Bull. Korean Math. Soc. **61** (2024), No. 2, pp. 519–527
https://doi.org/10.4134/BKMS.b230215
pISSN: 1015-8634 / eISSN: 2234-3016

RESOLUTION OF QUOTIENT SINGULARITIES VIA G-CONSTELLATIONS

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Abstract. For a finite subgroup G of $\mathrm{GL}_n(\mathbb{C})$, the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations is rarely smooth. This note shows that for a group G of type $\frac{1}{r}(1,a,b)$ with $r = abc + a + b$, there is a generic stability parameter $\theta \in \Theta$ such that the birational component Y_θ of θ-stable G-constellations provides a resolution of the quotient singularity $X := \mathbb{C}^3/G$.

1. Introduction

For a finite subgroup G of $\mathrm{GL}_n(\mathbb{C})$, a G-equivariant sheaf $\mathcal{F}$ on $\mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{F})$ isomorphic to the regular representation $\mathbb{C}[G]$ of G is called a *G-constellation.* The GIT parameter space for G-constellations is defined to be

$$\Theta = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right) = 0\right\},$$

where $R(G)$ is the representation space of G. For $\theta \in \Theta$, we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-(semi)stable* if $\theta(\mathcal{G}) \geq 0$ ($\theta(\mathcal{G}) > 0$) for every proper subsheaf $\mathcal{G}$ of $\mathcal{F}$.

(ii) θ is generic if every θ-semistable G-constellation is θ-stable.

The moduli spaces of θ-stable G-constellations provide good birational models of the quotient variety $\mathbb{C}^n/G$. For example, for $G \subset \mathrm{SL}_3(\mathbb{C})$ and generic θ, the moduli space of θ-stable G-constellations is a crepant resolution of $\mathbb{C}^n/G$ (see [1, 2, 7, 9]). However, it is very rare for the moduli space of stable G-constellations to be smooth.

This note proves that for some abelian groups in $\mathrm{GL}_3(\mathbb{C})$, there is a generic stability parameter $\theta \in \Theta$ such that Y_θ is a resolution of the quotient singularity $X := \mathbb{C}^3/G$.

Theorem 1.1 (Main Theorem)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1,a,b)$ with $r = abc+a+b$, where a, b, c are positive integers and b is coprime*

Received April 7, 2023; Accepted July 24, 2023.
2020 *Mathematics Subject Classification.* 14B05, 14J17.
Key words and phrases. G-constellations, quotient singularities.
This work was partially supported by NRF grant (NRF-2021R1C1C1004097) of the Korean government.

to a. Then the birational component Y_θ of θ-stable G-constellations is smooth for a suitable parameter θ.

Acknowledgement. I would like to thank the referees for many valuable comments and corrections.

2. Moduli spaces of G-constellations and G-bricks

Through this section, we consider the group G of type $\frac{1}{r}(1,a,b)$, that is,

$$G = \langle \operatorname{diag}(\epsilon, \epsilon^a, \epsilon^b) \,|\, \epsilon^r = 1 \rangle \subset \mathrm{GL}_3(\mathbb{C}).$$

As G is abelian, the set of irreducible representations of G can be identified with the character group $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ of G. With this identification, we say that an irreducible representation ρ is *of weight i* if ρ maps $\operatorname{diag}(\epsilon, \epsilon^a, \epsilon^b)$ to ϵ^i, where $0 \le i < r$. In this case, we let $\mathrm{wt}(\rho)$ denote the weight of ρ and ρ_i the representation of weight i.

Set $\overline{L} = \mathbb{Z}^3$ and define the lattice

$$L = \overline{L} + \mathbb{Z} \cdot \frac{1}{r}(1,a,b).$$

We may identify the dual lattices $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ with Laurent monomials and G-invariant Laurent monomials, respectively. There is a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee$$

induced by the embedding of G into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$.

Let $\overline{M}_{\ge 0}$ denote genuine monomials in $\overline{M}$, i.e.,

$$\overline{M}_{\ge 0} = \{x_1^{m_1} x_2^{m_2} x_3^{m_3} \in \overline{M} \,|\, m_i \ge 0 \text{ for all } i\}.$$

For a subset A of Laurent monomials, let $\langle A \rangle$ denote the $\mathbb{C}[x_1, x_2, x_3]$-submodule of $\mathbb{C}[x_1^\pm, x_2^\pm, x_3^\pm]$ generated by A.

Let σ_+ be the cone generated by $\{e_1, e_2, e_3\}$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\overline{L} = \mathbb{Z}^3$. By toric geometry, the quotient variety $X = \mathbb{C}^3/G$ is isomorphic to $U_{\sigma_+} = \operatorname{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$.

2.1. G-constellations

Definition 2.1. A *G-constellation* on $\mathbb{C}^3$ is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^3$ with $\mathrm{H}^0(\mathcal{F})$ isomorphic to $\mathbb{C}[G]$ as a G-representation.

With the following GIT stability parameter space

$$\Theta = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \,\middle|\, \theta(\mathbb{C}[G]) = 0\right\},$$

we define the stability of G-constellations as follows.

Definition 2.2. For $\theta \in \Theta$ and a G-constellation $\mathcal{F}$, we say that:

(i) $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \ge 0$ for every nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$.

(ii) $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0$ for every nonzero proper subsheaf $\mathcal{G}$ of $\mathcal{F}$.
(iii) $\theta \in \Theta$ is *generic* if every θ-semistable G-constellation is θ-stable.

Let θ be generic. By King's result in [6], there is a quasiprojective scheme $\mathcal{M}_\theta$ which is a fine moduli space of θ-stable G-constellations. Moreover by [3, Theorem 1.1], the moduli space $\mathcal{M}_\theta$ has a distinguished component Y_θ birational to $\mathbb{C}^3/G$ and that there is a projective morphism $Y_\theta \to \mathbb{C}^3/G$ obtained by variation of GIT quotient.

Definition 2.3. The unique irreducible component Y_θ is called the *birational component* of the moduli space $\mathcal{M}_\theta$.

2.2. *G*-bricks and the birational component Y_θ

To describe the affine local charts of the birational component Y_θ, we introduce the notion of G-bricks (see [4]).

Definition 2.4. A subset Γ of Laurent monomials is called a *G-prebrick* if:

(i) the monomial $\mathbf{1}$ is in Γ;
(ii) there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ of weight ρ for each weight $\rho \in G^\vee$.
(iii) if $\mathbf{p'} \cdot \mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{p}, \mathbf{p'} \in \overline{M}_{\geq 0}$, then $\mathbf{p} \cdot \mathbf{m}_\rho \in \Gamma$;
(iv) the set Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path in Γ from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x_i.

For $\mathbf{m} \in \overline{M}$, we define $\mathrm{wt}_\Gamma(\mathbf{m})$ to be the unique element $\mathbf{m}_\rho$ in Γ of the same weight as $\mathbf{m}$.

For a G-prebrick $\Gamma = \{\mathbf{m}_\rho\}$, let $S(\Gamma)$ denote the subsemigroup of M generated by

$$\frac{\mathbf{n} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{n} \cdot \mathbf{m}_\rho)}$$

for all $\mathbf{n} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. It turns out that the semigroup $S(\Gamma)$ is finitely generated. Thus $S(\Gamma)$ defines a (not-necessarily-normal) affine toric variety $U(\Gamma) := \operatorname{Spec} \mathbb{C}[S(\Gamma)]$.

Definition 2.5. A G-prebrick Γ is called a *G-brick* if $U(\Gamma)$ contains a torus fixed point.

For a G-brick Γ, we define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle,$$

where $B(\Gamma) := \left\{ x_i \cdot \mathbf{m}_\rho \,\middle|\, \mathbf{m}_\rho \in \Gamma \right\} \setminus \Gamma$. The module $C(\Gamma)$ becomes a torus invariant G-constellation.

Remark 2.6. The G-brick Γ is a $\mathbb{C}$-basis of G-constellations over $U(\Gamma)$. Moreover, a submodule $\mathcal{G}$ of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a $\mathbb{C}$-basis of $\mathcal{G}$. This makes it easy to determine whether $C(\Gamma)$ is θ-stable or not.

Definition 2.7. A G-brick Γ is said to be *θ-stable* if $C(\Gamma)$ is θ-stable.

Definition 2.8. Let G be a finite diagonal group $G \subset \mathrm{GL}_3(\mathbb{C})$. Assume that we have a proper birational morphism $Y \to X := \mathbb{C}^3/G$ with a normal toric variety Y. Let $\Sigma_{\max}$ be the set of the 3-dimensional cones in the fan of Y. A *G-brickset* for Y is a set $\mathfrak{S}$ of G-bricks satisfying:

(i) there is a bijective map $\Sigma_{\max} \to \mathfrak{S}$ sending σ to Γ_σ;
(ii) $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

Lemma 2.9 ([5, Proposition 2.16])**.** *Let Y be a normal toric variety with a proper birational morphism $Y \to X := \mathbb{C}^3/G$. Assume that there exist*:

(i) *a G-brickset $\mathfrak{S}$ for $Y \to X$, and*
(ii) *a generic stability parameter θ such that $\Gamma \in \mathfrak{S}$ is θ-stable.*

Then Y is isomorphic to Y_θ.

3. Main result

In this section, we assume that G is the finite group of type $\frac{1}{r}(1, a, b)$ with $r = abc + a + b$, where a, b, c are positive integers and b is coprime to a. We may assume that $a < b$. We apply the methods in [5] for this case to prove the main theorem. For the convenience, set $a_1 = 1$, $a_2 = a$, $a_3 = b$.

3.1. Star subdivisions at v and the resolution

Recall the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, b),$$

and the cone $\sigma_+ = \mathrm{Cone}(e_1, e_2, e_3)$. Let X_v be the corresponding to the star subdivision of σ_+ at the lattice point $v = \frac{1}{r}(1, a, b)$, i.e., the fan of X_v consists of the following cones

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v).$$

Let U_k denote the affine toric variety corresponding to σ_k.

Let L_1, L_2 and L_3 be the sublattices of L generated by $\{v, e_2, e_3\}$, $\{e_1, v, e_3\}$ and by $\{e_1, e_2, v\}$, respectively. Let M_k denote the dual lattice of L_k. Note that M_2 has the dual basis $\{\zeta_1, \zeta_2, \zeta_3\}$ and M_3 has the dual basis $\{\eta_1, \eta_2, \eta_3\}$, where

$$\xi_1 = x_1 x_2^{-\frac{1}{a}}, \quad \xi_2 = x_2^{\frac{r}{a}}, \quad \xi_3 = x_3 x_2^{-\frac{b}{a}},$$
$$\eta_1 = x_1 x_3^{-\frac{1}{b}}, \quad \eta_2 = x_2 x_3^{-\frac{a}{b}}, \quad \eta_3 = x_3^{-\frac{r}{b}}.$$

Note that the lattice inclusion $L_k \hookrightarrow L$ induces a toric morphism

$$\overline{\pi}_k \colon \mathrm{Spec}\, \mathbb{C}[\sigma_k^\vee \cap M_k] \to U_k := \mathrm{Spec}\, \mathbb{C}[\sigma_2^\vee \cap M]$$

with $\mathrm{Spec}\, \mathbb{C}[\sigma_k^\vee \cap M_k] \cong \mathbb{C}^3$ and $U_k \cong \mathbb{C}^3/G_k$, where $G_k := L/L_k$. Thus we have:

(i) U_1 is smooth.

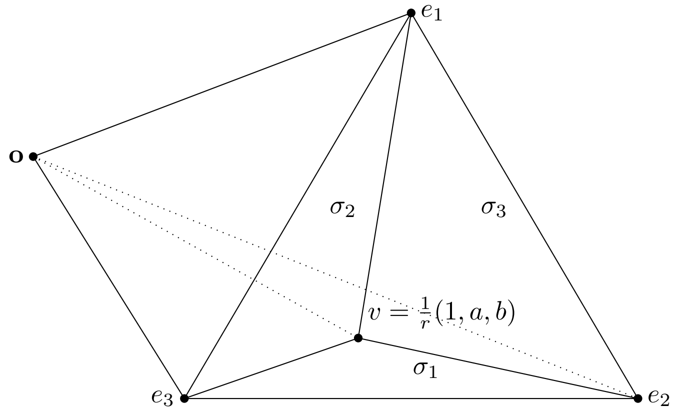

FIGURE 3.1. Star subdivision at v

(ii) U_2 is $\mathbb{C}^3/G_2$, where G_2 is of type $\frac{1}{a}(1, -b, b)$.
(iii) U_3 is $\mathbb{C}^3/G_3$, where G_3 is of type $\frac{1}{b}(1, a, -a)$.

Note that U_2 and U_3 are terminal 3-fold quotient singularities. Since X_v has only terminal singularities, X_v is the relative minimal model over $X = \mathbb{C}^3/G$.

Since each terminal 3-fold quotient singularity has an economic resolution (see [8]), we have the following three resolutions of singularities:

(i) $\overline{\varphi}_1 : Y_1 \to U_1 = \mathbb{C}^3$;
(ii) $\overline{\varphi}_2 : Y_2 \to U_2 = \mathbb{C}^3/G_2$;
(iii) $\overline{\varphi}_3 : Y_3 \to U_3 = \mathbb{C}^3/G_3$,

which induce a projective birational morphism $\varphi : Y \to X$ factoring through X_v fitting into the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\overline{\varphi}} & X_v \\ & \searrow^{\varphi} & \downarrow \\ & & X. \end{array}$$

In what follows, we find a moduli description of the resolution Y in terms of G-constellations.

3.2. Round down functions for star subdivisions

Definition 3.1 (Round down functions)**.** The *round down functions*

$$\phi_2 : \overline{M} \to M_2, \quad \phi_3 : \overline{M} \to M_3$$

for the star subdivision at $\frac{1}{r}(1, a, b)$ are defined by

$$\phi_2(x_1^{m_1} x_2^{m_2} x_3^{m_3}) = \xi_1^{m_1} \xi_2^{\lfloor \frac{1}{r}(m_1 + am_2 + bm_3) \rfloor} \xi_3^{m_3},$$

$$\phi_3(x_1^{m_1}x_2^{m_2}x_3^{m_3}) = \eta_1^{m_1}\eta_2^{m_2}\eta_3^{\lfloor \frac{1}{r}(m_1+am_2+bm_3)\rfloor},$$

where $\lfloor\ \rfloor$ is the floor function. For a G_k-brick Γ', we define

$$\phi_k^\star(\Gamma') := \{\mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \Gamma'\}.$$

Proposition 3.2 ([4, Proposition 4.5])**.** *For a G_k-brick Γ', the set $\Gamma := \phi_k^\star(\Gamma')$ is a G-brick with $S(\Gamma) = S(\Gamma')$. For $\mathbf{m} \in \overline{M}$, we have*

$$\mathrm{wt}_{\Gamma'}\big(\phi_k(\mathbf{m})\big) = \phi_k\big(\mathrm{wt}_\Gamma(\mathbf{m})\big).$$

The key theorem in [5] is the following.

Theorem 3.3 ([5, Theorem 3.12])**.** *Assume that each $Y_k \to U_k$ has a G_k-brickset $\mathfrak{S}_k$. Define*

$$\mathfrak{S} := \bigcup_k \big\{\phi_k^\star(\Gamma') \,\big|\, \Gamma' \in \mathfrak{S}_k\big\}.$$

Then $\mathfrak{S}$ is a G-brickset for the morphism $Y \to X$.

For a monomial $\mathbf{m}$ of weight $\rho \in G^\vee$, we define $\phi_k(\rho)$ is the weight of $\phi_k(\mathbf{m})$. This induces a well-defined surjective map

$$\phi_k\colon G^\vee \to G_k^\vee, \quad \mathrm{wt}(\mathbf{m}) \mapsto \mathrm{wt}(\phi_k(\mathbf{m})). \tag{3.4}$$

From this, we define the linear map

$$(\phi_k)_\star\colon \Theta \to \Theta^{(k)}$$

defined by

$$[(\phi_k)_\star(\theta)](\chi) = \sum_{\phi_k(\rho)=\chi} \theta(\rho) \text{ for } \chi \in G_k^\vee. \tag{3.5}$$

Lemma 3.6 ([5, Lemma 3.16])**.** *Suppose that we have $\theta^{(2)} \in \Theta^{(2)}$ and $\theta^{(3)} \in \Theta^{(3)}$. Then there is $\theta_P \in \Theta$ satisfying*

$$(\phi_k)_\star(\theta_P) \equiv \theta^{(k)} \quad \textit{for all } k. \tag{3.7}$$

3.3. Proof of the main theorem

Through this section, we prove the following theorem. Recall we set $a_1 = 1$, $a_2 = a$, $a_3 = b$.

Theorem 3.8 (Main Theorem)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1, a, b)$ with $r = abc+a+b$, where a, b, c are positive integers and b is coprime to a. Then the birational component Y_θ of θ-stable G-constellations is smooth for a suitable parameter θ.*

For the smooth model $Y \to X$ constructed in Section 3.1 above, recall we have

$$\overline{\varphi}_2\colon Y_2 \to U_2 = \mathbb{C}^3/G_2, \quad \overline{\varphi}_3\colon Y_3 \to U_3 = \mathbb{C}^3/G_3$$

which are the economic resolutions of U_2 and U_3, respectively.

By [4, Theorem 4.19], the economic resolution of a terminal 3-fold quotient singularity $\mathbb{C}^3/G'$ is isomorphic to the birational components of stable G'-constellations. This means for $k \in \{2,3\}$, we have a G_k-brickset $\mathfrak{S}_k$ for $Y_k \to U_k$ such that $\Gamma \in \mathfrak{S}_k$ is $\theta^{(k)}$-stable.

Combining this with Theorem 3.3, we have a G-brickset

$$\mathfrak{S} := \bigcup_k \{\phi_k^\star(\Gamma') \mid \Gamma' \in \mathfrak{S}_k\} \tag{3.9}$$

for the morphism $Y \to X$. Thus by Lemma 2.9, it is enough to find generic $\theta \in \Theta$ such that every $\Gamma \in \mathfrak{S}$ is θ-stable.

Lemma 3.10. *For a G-brick Γ in $\mathfrak{S}$, let $A \subset \Gamma$ be a $\mathbb{C}$-basis of a submodule $\mathcal{G}$ of $C(\Gamma)$. Assume that*

$$A \neq \phi_k^{-1}\big(\phi_k(A)\big) := \big\{\mathbf{m} \in \overline{M} \mid \phi_k(\mathbf{m}) \in \phi_k(A)\big\}.$$

(i) *If $\mathbf{m}_\rho$ is in $\phi_k^{-1}\big(\phi_k(A)\big) \setminus A$, then $0 \le \mathrm{wt}(\rho) < r - a_k$.*
(ii) *There is a monomial $\mathbf{m}_\rho$ of weight ρ in $\phi_k^{-1}\big(\phi_k(A)\big) \setminus A$ with $0 \le \mathrm{wt}(\rho) < a_k$.*

Proof. For (i), assume that $\mathbf{m}_\rho$ is in $\phi_k^{-1}\big(\phi_k(A)\big) \setminus A$. From the definition of ϕ_k, we have $\phi_k(x_k^l \cdot \mathbf{m}_\rho) = \phi_k(\mathbf{m}_\rho)$ for $l > 0$. This is equivalent to that $0 \le \mathrm{wt}(\rho) < r - a_k l$.

For (ii), assume that $\mathbf{m}$ is in $\phi_k^{-1}\big(\phi_k(A)\big) \setminus A$. Choose $l := \lfloor \frac{\mathrm{wt}(\mathbf{m})}{a_k} \rfloor$ and set $\mathbf{m}_\rho$ to be the monomial with $\mathbf{m} = x_k^l \cdot \mathbf{m}_\rho$. Then we have $\phi_k(\mathbf{m}_\rho) = \phi_k(\mathbf{m})$ with $\mathrm{wt}(\mathbf{m}_\rho) < a_k$. □

Lemma 3.11. *Let $\psi \in \Theta$ be the GIT parameter defined by*

$$\psi(\rho) = \begin{cases} -1 & \text{if } 0 \le \mathrm{wt}(\rho) < b, \\ 1 & \text{if } r - b \le \mathrm{wt}(\rho) < r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Then ψ satisfies the following:

(i) *For every ρ with $0 \le \mathrm{wt}(\rho) < b$, we have $\psi(\rho) < 0$.*
(ii) *For each k and for every $\chi \in G_k^\vee$, we have*

$$\sum_{\phi_k(\rho)=\chi} \psi(\rho) = 0.$$

Proof. From the definition of ψ, (i) follows. We prove (ii) for the case $k = 2$. First note that $\phi_2(\rho) = \chi$ if and only if $\mathrm{wt}(\rho) \equiv \mathrm{wt}(\chi) \pmod a$. Since $r = abc + a + b$, we have i and $i + (r - b)$ are equivalent modulo a. Thus for $0 \le i < b$, we get $\phi_2(\rho_i) = \phi_2(\rho_{i+r-b})$ and $\psi(\rho_i) + \psi(\rho_{i+r-b}) = 0$. This proves (ii). □

Remark 3.13. Here, (ii) means that for any monomial $\mathbf{m}$ we have

$$\sum_{\phi_k(\rho)=\phi_k(\mathbf{m})} \psi(\rho) = 0.$$

Theorem 3.14. *Fix $\theta_P \in \Theta$ satisfying* (3.7). *For a sufficiently large m, set*

$$\theta := \theta_P + m\psi.$$

Then every G-brick $\Gamma \in \mathfrak{S}$ defined by (3.9) *is θ-stable.*

Proof. Let σ be the corresponding cone to the G-brick Γ. Then σ is contained in one of the cones σ_1, σ_2, σ_3.

Suppose that $\sigma \subset \sigma_1$. This means that $\sigma = \sigma_1$ and

$$\Gamma = \{1, x_1, x_1^2, \dots, x_1^{r-2}, x_1^{r-1}\}.$$

Any nonzero proper submodule of $C(\Gamma)$ has a $\mathbb{C}$-basis of the form

$$A = \{x_1^j, x_1^{j+1}, \dots, x_1^{r-2}, x_1^{r-1}\}$$

for $1 \leq j < r$. Since $\psi(A) > 0$, Γ is θ-stable for $m \gg 0$.

Let us assume that $\sigma \subset \sigma_k$. Then we have the corresponding G_k-brick Γ' to Γ, i.e., $\Gamma = \phi_k^\star(\Gamma') = \{\mathbf{m} \in \overline{M} \,|\, \phi_k(\mathbf{m}) \in \Gamma'\}$. Let $\mathcal{G}$ be a nonzero proper submodule $\mathcal{G}$ of $C(\Gamma)$. We need to show that $\theta(\mathcal{G}) > 0$. Let $A \subsetneq \Gamma$ be a $\mathbb{C}$-basis of $\mathcal{G}$. We have the following two cases:

$$(1)\text{: } A \subsetneq \phi_k^{-1}\big(\phi_k(A)\big) \quad \text{or} \quad (2)\text{: } A = \phi_k^{-1}\big(\phi_k(A)\big).$$

In case (1), by Remark 3.13 and Lemma 3.10, we have

$$\sum_{\rho \in \phi_k^{-1}(\phi_k(A))} \psi(\rho) = 0, \text{ and } \sum_{\rho \in \phi_k^{-1}(\phi_k(A)) \setminus A} \psi(\rho) < 0.$$

Therefore $\psi(\mathcal{G}) > 0$ and $\theta(\mathcal{G}) > 0$ for sufficiently large m.

In case (2), by (ii) of Lemma 3.11, we have $\psi(\mathcal{G}) = 0$. Note that $\phi_k(A)$ defines a nonzero proper submodule $\mathcal{G}'$ of $C(\Gamma')$. Since $C(\Gamma')$ is $\theta^{(k)}$-stable, we have $\theta^{(k)}(\mathcal{G}') > 0$. From (3.7), we have $\theta(\mathcal{G}) = \theta^{(k)}(\mathcal{G}') > 0$. □

Corollary 3.15. *With the notation above, the birational component Y_θ is isomorphic to Y defined in Section 3.1.*

This completes the proof of Theorem 3.8.

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