for} \quad |y| \geqslant |x|.$$

The function $\rho_p$ is defined as follows:

$$\rho_p(s) := \begin{cases} s^p, & \text{if } p \in (0, \infty), \\ 0, & \text{if } p = \infty \text{ and } s \leqslant 1, \\ \infty, & \text{if } p = \infty \text{ and } s > 1. \end{cases}$$

The convention $1^\infty = 0$ is adopted in order for $\rho_p$ to be left-continuous. The variable exponent modular is defined by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \rho_{p(x)}(f(x))dx.$$

The Luxemburg norm of a function $f \in L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(|f|/\lambda) \leqslant 1\}.$$

When $p(\cdot) \geqslant 1$, it is a norm otherwise it is always a quasi-norm.

Let $\{h_j\}_{j=0}^\infty$ be a sequence of measurable functions on $\mathbb{R}^n$. Then the quasi-norm $\|\cdot\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$ is given by

$$\big\|\{h_j\}_{j=0}^\infty\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \equiv \left\| \left( \sum_{j=0}^\infty |h_j(\cdot)|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}.$$

The quasi-norm $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is denoted by

$$\big\|\{h_j\}_{j=0}^\infty\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \equiv \inf \left\{ \lambda > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left( \left\{ \frac{h_j}{\lambda} \right\}_{j=0}^\infty \right) \leqslant 1 \right\} < \infty,$$

where

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\{h_j\}_{j=0}^{\infty}\big) \equiv \sum_{j=0}^{\infty} \inf\left\{\lambda_j > 0 : \int_{\mathbb{R}^n} \left(\frac{|h_j(x)|}{\lambda_j^{1/q(x)}}\right)^{p(x)} dx \leqslant 1\right\}.$$

If $q^+ < \infty$, then

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\big(\{h_j\}_{j=0}^{\infty}\big) = \sum_{j=0}^{\infty} \big\||h_j|^{q(\cdot)}\big\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Since the above right-hand side expression is much simper, we use this notation to represent the above left-hand side even when $q^+ = \infty$.

The following generalized Hölder inequality will be used in the sequel.

Let $q, q_1, q_2 \in \mathcal{P}_0(\mathbb{R}^n)$, $q_1^+, q_2^+ < \infty$ and $\frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} = \frac{1}{q(\cdot)}$. Then there exists a constant $C_{q,q_1}$ independent of the functions $f \in L^{q_1(\cdot)}$ and $h \in L^{q_2(\cdot)}$ such that

$$\|fh\|_{L^{q(\cdot)}} \leqslant C_{q,q_1}\|f\|_{L^{q_1(\cdot)}}\|h\|_{L^{q_2(\cdot)}}.$$

We start with the log-Hölder continuity [9, 10], which is the cornerstone of the variable exponent function spaces.

**Definition 2.1.** Let $f(\cdot)$ be a real function on $\mathbb{R}^n$.

(i) If there exists $C_{\log}(f) > 0$ such that

$$|f(x) - f(y)| \lesssim \frac{C_{\log}(f)}{\log(e + \frac{1}{|x-y|})}$$

holds for all $x, y \in \mathbb{R}^n$, then the function $f(\cdot)$ is called locally log-Hölder continuous, abbreviated $f(\cdot) \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$,

(ii) If $f(\cdot)$ is locally log-Hölder continuous and there exist a positive constant $C_\infty$ and $f_\infty \in \mathbb{R}$ such that

$$|f(x) - f_\infty| \lesssim \frac{C_\infty}{\log(e + |x|)}$$

holds for all $x \in \mathbb{R}^n$, then the function $f(\cdot)$ is called globally log-Hölder continuous, abbreviated $f(\cdot) \in C^{\log}(\mathbb{R}^n)$.

Let $\mathcal{P}^{\log}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying $\frac{1}{p(\cdot)} \in C^{\log}(\mathbb{R}^n)$. The class $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ is defined analogously.

The convolution $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

We recall the definition of Local Hardy spaces with variable exponents $h^{p(\cdot)}$ [20, 22].

**Definition 2.2.** Let $f \in \mathcal{S}'$, $p(\cdot) \in \mathcal{P}_0^{\log} \cap LH$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $M$ the grand maximal operator given by $M_{loc}f(x) = \sup\{|\psi_t * f(x)| : 0 < t < 1, \psi \in \mathcal{F}_N\}$ for any fixed large integer $N$, where

$\mathcal{F}_N = \{\psi \in \mathcal{S} : \int \psi(x)dx = 1, \sum_{|\beta|\leqslant N} \sup(1+|x|)^N |\partial^\beta \psi(x)| \leqslant 1\}$. The local Hardy space with variable exponent $h^{p(\cdot)}$ is the set of all $f \in \mathcal{S}'$ for which the quantity

$$\|f\|_{h^{p(\cdot)}} = \|M_{loc} f\|_{L^{p(\cdot)}} < \infty.$$

From [20], we know that

$$\|f\|_{h^{p(\cdot)}} \approx \|\sup_{j\in\mathbb{N}} |\varphi_j(D)f|\|_{L^{p(\cdot)}},$$

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $j \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Now, we will recall the definition of Triebel-Lizorkin spaces with variable exponents and Besov spaces with variable exponents. First, we give a definition [19] which will be used in the sequel.

**Definition 2.3** (Littlewood-Paley Partitions of Unity)**.** $\{\phi_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ is a Littlewood-Paley partition of unity in $\mathbb{R}^n$ if $\mathrm{supp}(\phi_0) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$ and $\phi_0(\xi) = 1$ in $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$ and for $k \in \mathbb{N}$,

$$\phi_k(\xi) = \phi(2^{-k}\xi),\ \xi \in \mathbb{R}^n,$$

where $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ for every $\xi \in \mathbb{R}^n$.

From the definition we know that

$$\mathrm{supp}(\phi_k) \subset \left\{\xi \in \mathbb{R}^n : 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\right\} \text{ for } k \in \mathbb{N} \text{ and } \sum_{k\in\mathbb{N}_0} \phi_k \equiv 1.$$

**Definition 2.4.** Let $\{\psi_k\}_{k\in\mathbb{N}_0}$ be a Littlewood-Paley partition of unity, $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$, and $s(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. The Triebel-Lizorkin space with variable exponents $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as

$$F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\},$$

where

$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\{2^{ks(\cdot)}\psi_k(D)f\}_{k\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \left\| \left(\sum_{k=0}^{\infty} \left|2^{ks(\cdot)}\psi_k(D)f\right|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}.$$

**Definition 2.5.** Let $\{\psi_k\}_{k\in\mathbb{N}_0}$ be a Littlewood-Paley partition of unity, $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$, and $s(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. The Besov space with variable exponents $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined as

$$B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\},$$

where

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\{2^{ks(\cdot)}\psi_k(D)f\}_{k\in\mathbb{N}_0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Next, we give two lemmas which will be used later [21].

**Lemma 2.6.** *Let $f_k$ and $h_k$ be measurable functions on $\mathbb{R}^n$.*

(i) *If $p(\cdot), q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$, then*

$$\|\{f_k + h_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-,q^-,1)} \leqslant \|\{f_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-,q^-,1)} + \|\{h_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^{\min(p^-,q^-,1)}.$$

(ii) *If $p(\cdot), q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$ and*

$$\alpha \equiv \min(q^-, 1) \min\left(1, \left(\frac{p}{q}\right)^-\right),$$

*then*

$$\|\{f_k + h_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{\alpha} \leqslant \|\{f_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{\alpha} + \|\{h_k\}_{k=0}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{\alpha}.$$

**Lemma 2.7.** *Let $p(\cdot), q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$, and $s(\cdot) \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. Then*

(i) *If $\nu > \frac{n}{2} + \frac{n+3C_{\log}(s)\min(p^-,q^-)}{\min(p^-,q^-)}$, then*

$$\left\|\{2^{ks(\cdot)} m_k(D) f_k\}_{k=0}^{\infty}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \sup_{j\geqslant 0} \|m_k(2^k \cdot)\|_{L_\nu^2} \cdot \left\|\{2^{ks(\cdot)} f_k\}_{k=0}^{\infty}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

*for all $\{m_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=0}^{\infty}$ satisfying* $\operatorname{supp} \widehat{f_k} \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{k+1}\right\}$.

(ii) *If $\nu > \frac{n}{2} + \frac{2n+3C_{\log}(s)\min(p^-,q^-)}{\min(p^-,q^-)}$, then*

$$\left\|\{2^{ks(\cdot)} m_k(D) f_k\}_{k=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \sup_{j\geqslant 0} \|m_k(2^k \cdot)\|_{L_\nu^2} \cdot \left\|\{2^{ks(\cdot)} f_k\}_{k=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

*for all $\{m_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=0}^{\infty}$ satisfying* $\operatorname{supp} \widehat{f_k} \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{k+1}\right\}$.

*Remark* 2.8. Since $m_k(D)f_k(x) = m_k(BD)[f_k(B^{-1}\cdot)](Bx)$ for $B > 0$, using change of variables, if we replace $\sup_{j\geqslant 0} \|m_k(2^k\cdot)\|_{L_\nu^2}$ by $\sup_{j\geqslant 0} \|m_k(2^k B\cdot)\|_{L_\nu^2}$ for all $\{m_k\}_{k=0}^{\infty} \in L_\nu^2(\mathbb{R}^n)$ and $\{f_k\}_{k=0}^{\infty}$ satisfying $\operatorname{supp} \widehat{f_k} \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leqslant B2^k\right\}$, then Lemma 2.7 also holds, where the implicit constant in the inequality is independent of $B$.

## 3. Proof of Theorem 1.1 and Theorem 1.2

Firstly, we will recall the argument of the reduction to elementary symbols given by Bényi-Torres [7, 16]. Let $\sigma \in BS_{1,1}^m$, and $\{\varphi_j\}_{j=0}^{\infty}$ be a Littlewood-Paley partition of unity. Following the decomposition outlined in [16] (pages 313 to 314), we know that

$$\begin{aligned}\sigma(x,\xi,\eta) &= \left(\sum_{j=0}^{\infty}\sum_{k=0}^{j} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\right) \sigma(x,\xi,\eta)\varphi_j(\xi)\varphi_k(\eta) \\ &= \sum_{j=0}^{\infty} m_j(x)\psi_j(\xi)\chi_j(\eta) + \sum_{j=0}^{\infty} m_j(x)\chi_j(\xi)\psi_j(\eta),\end{aligned}$$

where

$$m_j(x) \approx c_{j,k,\ell}(x),\ \|\partial^\alpha m_j\|_{L^\infty} \lesssim 2^{j(m+|\alpha|)}\|\sigma\|_{BS^m_{1,1,N'}},\ |\alpha| \leqslant N;$$

$$\psi_j(\xi) \approx \varphi_j(\xi),\ \text{supp}\ \psi_0 \subset \{|\xi| \lesssim 1\},\ \text{supp}\ \psi_j \subset \{|\xi| \approx 2^j\},\ j \geqslant 1,$$
$$\|\partial^\alpha \psi_j\|_{L^\infty} \lesssim 2^{-j|\alpha|},\ |\alpha| \leqslant N;$$

$$\chi_j(\eta) \approx \varphi_0(2^{-j}\eta),\ \text{supp}\ \chi_j \subset \{|\xi| \lesssim 2^j\},\ j \geqslant 0,\ \|\partial^\alpha \chi_j\|_{L^\infty} \lesssim 2^{-j|\alpha|},\ |\alpha| \leqslant N;$$

and

$$c_{j,k,\ell}(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \sigma(x, 2^j\xi, 2^j\eta)\phi(\xi)\phi_0(\eta)e^{-i(k\cdot\xi+\ell\cdot\eta)}d\xi d\eta,$$

where, $\{\phi_j\}_{j=0}^\infty$ satisfying $\phi_j = 1$ on supp $\varphi_j$, $\phi_j = \phi(\xi/2^j)$, $j \geqslant 1$, supp $\phi_0 \subset \{|\xi| \leqslant 3\}$, supp $\phi \subset \{1/3 \leqslant |\xi| \leqslant 3\}$, and $N$ can be chosen arbitrarily large and $N'$ is determined from $N$.

Since the estimate of $\sum_{j=0}^\infty m_j(x)\, \chi_j(\xi)\psi_j(\eta)$ is similar to that of $\sum_{j=0}^\infty m_j(x)$ $\psi_j(\xi)\chi_j(\eta)$, it is enough to focus on $\sum_{j=0}^\infty m_j(x)\psi_j(\xi)\chi_j(\eta)$. For the sake of simplicity of the notation, we assume that $\|\sigma\|_{BS^m_{1,1,N'}}$ is equal to one.

Continuing as in [16], we have that

$$T_\sigma(f,g)(x) = \sum_{j=0}^\infty \sum_{k=0}^\infty m_{j,k}(x)\psi_j(D)f(x)\chi_j(D)g(x),$$

and

$$\begin{aligned}\varphi_\ell(D)T_\sigma(f,g) &= \sum_{k=0}^\infty \sum_{j+k+L\geqslant \ell}^\infty \varphi_\ell(D)[m_{j,k}\psi_j(D)f\chi_j(D)g] \\ &= \sum_{k=0}^\infty \sum_{j=0}^\infty \varphi_\ell(D)[m_{j_{(k,\ell)},k}\psi_{j_{(k,\ell)}}(D)f\chi_{j_{(k,\ell)}}(D)g],\end{aligned}$$

where $j_{(k,\ell)} = j - k + \ell - L$, and

$$m_{j,0} = \varphi_0(D/2^j)m_j,\ m_{j,k} = \varphi(D/2^{j+k})m_j,\ k \geqslant 1,$$

$$\|m_{j,k}\|_{L^\infty} \lesssim 2^{jm-kN},\ j,k \geqslant 0, \tag{3}$$

$$\text{supp}\ \widehat{m_{j,k}} \subset \left\{|\xi| \lesssim 2^{j+k}\right\},\ j,k \geqslant 0,$$

$$\text{supp}\ \mathcal{F}[m_{j,k}\psi_j(D)f\chi_j(D)g] \subset \left\{|\xi| \lesssim 2^{j+k}\right\},\ j,k \geqslant 0,$$

and $L$ is a fixed integer determined from the implicit constants for the sizes of the supports of $\varphi_\ell, \widehat{m_{j,k}}, \psi_j, \chi_j$.

Let $r = \min(p^-, q^-, 1)$. Using (i) of Lemma 2.6, we have

$$\begin{aligned}&\|T_\sigma(f,g)\|^r_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \\ &= \left\|\{2^{\ell s(\cdot)}\varphi_\ell(D)T_\sigma(f,g)\}_{\ell=0}^\infty\right\|^r_{L^{p(\cdot)}(\ell^{q(\cdot)})}\end{aligned}$$

$$\leqslant \sum_{j,k=0}^{\infty} \left\| \left\{ 2^{\ell s(\cdot)} \varphi_\ell(D)[m_{j_{(k,\ell)},k} \psi_{j_{(k,\ell)}}(D) f \chi_{j_{(k,\ell)}}(D) g] \right\}_{\ell=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}^r .$$

If $\nu > \frac{n}{2} + \frac{n+3C_{\log}(s)\min(p^-,q^-)}{\min(p^-,q^-)}$, then using (i) of Lemma 2.7 and Remark 2.8 with $B = 2^j$, we obtain that

$$\left\| \left\{ 2^{\ell s(\cdot)} \varphi_\ell(D)[m_{j_{(k,\ell)},k} \psi_{j_{(k,\ell)}}(D) f \chi_{j_{(k,\ell)}}(D) g] \right\}_{\ell=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

$$\lesssim \sup_{\ell \geqslant 0} \|\varphi_\ell(2^{j+\ell} \cdot)\|_{L^2_\nu} \cdot \left\| \left\{ 2^{\ell s(\cdot)} [m_{j_{(k,\ell)},k} \psi_{j_{(k,\ell)}}(D) f \chi_{j_{(k,\ell)}}(D) g] \right\}_{\ell=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} .$$

Since supp $\varphi_\ell(2^{j+\ell}\cdot) \subset \left\{ |\xi| \lesssim 2^{-j} \right\}$ and $\|\partial^\alpha \varphi_\ell(2^{j+\ell}\cdot)\|_{L^\infty} \lesssim 2^{j|\alpha|}$, we get that

$$\sup_{\ell \geqslant 0} \|\varphi_\ell(2^{j+\ell}\cdot)\|_{L^2_\nu} \lesssim \sup_{\ell \geqslant 0} \|\varphi_\ell(2^{j+\ell}\cdot)\|_{L^2}^{1-\varsigma} \|\varphi_\ell(2^{j+\ell}\cdot)\|_{L^2_{[\nu]+1}}^{\varsigma} \lesssim 2^{j(\nu - n/2)},$$

where $[\nu]$ is the integer part of $\nu$ and $\nu = \varsigma([\nu]+1)$. Moreover, by change of variables, (3) and Hölder inequality, we have

$$\left\| \left\{ 2^{\ell s(\cdot)} [m_{j_{(k,\ell)},k} \psi_{j_{(k,\ell)}}(D) f \chi_{j_{(k,\ell)}}(D) g] \right\}_{\ell=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

$$= \left\| \left( \sum_{\ell=j-k-L}^{\infty} \left| 2^{(-j+k+\ell-L)s(\cdot)} [m_{\ell,k} \psi_\ell(D) f \chi_\ell(D) g](x) \right|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}$$

$$\lesssim 2^{-(j-k)s(\cdot)} \left\| \left( \sum_{\ell=0}^{\infty} \left| 2^{\ell s(\cdot)} [m_{\ell,k} \psi_\ell(D) f \chi_\ell(D) g](x) \right|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}}$$

$$\lesssim 2^{-kN-(j-k)s^-} \left\| \left( \sum_{\ell=0}^{\infty} \left| 2^{\ell(m+s(\cdot))} \psi_\ell(D) f(x) \right|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right\|_{L^{p_1(\cdot)}} \left\| \sup_{\ell \geqslant 0} |\chi_\ell(D) g| \right\|_{L^{p_2(\cdot)}}$$

$$\lesssim 2^{-kN-(j-k)s^-} \|f\|_{F^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}} .$$

Therefore, when $s^- > \nu - n/2$, we obtain that

$$\|T_\sigma(f,g)\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}$$

$$\lesssim \left( \sum_{j,k=0}^{\infty} 2^{jr(\nu-n/2)} 2^{-krN-(j-k)rs^-} \right)^{\frac{1}{r}} \|f\|_{F^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}}$$

$$= \left( \sum_{j=0}^{\infty} 2^{-jr(s^- - \nu + n/2)} \right)^{\frac{1}{r}} \left( \sum_{k=0}^{\infty} 2^{-kr(N-s^-)} \right)^{\frac{1}{r}} \|f\|_{F^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}}$$

$$\lesssim \|f\|_{F^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}} \|g\|_{h^{p_2(\cdot)}},$$

since $N$ can be chosen arbitrarily large.

The proof for (1) is complete.

As for (2), let $\alpha = \min(q^-, 1) \min\left(1, \left(\frac{p}{q}\right)^-\right)$. Using (ii) of Lemma 2.6, we have

$$\begin{aligned}\|T_\sigma(f,g)\|^\alpha_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} &= \left\|\{2^{\ell s(\cdot)}\varphi_\ell(D)T_\sigma(f,g)\}_{\ell=0}^\infty\right\|^\alpha_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ &\leqslant \sum_{j,k=0}^\infty \left\|\left\{2^{\ell s(\cdot)}\varphi_\ell(D)[m_{j_{(k,\ell)},k}\psi_{j_{(k,\ell)}}(D)f\chi_{j_{(k,\ell)}}(D)g]\right\}_{\ell=0}^\infty\right\|^\alpha_{\ell^{q(\cdot)}(L^{p(\cdot)})}.\end{aligned}$$

If $\nu > \frac{n}{2} + \frac{2n+3C_{\log}(s)\min(p^-,q^-)}{\min(p^-,q^-)}$, then using (ii) of Lemma 2.7 and Remark 2.8 with $B = 2^j$, we obtain that

$$\begin{aligned}&\left\|\left\{2^{\ell s(\cdot)}\varphi_\ell(D)[m_{j_{(k,\ell)},k}\psi_{j_{(k,\ell)}}(D)f\chi_{j_{(k,\ell)}}(D)g]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ \lesssim &\sup_{\ell\geqslant 0}\|\varphi_\ell(2^{j+\ell}\cdot)\|_{L^2_\nu} \cdot \left\|\left\{2^{\ell s(\cdot)}[m_{j_{(k,\ell)},k}\psi_{j_{(k,\ell)}}(D)f\chi_{j_{(k,\ell)}}(D)g]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\left\|\left\{2^{\ell s(\cdot)}[m_{j_{(k,\ell)},k}\psi_{j_{(k,\ell)}}(D)f\chi_{j_{(k,\ell)}}(D)g]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ = &\left\|\left\{2^{(-j+k+\ell-L)s(\cdot)}[m_{\ell,k}\psi_\ell(D)f\chi_\ell(D)g]\right\}_{\ell=j-k-L}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ \lesssim &\, 2^{-(j-k)s(\cdot)}\left\|\left\{2^{\ell s(\cdot)}[m_{\ell,k}\psi_\ell(D)f\chi_\ell(D)g]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ \lesssim &\, 2^{-kN-(j-k)s^-}\left\|\left\{2^{\ell(m+s(\cdot))}[\psi_\ell(D)f\chi_\ell(D)g]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\ \lesssim &\, 2^{-kN-(j-k)s^-}\left\|\left\{2^{\ell(m+s(\cdot))}[\psi_\ell(D)f]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}\sup_{\ell\geqslant 0}\|\chi_\ell(D)g\|_{L^{p_2(\cdot)}} \\ \lesssim &\, 2^{-kN-(j-k)s^-}\left\|\left\{2^{\ell(m+s(\cdot))}[\psi_\ell(D)f]\right\}_{\ell=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p_1(\cdot)})}\left\|\sup_{\ell\geqslant 0}|\chi_\ell(D)g|\right\|_{L^{p_2(\cdot)}} \\ \lesssim &\, 2^{-kN-(j-k)s^-}\|f\|_{B^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{h^{p_2(\cdot)}}.\end{aligned}$$

Therefore, when $s^- > \nu - n/2$, we can get that

$$\begin{aligned}&\|T_\sigma(f,g)\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \\ \lesssim &\left(\sum_{j=0}^\infty 2^{-jr(s^- -\nu+n/2)}\right)^{\frac{1}{\alpha}}\left(\sum_{k=0}^\infty 2^{-kr(N-s^-)}\right)^{\frac{1}{\alpha}}\|f\|_{B^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{h^{p_2(\cdot)}} \\ \lesssim &\|f\|_{B^{m+s(\cdot)}_{p_1(\cdot),q(\cdot)}}\|g\|_{h^{p_2(\cdot)}}.\end{aligned}$$

The proof for (2) is complete.

This completes the proofs of Theorems 1.1 and 1.2.

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