Bull. Korean Math. Soc. **61** (2024), No. 2, pp. 433–450
https://doi.org/10.4134/BKMS.b230136
pISSN: 1015-8634 / eISSN: 2234-3016

GEOMETRIC INEQUALITIES FOR AFFINE CONNECTIONS ON RIEMANNIAN MANIFOLDS

Huiting Chang and Fanqi Zeng

Abstract. Using a Reilly type integral formula due to Li and Xia [23], we prove several geometric inequalities for affine connections on Riemannian manifolds. We obtain some general De Lellis-Topping type inequalities associated with affine connections. These not only permit to derive quickly many well-known De Lellis-Topping type inequalities, but also supply a new De Lellis-Topping type inequality when the 1-Bakry-Émery Ricci curvature is bounded from below by a negative function. On the other hand, we also achieve some Lichnerowicz type estimate for the first (nonzero) eigenvalue of the affine Laplacian with the Robin boundary condition on Riemannian manifolds.

1. Introduction

The classical Reilly formula goes back to R. C. Reilly, who showed in [28] an integral version of a Bochner type formula. Since then, the Reilly type formulas have been enriched extensively and broadly, and they play important roles in many branches of analysis and geometry. A huge literature exists on the Reilly type formulas, and it's just impossible for our intention to mention all the progress. So we refer to the classical and recent works [11,15,17,19–22,24,25,27] for interested readers.

More recently, in the important works of Li and Xia [23], the authors introduced a 2-parameter family of affine connections and derived the corresponding Ricci curvature. Then they established an integral Bochner type formula and obtained from it various geometric inequalities under more general Ricci curvature conditions, for example, Heintze-Karchner type and Minkowski type inequalities. They also obtained from it the closed, the Dirichlet and the Neumann first (nonzero) eigenvalue estimates. Later, Huang, Ma and Zhu [16]

Received March 8, 2023; Accepted September 22, 2023.
2010 *Mathematics Subject Classification.* Primary 53C21; Secondary 58J32.
Key words and phrases. Reilly type formula, affine connection, De Lellis-Topping inequality, eigenvalue.
The research of authors is supported by NSFC (No.12101530), the Science and Technology Project of Henan Province (No.232102310321), and the Key Scientific Research Program in Universities of Henan Province (Nos.21A110021, 22A110021) and Nanhu Scholars Program for Young Scholars of XYNU (No.2019).

achieved some Colesanti type inequality by using such new Reilly type integral formula with respect to affine connections. In this paper, we use this generalized Reilly formula and suitable elliptic PDEs to prove De Lellis-Topping type inequalities and Lichnerowicz type inequalities associated with affine connections under more general Ricci curvature conditions.

Let us provide the rigorous definitions we are going to employ for affine connections on Riemannian manifolds (see [16, 23]).

Let $(M, \widetilde{g})$ be an n-dimensional Riemannian manifold with the boundary ∂M and $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{g}$. Let $V = e^f$, where f is a smooth function defined on M. We call $(M, \widetilde{g}, V)$ a *Riemannian triple*.

For two real constants α, γ and two vector fields X, Y on M, we define a 2-parameter family of *affine connections* by

$$D_X^{\alpha,\gamma} Y := \widetilde{\nabla}_X Y + \alpha \mathrm{d}f(X)Y + \alpha \mathrm{d}f(Y)X + \gamma \widetilde{g}(X,Y)\widetilde{\nabla} f. \tag{1.1}$$

As Li and Xia [23] have already noted, when $\alpha = \gamma = 0$, $D^{\alpha,\gamma}$ is a Levi-Civita connection for $\widetilde{g}$; when $\alpha = -\gamma$, $D^{\alpha,\gamma}$ is a Levi-Civita connection for the conformal metric $e^{2\alpha f}\widetilde{g}$. Except for $\alpha = \gamma = 0$ and $\alpha = -\gamma$, $D^{\alpha,\gamma}$ may not be a Levi-Civita connection for any Riemannian metric. Then the Ricci curvature with respect to the affine connection $D^{\alpha,\gamma}$ is given by

$$\begin{aligned}\mathrm{Ric}^{D^{\alpha,\gamma}} := \mathrm{Ric} - [(n-1)\alpha + \gamma]\widetilde{\nabla}^2 f + \left[(n-1)\alpha^2 - \gamma^2\right] \mathrm{d}f \otimes \mathrm{d}f \\ + \left[\gamma \widetilde{\Delta} f + (\gamma^2 + (n-1)\alpha\gamma)|\widetilde{\nabla} f|^2\right] \widetilde{g}.\end{aligned} \tag{1.2}$$

$\mathrm{Ric}^{D^{\alpha,\gamma}}$ unifies various particular cases recently studied in the literature, such as Ricci tensor, static Ricci tensor, 1-Bakry-Émery Ricci tensor and conformal Ricci tensor.

Throughout this paper, we use $\widetilde{\nabla}$ and $\widetilde{\Delta}$ to denote the gradient and the Laplacian on M, respectively; ∇ and Δ to denote the gradient and the Laplacian on ∂M, respectively, with respect to the induced metric g. The volume form of $\widetilde{g}$ is $\mathrm{d}\Omega$ and the volume form of induced metric g on ∂M is $\mathrm{d}A$, respectively. The mean curvature H of ∂M is given by $H = \mathrm{tr}_{\widetilde{g}}(II)$, where $II(X,Y) = \widetilde{g}(\widetilde{\nabla}_X \mathbf{n}, Y)$ denotes the second fundamental form with $\mathbf{n}$ the outward unit normal on ∂M. We use $\widetilde{\nabla}^{D^{\alpha,\gamma}}$, $\widetilde{\nabla}^{2,D^{\alpha,\gamma}}$ and $\widetilde{\Delta}^{D^{\alpha,\gamma}}$ to denote the *affine gradient*, the *affine Hessian*, and the *affine Laplacian* with respect to $D^{\alpha,\gamma}$; the exact definition will be given in Section 2. We also make the following conventions:

$$II^{D^{\alpha,\gamma}} = II - \gamma(\ln V)_{\mathbf{n}} g$$

and

$$H^{D^{\alpha,\gamma}} = H + (n-1)\alpha(\ln V)_{\mathbf{n}}.$$

In celebrated paper [10], De Lellis and Topping proved (and independently by Andrews, cf. [8, Corollary B. 20]) a classical inequality on closed manifolds. Later, this inequality was called the De Lellis-Topping type inequality. In recent years, De Lellis-Topping type inequalities in other spaces (for instance, CR manifolds, smooth metric measure spaces, sub-static manifolds, asymptotically Euclidean manifolds, etc.) have also been paid attention to. For instance, Chen, Saotome and Wu [5] gave a CR version of De Lellis-Topping type inequality; Wu [30] showed a De Lellis-Topping type inequality in smooth metric measure spaces; Li and Xia [24] obtained a weighted De Lellis-Topping type inequality for closed sub-static manifolds; Avalos and Freitas [1] proved a De Lellis-Topping type inequality in asymptotically Euclidean manifolds which does not need restrictions on the Ricci curvature. Cheng proved in [7] a De Lellis-Topping type inequality for symmetric $(0,2)$-tensors satisfying a second Bianchi type identity on closed manifolds. Another recent generalization of the De Lellis-Topping inequality, see, e.g., [3,4,6,9,11,12,14,15,18,26,31].

In view of these developments, inspired by the important works of Li and Xia [23], it is natural to investigate the more general De Lellis-Topping type inequalities for affine connections. In this paper, we will give a more general De Lellis-Topping type inequality for affine connections. We note that our results include many previously known results as special cases.

The main result of this paper is the following De Lellis-Topping type inequalities.

Theorem 1.1. *Let $(M^n, \widetilde{g}, V = e^f)$ be an n-dimensional compact Riemannian triple with $n \geq 3$ and boundary ∂M. Let $D^{\alpha,\gamma}$ be the affine connection defined as in* (1.1) *and $\tau = (n+1)\alpha + \gamma$. Let T be a symmetric $(0,2)$-tensor such that $T(\mathbf{n}, \cdot) \geq 0$ along the boundary and $\mathrm{div} T = c\widetilde{\nabla} B$, where $B = \mathrm{tr}_{\widetilde{g}} T$ denotes its trace and c is a constant. If $V^{\gamma-\alpha}\mathrm{Ric}^{D^{\alpha,\gamma}} \geq -(n-1)K_1\widetilde{g}$ for some nonnegative constant K_1, and $K_2 = \max(V^{\alpha-\gamma}|\widetilde{\nabla}^{D^{\alpha,\gamma}} f|^2)$, and $II^{D^{\alpha,\gamma}} \geq 0$, then*

$$\frac{(nc-1)^2}{n^2} \int_M V^\tau (B - \overline{B}^{V^\tau})^2 \,\mathrm{d}\Omega \leq C_{n,K_1,K_2,\lambda_1} \int_M V^\tau \left| T - \frac{B}{n}\widetilde{g} \right|^2 \mathrm{d}\Omega \tag{1.3}$$

and, equivalently,

$$\frac{(nc-1)^2}{n} \int_M V^\tau \left| T - \frac{\overline{B}^{V^\tau}}{n}\widetilde{g} \right|^2 \mathrm{d}\Omega \leq \widehat{C}_{n,K_1,K_2,\lambda_1} \int_M V^\tau \left| T - \frac{B}{n}\widetilde{g} \right|^2 \mathrm{d}\Omega, \tag{1.4}$$

where

$$\begin{aligned} C_{n,K_1,K_2,\lambda_1} = {} & \frac{n-1}{n} + \frac{1}{\lambda_1}\Big[(n-1)K_1 + [(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2 \\ & + 2\sqrt{[\lambda_1 + (n-1)K_1][(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}\Big], \\ \widehat{C}_{n,K_1,K_2,\lambda_1} = {} & \frac{(nc-1)^2 + n - 1}{n} + \frac{1}{\lambda_1}\Big[(n-1)K_1 + [(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2 \\ & + 2\sqrt{[\lambda_1 + (n-1)K_1][(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}\Big], \end{aligned}$$

$\overline{B}^{V^\tau} = \frac{\int_M BV^\tau \,\mathrm{d}\Omega}{\int_M V^\tau \,\mathrm{d}\Omega}$, $II^{D^{\alpha,\gamma}} = II - \gamma(\ln V)_{\mathbf{n}} g$ *and* λ_1 *indicates the Neumann first* (*nonzero*) *eigenvalue of the affine Laplacian* $\widetilde{\Delta}^{D^{\alpha,\gamma}}$, *i.e., there exists some non-trivial* ϕ *such that* $\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi = -\lambda_1\phi$ *with Neumann boundary condition* $\phi_{\mathbf{n}} = 0$.

Remark 1.1. The following are some classical or more recent De Lellis-Topping type inequalities which can be derived by the inequalities (1.3)-(1.4).

(I) In the case $\alpha = \gamma = 0$, it is easy to see from (1.2) that

$$V^{\gamma-\alpha}\mathrm{Ric}^{D^{\alpha,\gamma}} = \mathrm{Ric}.$$

For the case $\alpha = \gamma = 0$, taking $T = \mathrm{Ric}$, $K_1 = 0$ and $\partial M = \emptyset$ in Theorem 1.1, we get classical De Lellis-Topping inequalities shown by De Lellis and Topping [10, Theorem 1.1]. On the other hand, for the case $\alpha = \gamma = 0$, taking $\partial M = \emptyset$ in Theorem 1.1, we get De Lellis-Topping type inequalities shown by Cheng [7, Theorem 1.7].

(II) In the case $\alpha = 0$, $\gamma = 1$, it is easy to see from (1.2) that

$$V^{\gamma-\alpha}\mathrm{Ric}^{D^{\alpha,\gamma}} = V\left(\mathrm{Ric} - \frac{\widetilde{\nabla}^2 V}{V} + \frac{(\widetilde{\Delta}V)}{V}\widetilde{g}\right),$$

where $\mathrm{Ric} - \frac{\widetilde{\nabla}^2 V}{V} + \frac{(\widetilde{\Delta}V)}{V}\widetilde{g}$ is exactly the static Ricci tensor, see [24]. For the case $\alpha = 0$, $\gamma = 1$, taking $T = \mathrm{Ric}$, $K_1 = 0$ and $\partial M = \emptyset$ in Theorem 1.1, we get a weighted De Lellis-Topping type inequality shown by Li and Xia [24, Theorem 6.1]. On the other hand, for the case $\alpha = 0$, $\gamma = 1$, taking $\partial M = \emptyset$ in Theorem 1.1, we get De Lellis-Topping type inequalities shown by Zeng [31, Theorem 1.2].

(III) In the case $\alpha = \frac{1}{n-1}$, $\gamma = 0$, it is easy to see from (1.2) that

$$V^{\gamma-\alpha}\mathrm{Ric}^{D^{\alpha,\gamma}} = e^{-\frac{1}{n-1}f}(\mathrm{Ric} - \widetilde{\nabla}^2 f + \frac{1}{n-1}\mathrm{d}f \otimes \mathrm{d}f),$$

where $\mathrm{Ric} - \widetilde{\nabla}^2 f + \frac{1}{n-1}\mathrm{d}f \otimes \mathrm{d}f$ is 1-Bakry-Émery Ricci tensor in the literature which was introduced by Bakry and Émery [2]. For the case $\alpha = \frac{1}{n-1}$, $\gamma = 0$, Theorem 1.1 gives new De Lellis-Topping type inequalities when the 1-Bakry-Émery Ricci curvature is bounded from below by a negative function. We list the inequalities as follows:

Corollary 1.2. *Let* $(M^n, \widetilde{g}, e^f \mathrm{d}\Omega)$ *be a smooth weighted Riemannian manifold with* $n \geq 3$ *and boundary* ∂M. *Let* $D^{\frac{1}{n-1},0}$ *be the affine connection defined as in* (1.1) *in the case* $\alpha = \frac{1}{n-1}$, $\gamma = 0$. *Let* T *be a symmetric* $(0,2)$*-tensor such that* $T(\mathbf{n},\cdot) \geq 0$ *along the boundary and* $\mathrm{div}T = c\widetilde{\nabla}B$, *where* $B = \mathrm{tr}_{\widetilde{g}}T$ *denotes its trace and* c *is a constant. If*

$$\mathrm{Ric} - \widetilde{\nabla}^2 f + \frac{1}{n-1}\mathrm{d}f \otimes \mathrm{d}f \geq -(n-1)K_1 e^{\frac{1}{n-1}f}\widetilde{g}$$

for some nonnegative constant K_1*, and* $K_2 = \max(e^{\frac{1}{n-1}f}|\widetilde{\nabla}^{D^{\frac{1}{n-1},0}} f|^2)$*, and* $II^{D^{\frac{1}{n-1},0}} \geq 0$*, then*

$$\frac{(nc-1)^2}{n^2}\int_M V^\tau (B-\overline{B}^{V^\tau})^2\,\mathrm{d}\Omega \leq \overline{C}_{n,K_1,K_2,\lambda_1}\int_M V^\tau \left|T-\frac{B}{n}\widetilde{g}\right|^2\mathrm{d}\Omega \tag{1.5}$$

and, equivalently,

$$\frac{(nc-1)^2}{n}\int_M V^\tau \left|T-\frac{\overline{B}^{V^\tau}}{n}\widetilde{g}\right|^2\mathrm{d}\Omega \leq \widetilde{C}_{n,K_1,K_2,\lambda_1}\int_M V^\tau \left|T-\frac{B}{n}\widetilde{g}\right|^2\mathrm{d}\Omega, \tag{1.6}$$

where

$$\begin{aligned}\overline{C}_{n,K_1,K_2,\lambda_1} &= \frac{n-1}{n}+\frac{1}{\lambda_1}\left[(n-1)K_1+\frac{n^2+n}{(n-1)^2}K_2\right.\\ &\quad\left.+\frac{2}{n-1}\sqrt{[\lambda_1+(n-1)K_1](n^2+n)K_2}\right],\\ \widetilde{C}_{n,K_1,K_2,\lambda_1} &= \frac{(nc-1)^2+n-1}{n}+\frac{1}{\lambda_1}\left[(n-1)K_1+\frac{n^2+n}{(n-1)^2}K_2\right.\\ &\quad\left.+\frac{2}{n-1}\sqrt{[\lambda_1+(n-1)K_1](n^2+n)K_2}\right],\end{aligned}$$

$\overline{B}^{V^\tau} = \frac{\int_M BV^\tau\,\mathrm{d}\Omega}{\int_M V^\tau\,\mathrm{d}\Omega}$, $\tau = \frac{n+1}{n-1}$ *and* λ_1 *indicates the Neumann first* (*nonzero*) *eigenvalue of the affine Laplacian* $\widetilde{\Delta}^{D^{\frac{1}{n-1},0}}$*, i.e., there exists some non-trivial* ϕ *such that* $\widetilde{\Delta}^{D^{\frac{1}{n-1},0}}\phi = -\lambda_1\phi$ *with Neumann boundary condition* $\phi_{\mathbf{n}} = 0$*.*

In the following, we consider the eigenvalue problem with Robin boundary condition:

$$\begin{cases}\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi = -\tau\phi, & \text{in } M,\\ \phi\cos\theta + V^\gamma\phi_{\mathbf{n}}\sin\theta = 0, & \text{on } \partial M,\end{cases} \tag{1.7}$$

where $\theta \in [0,\pi)$ is a constant.

Clearly, if $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively, we see that the Robin boundary problem (1.7) reduces to the Dirichlet boundary problem for PDE

$$\begin{cases}\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi = -\xi\phi, & \text{in } M,\\ \phi = 0, & \text{on } \partial M,\end{cases} \tag{1.8}$$

and the Neumann boundary problem for PDE

$$\begin{cases}\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi = -\lambda\phi, & \text{in } M,\\ \phi_{\mathbf{n}} = 0, & \text{on } \partial M.\end{cases} \tag{1.9}$$

Denote by τ_1 the first (nonzero) eigenvalues of (1.7). Then, we obtain the following:

Theorem 1.3. *Let* $(M^n, \widetilde{g}, V = e^f)$ *be an n-dimensional compact Riemannian triple with* $\mathrm{Ric}^{D^{\alpha,\gamma}} \geq (n-1)V^{\alpha-\gamma}\widetilde{g}$. *Then the first* (*nonzero*) *eigenvalues of eigenvalue problem* (1.7) *satisfy*

$$\tau_1^2 - n\tau_1 \geq \begin{cases} C_1(M, f, \phi, \theta), & \text{if } \theta \neq \frac{\pi}{2}, \\ C_2(M, f, \phi, \theta), & \text{if } \theta \neq 0, \end{cases}$$

where

$$\begin{aligned} &C_1(M, f, \phi, \theta) \\ &= \frac{n}{n-1}\int_{\partial M} V^\tau \Big[2V^{-\gamma}\tan\theta \big|\nabla^{D^{\alpha,\gamma}}(V^\gamma \phi_{\mathbf{n}})\big|^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) \\ &\quad + \big(H^{D^{\alpha,\gamma}}V^{-\alpha} - (n-1)\tan\theta\big)V^{2\gamma-\alpha}\phi_{\mathbf{n}}^2\Big]\,\mathrm{d}A \cdot \left(\int_M V^\tau \phi^2\,\mathrm{d}\Omega\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} &C_2(M, f, \phi, \theta) \\ &= \frac{n}{n-1}\int_{\partial M} V^\tau \Big[2V^{-\gamma}\cot\theta \big|\nabla^{D^{\alpha,\gamma}}\phi\big|^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) \\ &\quad + \big(H^{D^{\alpha,\gamma}}V^{-\alpha}\cot^2\theta - (n-1)\cot\theta\big)V^{-\alpha}\phi^2\Big]\,\mathrm{d}A \cdot \left(\int_M V^\tau \phi^2\,\mathrm{d}\Omega\right)^{-1}. \end{aligned}$$

Consequently, by taking $\partial M = \emptyset$, $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively, we have the following:

Corollary 1.4 ([23])**.** *Let* $(M^n, \widetilde{g}, V = e^f)$ *be an n-dimensional compact Riemannian triple with* $\mathrm{Ric}^{D^{\alpha,\gamma}} \geq (n-1)V^{\alpha-\gamma}\widetilde{g}$. *Then we have the following results.*

(a) *Assume that* $\partial M = \emptyset$. *Then*

$$\eta_1 \geq n.$$

(b) *Assume that* $\partial M \neq \emptyset$ *and* ∂M *satisfies* $H^{D^{\alpha,\gamma}} \geq 0$. *Then*

$$\xi_1 \geq n.$$

(c) *Assume that* $\partial M \neq \emptyset$ *and* ∂M *satisfies* $II^{D^{\alpha,\gamma}} \geq 0$. *Then*

$$\lambda_1 \geq n.$$

Here η_1, ξ_1 *and* λ_1 *indicate the closed, the Dirichlet and the Neumann first* (*nonzero*) *eigenvalue of the affine Laplacian* $\widetilde{\Delta}^{D^{\alpha,\gamma}}$.

From Theorem 1.3, we have the following Lichnerowicz type estimate for the first (nonzero) eigenvalue of the affine Laplacian $\widetilde{\Delta}^{D^{\alpha,\gamma}}$ with the Robin boundary condition on Riemannian manifolds.

Theorem 1.5. *Let* $(M^n, \widetilde{g}, V = e^f)$ *be an n-dimensional compact Riemannian triple with* $\mathrm{Ric}^{D^{\alpha,\gamma}} \geq (n-1)V^{\alpha-\gamma}\widetilde{g}$. *Then we have the following results.*

(i) *Assume that* ∂M *satisfies* $II^{D^{\alpha,\gamma}} \geq 0$ *and* $H^{D^{\alpha,\gamma}} \geq [(n-1)\tan\theta]V^{\alpha}$, $\theta \in [0, \frac{\pi}{2})$. *Then*

$$\tau_1 \geq n.$$

(ii) *Assume that there is a nonzero eigenfunction* ϕ *with respect to* τ_1 *satisfying*

$$\phi = const., \ \phi\cos\theta + V^{\gamma}\phi_{\mathbf{n}}\sin\theta = 0, \ \text{on } \partial M,$$

and $H^{D^{\alpha,\gamma}}\cot^2\theta \geq [(n-1)\cot\theta]V^{\alpha}$, $\theta \in (0, \frac{\pi}{2}]$. *Then*

$$\tau_1 \geq n.$$

Remark 1.2. In particular, if $\alpha = \gamma = 0$, then Theorems 1.3 and 1.5 reduce to [29, Theorem 1.1], [29, Theorem 1.2] and [29, Theorem 1.3], respectively.

The rest of this paper is organized as follows. In Section 2, we recall the notations and the Reilly type integral formula under affine connections. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorems 1.3 and 1.5.

2. Preliminaries

In this section, we recall some preliminary results in Li and Xia [23].

2.1. Torsion-free affine connections $D^{\alpha,\gamma}$

For two real constants α, γ and two vector fields X, Y on M, a 2-parameter family of affine connections $D^{\alpha,\gamma}$ is defined by

$$D^{\alpha,\gamma}_X Y := \widetilde{\nabla}_X Y + \alpha \mathrm{d}f(X)Y + \alpha \mathrm{d}f(Y)X + \gamma\widetilde{g}(X,Y)\widetilde{\nabla}f,$$

where f is a smooth function defined on M. One can check that $D^{\alpha,\gamma}_X Y$ is *torsion-free*.

2.2. Affine gradient, affine Hessian and affine Laplacian

For a bounded domain Ω with boundary Σ in an n-dimensional smooth Riemannian triple $(M^n, \widetilde{g}, V = e^f)$.

(1) The affine gradient on Ω and Σ are defined, respectively, by

$$\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi := V^{\gamma-\alpha}\widetilde{\nabla}\phi, \ \nabla^{D^{\alpha,\gamma}}\phi := V^{\gamma-\alpha}\nabla\phi.$$

(2) The affine Hessian $\widetilde{\nabla}^{2,D^{\alpha,\gamma}}\phi$ and affine Laplacian $\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi$ on Ω are defined, respectively, by

$$\begin{aligned}\widetilde{\nabla}^{2,D^{\alpha,\gamma}}\phi &:= D^{\alpha,\gamma}(V^{\gamma-\alpha}\widetilde{\nabla}\phi)\\ &= V^{\gamma-\alpha}[\widetilde{\nabla}^2\phi + \gamma \mathrm{d}f\otimes \mathrm{d}\phi + \gamma \mathrm{d}\phi\otimes \mathrm{d}f + \alpha\langle\widetilde{\nabla}f, \widetilde{\nabla}\phi\rangle\widetilde{g}],\end{aligned}$$

and

$$\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi := \mathrm{tr}_{\widetilde{g}}(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}\phi) = V^{\gamma-\alpha}[\widetilde{\Delta}\phi + (2\gamma + n\alpha)\langle\widetilde{\nabla}f, \widetilde{\nabla}\phi\rangle].$$

2.3. Reilly type integral formula

Lemma 2.1 ([23]). *Let W be any smooth vector field on M. Then*

$$V^\tau D_i^{\alpha,\gamma} W^i = \widetilde{\nabla}_i (V^\tau W^i) \tag{2.1}$$

is a divergent form with respect to the Riemannian volume form $\mathrm{d}\Omega$, *where* $\tau = (n+1)\alpha + \gamma$ *and we adopt the Einstein convention.*

The following Reilly type integral formula with respect to $D^{\alpha,\gamma}$ has been proved:

Lemma 2.2 ([23]). *Let $(M^n, \widetilde{g}, V = e^f)$ be an n-dimensional compact Riemannian triple with $n \geq 3$ and boundary ∂M. Let $D^{\alpha,\gamma}$ be the affine connection defined as in* (1.1) *and $\tau = (n+1)\alpha + \gamma$. Let ϕ be a smooth function on a bounded domain $\Omega \subset M$ with smooth boundary Σ. Then the following integral formula holds*:

$$\begin{aligned} &\int_\Omega V^\tau \left[\left(\widetilde{\Delta}^{D^{\alpha,\gamma}} \phi \right)^2 - \left| \widetilde{\nabla}^{2,D^{\alpha,\gamma}} \phi \right|_{\widetilde{g}}^2 - \mathrm{Ric}^{D^{\alpha,\gamma}} (\widetilde{\nabla}^{D^{\alpha,\gamma}} \phi, \widetilde{\nabla}^{D^{\alpha,\gamma}} \phi) \right] \mathrm{d}\Omega \\ = &\int_\Sigma V^\tau \Big[H^{D^{\alpha,\gamma}} \langle \widetilde{\nabla}^{D^{\alpha,\gamma}} \phi, \mathbf{n} \rangle^2 + II^{D^{\alpha,\gamma}} \left(\nabla^{D^{\alpha,\gamma}} \phi, \nabla^{D^{\alpha,\gamma}} \phi \right) \\ &\qquad - 2V^{-\gamma} \left\langle \nabla^{D^{\alpha,\gamma}} \phi, \nabla^{D^{\alpha,\gamma}} (V^\gamma \phi_{\mathbf{n}}) \right\rangle \Big] \mathrm{d}A, \end{aligned} \tag{2.2}$$

where

$$II^{D^{\alpha,\gamma}} = II - \gamma (\ln V)_{\mathbf{n}} g$$

and

$$H^{D^{\alpha,\gamma}} = H + (n-1)\alpha (\ln V)_{\mathbf{n}}.$$

3. Proof of Theorem 1.1

Let $\widetilde{\nabla}$ denote the Levi-Civita connection on $(M, \widetilde{g})$ and also the induced connections on tensor bundles on M. Let T be a symmetric $(0,2)$-tensor field on M and denote by B the trace of T. Denote by $\overline{B}^{V^\tau} = \frac{\int_M BV^\tau \,\mathrm{d}\Omega}{\int_M V^\tau \,\mathrm{d}\Omega}$ and set $\mathring{T} = T - \frac{B}{n}\widetilde{g}$. We adopt the Einstein convention.

We note that if $c = \frac{1}{n}$, then

$$(nc-1)^2 \int_M V^\tau (B - \overline{B}^{V^\tau})^2 \,\mathrm{d}\Omega = 0$$

and inequality (1.3) or (1.4) follows trivially. Hence, it suffices to prove the case $c \neq \frac{1}{n}$.

Now, let us suppose that $c \neq \frac{1}{n}$. By the assumption $\mathrm{div} T = c\widetilde{\nabla} B$,

$$\mathrm{div} \mathring{T} = \mathrm{div} \left(T - \frac{B}{n} \widetilde{g} \right) = \frac{nc-1}{n} \widetilde{\nabla} B. \tag{3.1}$$

We let $u : M \to \mathbb{R}$ be the unique solution to the following PDE with the Neumann boundary condition

$$\begin{cases} \widetilde{\Delta}^{D^{\alpha,\gamma}} u = B - \overline{B}^{V^\tau}, & \text{in } M, \\ u_{\mathbf{n}} = 0, & \text{on } \partial M. \end{cases} \tag{3.2}$$

Recall that $\widetilde{\Delta}^{D^{\alpha,\gamma}} u = V^{\gamma-\alpha}[\widetilde{\Delta} u + (2\gamma + n\alpha)\langle \widetilde{\nabla} f, \widetilde{\nabla} u\rangle]$. The existence and uniqueness of equation (3.2) is due to the standard elliptic PDE theory (refer to [13]).

Under the local coordinates $\{\partial_i\}$, by using (3.2) and the divergence theorem, we have

$$\begin{aligned} & (nc-1)\int_M \left(B - \overline{B}^{V^\tau}\right)^2 V^\tau \, \mathrm{d}\Omega \\ = & (nc-1)\int_M \left(B - \overline{B}^{V^\tau}\right) V^\tau \widetilde{\Delta}^{D^{\alpha,\gamma}} u \, \mathrm{d}\Omega \\ = & (nc-1)\int_M \left(B - \overline{B}^{V^\tau}\right) \widetilde{\nabla}_i \left(V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u\right) \mathrm{d}\Omega \\ = & -(nc-1)\int_M V^\tau \widetilde{\nabla}_i B \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u \, \mathrm{d}\Omega, \end{aligned} \tag{3.3}$$

where we have used in the second equality the fact that

$$V^\tau \widetilde{\Delta}^{D^{\alpha,\gamma}} u = \widetilde{\nabla}_i (V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u)$$

(see Lemma 2.1).

Using (3.1) and (3.3), we have

$$\begin{aligned} & (nc-1)\int_M \left(B - \overline{B}^{V^\tau}\right)^2 V^\tau \, \mathrm{d}\Omega \\ = & -n\int_M \mathring{T}_{ij,j} V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u \, \mathrm{d}\Omega \\ = & -n\int_M \mathring{T}_{ij,j} V^{\tau+\gamma-\alpha} \widetilde{\nabla}_i u \, \mathrm{d}\Omega \\ = & \, n\int_M \mathring{T}_{ij} \widetilde{\nabla}_j \left(V^{\tau+\gamma-\alpha} \widetilde{\nabla}_i u\right) \mathrm{d}\Omega - n\int_{\partial M} \mathring{T}\left(V^{\tau+\gamma-\alpha} \widetilde{\nabla} u, \mathbf{n}\right) \mathrm{d}A \\ = & \, n\int_M \mathring{T}_{ij} \widetilde{\nabla}_j \left(V^{\tau+\gamma-\alpha} \widetilde{\nabla}_i u\right) \mathrm{d}\Omega - n\int_{\partial M} (T - \frac{B}{n}\widetilde{g})\left(V^{\tau+\gamma-\alpha} \widetilde{\nabla} u, \mathbf{n}\right) \mathrm{d}A \\ \leq & \, n\int_M \mathring{T}_{ij} \widetilde{\nabla}_j \left(V^{\tau+\gamma-\alpha} \widetilde{\nabla}_i u\right) \mathrm{d}\Omega, \end{aligned} \tag{3.4}$$

where in the last inequality we have used $u_{\mathbf{n}} = 0$ and $T(\mathbf{n}, \cdot) \geq 0$ on ∂M.

By a direct calculation, we know

$$\widetilde{\nabla}_j \left(V^{\tau+\gamma-\alpha} \widetilde{\nabla}_i u\right) = V^{\tau+\gamma-\alpha}\left[u_{ij} + (\tau + \gamma - \alpha) u_i f_j\right]. \tag{3.5}$$

Recall that

$$\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u = V^{\gamma-\alpha}\left[\widetilde{\nabla}^2 u + \gamma \mathrm{d}f \otimes \mathrm{d}u + \gamma \mathrm{d}u \otimes \mathrm{d}f + \alpha\langle\widetilde{\nabla}u, \widetilde{\nabla}f\rangle\widetilde{g}\right].$$

Under the local coordinates $\{\partial_i\}$, we have

$$u_{ij} = V^{\alpha-\gamma}(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij} - \gamma u_i f_j - \gamma f_i u_j - \alpha u_k f_k \widetilde{g}_{ij}. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\begin{aligned}\widetilde{\nabla}_j\left(V^{\tau+\gamma-\alpha}\widetilde{\nabla}_i u\right) = V^{\tau+\gamma-\alpha}\big[&V^{\alpha-\gamma}(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij} - \gamma u_i f_j - \gamma f_i u_j \\ &- \alpha u_k f_k \widetilde{g}_{ij} + (\tau+\gamma-\alpha)u_i f_j\big].\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.4), we have

$$\begin{aligned}(nc-1)&\int_M \left(B - \overline{B}^{V^\tau}\right)^2 V^\tau \,\mathrm{d}\Omega \\ &\le n\int_M \mathring{T}_{ij}\widetilde{\nabla}_j\left(V^{\tau+\gamma-\alpha}\widetilde{\nabla}_i u\right)\mathrm{d}\Omega \\ &= n\int_M \mathring{T}_{ij}V^\tau\left((\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij} + (\tau-\gamma-\alpha)V^{\gamma-\alpha}u_i f_j - \alpha V^{\gamma-\alpha}u_k f_k \widetilde{g}_{ij}\right)\mathrm{d}\Omega \\ &= n\int_M \mathring{T}_{ij}V^\tau D_{ij}\,\mathrm{d}\Omega \\ &= n\int_M \mathring{T}_{ij}V^\tau \mathring{D}_{ij}\,\mathrm{d}\Omega \\ &\le n\left(\int_M V^\tau|\mathring{T}_{ij}|^2\,\mathrm{d}\Omega\right)^{\frac{1}{2}}\left(\int_M V^\tau|\mathring{D}_{ij}|^2\,\mathrm{d}\Omega\right)^{\frac{1}{2}},\end{aligned} \tag{3.8}$$

where

$$D_{ij} = (\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij} + V^{\gamma-\alpha}\left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]$$

and

$$\mathring{D}_{ij} = D_{ij} - \frac{1}{n}\mathrm{tr}_{\widetilde{g}}(D_{ij})\widetilde{g}_{ij}.$$

It is easy to check that $\mathrm{tr}_{\widetilde{g}}(D_{ij}) = \widetilde{\Delta}^{D^{\alpha,\gamma}}u$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathring{D}_{ij}|^2 &= |D_{ij}|^2 - \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}}u\right)^2 \\ &= \left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij} + V^{\gamma-\alpha}\left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]\right|^2 - \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}}u\right)^2 \\ &= \left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij}\right|^2 + 2V^{\gamma-\alpha}(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij}\left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right] \\ &\quad + V^{2(\gamma-\alpha)}\left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]^2 - \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}}u\right)^2 \\ &\le \left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij}\right|^2 + \delta\left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}}u)_{ij}\right|^2\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&+ \frac{1}{\delta} V^{2(\gamma-\alpha)} \left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]^2 \\
&+ V^{2(\gamma-\alpha)} \left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]^2 - \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \\
=& (1+\delta)\left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}} u)_{ij}\right|^2 + (1+\frac{1}{\delta}) V^{2(\gamma-\alpha)} \left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]^2 \\
&- \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2,
\end{aligned}
$$

where δ is a positive constant to be determined. Since

$$
\begin{aligned}
&\left[(\tau-\gamma-\alpha)u_i f_j - \alpha u_k f_k \widetilde{g}_{ij}\right]^2 \\
=& (\tau-\gamma-\alpha)^2 u_i^2 f_j^2 + n\alpha^2 u_k^2 f_k^2 - 2\alpha(\tau-\gamma-\alpha)u_k f_k u_i f_i \\
=& [(\tau-\gamma-\alpha)^2 + n\alpha^2] u_i^2 f_j^2 - 2\alpha(\tau-\gamma-\alpha)\langle \widetilde{\nabla} u, \widetilde{\nabla} f\rangle^2 \\
=& [(\tau-\gamma-\alpha)^2 + n\alpha^2] u_i^2 f_j^2 - 2n\alpha^2\langle \widetilde{\nabla} u, \widetilde{\nabla} f\rangle^2 \\
\le& [(\tau-\gamma-\alpha)^2 + n\alpha^2] u_i^2 f_j^2 .
\end{aligned}
$$

Combine this with (3.9), and we have

$$
\begin{aligned}
|\mathring{D}_{ij}|^2 \le& (1+\delta)\left|(\widetilde{\nabla}^{2,D^{\alpha,\gamma}} u)_{ij}\right|^2 - \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \\
&+ (1+\frac{1}{\delta})[(\tau-\gamma-\alpha)^2 + n\alpha^2] V^{2(\gamma-\alpha)} u_i^2 f_j^2 .
\end{aligned} \tag{3.10}
$$

And then

$$
\begin{aligned}
&\int_M V^\tau |\mathring{D}_{ij}|^2 \,\mathrm{d}\Omega \\
\le& (1+\delta)\int_M V^\tau |(\widetilde{\nabla}^{2,D^{\alpha,\gamma}} u)_{ij}|^2 \,\mathrm{d}\Omega \\
&+ (1+\frac{1}{\delta})[(\tau-\gamma-\alpha)^2 + n\alpha^2]\int_M V^\tau |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2 |\widetilde{\nabla} f|^2 \,\mathrm{d}\Omega \\
&- \frac{1}{n}\int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \,\mathrm{d}\Omega \\
\le& (1+\delta)\int_M V^\tau |(\widetilde{\nabla}^{2,D^{\alpha,\gamma}} u)_{ij}|^2 \,\mathrm{d}\Omega \\
&+ (1+\frac{1}{\delta})[(\tau-\gamma-\alpha)^2 + n\alpha^2] K_2 \int_M V^{\tau+\alpha-\gamma} |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2 \,\mathrm{d}\Omega \\
&- \frac{1}{n}\int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \,\mathrm{d}\Omega,
\end{aligned} \tag{3.11}
$$

where we use $K_2 = \max(V^{\alpha-\gamma}|\widetilde{\nabla}^{D^{\alpha,\gamma}} f|^2)$.

Since $V^{\gamma-\alpha}\mathrm{Ric}^{D^{\alpha,\gamma}} \geq -(n-1)K_1 g$, and $II^{D^{\alpha,\gamma}} \geq 0$, by the integral formula (2.2) and boundary condition in (3.2) we have

$$\begin{aligned}(3.12)\qquad & \int_M V^\tau \left|\widetilde{\nabla}^{2,D^{\alpha,\gamma}} u\right|^2 \mathrm{d}\Omega \\ \leq & -\int_M V^\tau \mathrm{Ric}^{D^{\alpha,\gamma}}(\widetilde{\nabla}^{D^{\alpha,\gamma}} u, \widetilde{\nabla}^{D^{\alpha,\gamma}} u)\,\mathrm{d}\Omega + \int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega \\ \leq & \int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega + (n-1)K_1 \int_M V^{\tau+\alpha-\gamma} \left|\widetilde{\nabla}^{D^{\alpha,\gamma}} u\right|^2 \mathrm{d}\Omega.\end{aligned}$$

Therefore, (3.11) becomes

$$\begin{aligned}(3.13)\qquad & \int_M V^\tau |\mathring{D}_{ij}|^2\,\mathrm{d}\Omega \\ \leq & \left[(n-1)(1+\delta)K_1 + (1+\frac{1}{\delta})[(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2\right] \\ & \times \int_M V^{\tau+\alpha-\gamma} |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2\,\mathrm{d}\Omega + \left(\frac{n-1}{n} + \delta\right) \int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega.\end{aligned}$$

Using the Rayleigh-Ritz principle, we note that the Neumann first (nonzero) eigenvalue λ_1 of the affine Laplacian $\widetilde{\Delta}^{D^{\alpha,\gamma}}$ can be characterized by

$$\lambda_1 = \inf\left\{\frac{\int_M V^{\tau+\alpha-\gamma}|\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2\,\mathrm{d}\Omega}{\int_M V^\tau u^2\,\mathrm{d}\Omega},\quad 0 \neq u \in C^\infty(M) \text{ and } u_{\mathbf{n}} = 0 \text{ on } \partial M\right\}.$$

Then, by using the divergence theorem, and the Cauchy-Schwarz inequality and the fact that $V^\tau \widetilde{\Delta}^{D^{\alpha,\gamma}} u = \widetilde{\nabla}_i(V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u)$, we have

$$\begin{aligned}(3.14)\qquad & \int_M V^{\tau+\alpha-\gamma} |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2\,\mathrm{d}\Omega \\ = & \int_M V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u \widetilde{\nabla}_i u\,\mathrm{d}\Omega \\ = & -\int_M u \widetilde{\nabla}_i (V^\tau \widetilde{\nabla}_i^{D^{\alpha,\gamma}} u)\,\mathrm{d}\Omega + \int_{\partial M} V^\tau \langle \widetilde{\nabla}^{D^{\alpha,\gamma}} u, \mathbf{n}\rangle\,\mathrm{d}A \\ = & -\int_M u V^\tau \widetilde{\Delta}^{D^{\alpha,\gamma}} u\,\mathrm{d}\Omega \\ \leq & \left(\int_M V^\tau u^2\,\mathrm{d}\Omega\right)^{\frac{1}{2}} \left(\int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega\right)^{\frac{1}{2}} \\ \leq & \left(\frac{1}{\lambda_1}\int_M V^{\tau+\alpha-\gamma} |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2\,\mathrm{d}\Omega\right)^{\frac{1}{2}} \left(\int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega\right)^{\frac{1}{2}},\end{aligned}$$

which gives

$$(3.15)\qquad \int_M V^{\tau+\alpha-\gamma} |\widetilde{\nabla}^{D^{\alpha,\gamma}} u|^2\,\mathrm{d}\Omega \leq \frac{1}{\lambda_1} \int_M V^\tau \left(\widetilde{\Delta}^{D^{\alpha,\gamma}} u\right)^2 \mathrm{d}\Omega$$

$$= \frac{1}{\lambda_1}\int_M V^\tau \left(B - \overline{B}^{V^\tau}\right)^2 \,\mathrm{d}\Omega.$$

Inserting (3.15) into (3.13), we have

$$\begin{aligned}
(3.16)\qquad & \int_M V^\tau |\mathring{D}_{ij}|^2 \,\mathrm{d}\Omega \\
&\le \frac{1}{\lambda_1}\left[(n-1)(1+\delta)K_1 + (1+\frac{1}{\delta})[(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2\right] \\
&\quad \times \int_M V^\tau \left(B - \overline{B}^{V^\tau}\right)^2 \,\mathrm{d}\Omega \\
&\quad + \left(\frac{n-1}{n} + \delta\right)\int_M V^\tau \left(B - \overline{B}^{V^\tau}\right)^2 \,\mathrm{d}\Omega \\
&= \frac{1}{\lambda_1}\left[\frac{n-1}{n}\lambda_1 + (n-1)K_1 + [(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2 + [\lambda_1 + (n-1)K_1]\delta\right. \\
&\quad \left. + \frac{[(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}{\delta}\right]\int_M V^\tau \left(B - \overline{B}^{V^\tau}\right)^2 \,\mathrm{d}\Omega.
\end{aligned}$$

Note that

$$\begin{aligned}
&[\lambda_1 + (n-1)K_1]\delta + \frac{[(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}{\delta} \\
&\ge 2\sqrt{[\lambda_1 + (n-1)K_1][(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}.
\end{aligned}$$

Moreover, the equality holds if and only if

$$\delta = \sqrt{\frac{[(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2}{\lambda_1 + (n-1)K_1}}.$$

Thus if we choose $\delta = \sqrt{\frac{[(\tau-\gamma-\alpha)^2+n\alpha^2]K_2}{\lambda_1+(n-1)K_1}}$, then (3.16) becomes

$$\begin{aligned}
(3.17)\qquad & \int_M V^\tau |\mathring{D}_{ij}|^2 \,\mathrm{d}\Omega \\
&\le \frac{1}{\lambda_1}\left[\frac{n-1}{n}\lambda_1 + (n-1)K_1 + [(\tau-\gamma-\alpha)^2 + n\alpha^2]K_2\right. \\
&\quad \left. + 2\sqrt{[\lambda_1+(n-1)K_1][(\tau-\gamma-\alpha)^2+n\alpha^2]K_2}\right]\int_M V^\tau \left(B - \overline{B}^{V^\tau}\right)^2 \,\mathrm{d}\Omega.
\end{aligned}$$

Therefore, combining (3.17) with (3.8) concludes the proof of (1.3). From the identity

$$\left|T - \frac{\overline{B}^{V^\tau}}{n}\widetilde{g}\right|^2 = \left|T - \frac{B}{n}\widetilde{g}\right|^2 + \frac{1}{n}(B - \overline{B}^{V^\tau})^2,$$

we obtain the inequality (1.4). Therefore, we complete the proof of Theorem 1.1.

4. Proof of Theorems 1.3 and 1.5

We first prove Theorem 1.3.

Proof of Theorem 1.3. We denote by τ_1 the first non-zero eigenvalue of eigenvalue problem (1.7). Let ϕ be a nonzero eigenfunction with respect to τ_1. Applying $\mathrm{Ric}^{D^{\alpha,\gamma}} \geq (n-1)V^{\alpha-\gamma}\widetilde{g}$ and the Cauchy-Schwarz inequality

$$\left|\widetilde{\nabla}^{2,D^{\alpha,\gamma}}\phi\right|^2_{\widetilde{g}} \geq \frac{1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi\right)^2$$

in (2.2) gives

$$\begin{aligned}(4.1)\qquad &\int_M V^\tau \left[\frac{n-1}{n}\left(\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi\right)^2 - (n-1)V^{\alpha-\gamma}|\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi|^2\right] \mathrm{d}\Omega \\ &\geq \int_{\partial M} V^\tau \Big[H^{D^{\alpha,\gamma}}\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \mathbf{n}\rangle^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) \\ &\quad - 2V^{-\gamma}\left\langle\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}(V^\gamma\phi_{\mathbf{n}})\right\rangle\Big] \mathrm{d}A.\end{aligned}$$

Using the divergence theorem, we obtain

$$\begin{aligned}(4.2)\qquad &\int_M V^\tau \left[V^{\alpha-\gamma}|\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi|^2\right] \mathrm{d}\Omega \\ &= \int_M V^{\tau+\gamma-\alpha}|\widetilde{\nabla}\phi|^2 \,\mathrm{d}\Omega \\ &= \int_M V^\tau\langle V^{\gamma-\alpha}\widetilde{\nabla}\phi, \widetilde{\nabla}\phi\rangle \,\mathrm{d}\Omega \\ &= \int_M V^\tau\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \widetilde{\nabla}\phi\rangle \,\mathrm{d}\Omega \\ &= \int_{\partial M} \phi V^\tau\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \mathbf{n}\rangle \,\mathrm{d}A - \int_M \phi\widetilde{\nabla}_i(V^\tau\widetilde{\nabla}_i^{D^{\alpha,\gamma}}\phi) \,\mathrm{d}\Omega \\ &= \int_{\partial M} \phi V^\tau\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \mathbf{n}\rangle \,\mathrm{d}A - \int_M \phi V^\tau\widetilde{\Delta}^{D^{\alpha,\gamma}}\phi \,\mathrm{d}\Omega.\end{aligned}$$

Plugging (4.2) into (4.1) yields

$$\begin{aligned}(4.3)\qquad &(\tau_1^2 - n\tau_1)\int_M V^\tau\phi^2 \,\mathrm{d}\Omega \\ &\geq \frac{n}{n-1}\int_{\partial M} V^\tau\Big[H^{D^{\alpha,\gamma}}\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \mathbf{n}\rangle^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) \\ &\quad + (n-1)\phi\langle\widetilde{\nabla}^{D^{\alpha,\gamma}}\phi, \mathbf{n}\rangle - 2V^{-\gamma}\left\langle\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}(V^\gamma\phi_{\mathbf{n}})\right\rangle\Big] \mathrm{d}A.\end{aligned}$$

If $\theta \neq \frac{\pi}{2}$, it follows from the boundary condition that

$$\phi + V^\gamma\phi_{\mathbf{n}}\tan\theta = 0.$$

This together with (4.3) implies

$$(\tau_1^2 - n\tau_1)\int_M V^\tau \phi^2 \,\mathrm{d}\Omega \tag{4.4}$$
$$\geq \frac{n}{n-1}\int_{\partial M} V^\tau \Big[H^{D^{\alpha,\gamma}} V^{2(\gamma-\alpha)}\phi_{\mathbf{n}}^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right)$$
$$- [(n-1)\tan\theta] V^{2\gamma-\alpha}\phi_{\mathbf{n}}^2 + 2V^{-\gamma}\tan\theta \left|\nabla^{D^{\alpha,\gamma}}(V^\gamma \phi_{\mathbf{n}})\right|^2 \Big] \,\mathrm{d}A.$$

Hence, we have

$$(\tau_1^2 - n\tau_1) \tag{4.5}$$
$$\geq \frac{n}{n-1}\int_{\partial M} V^\tau \Big[2V^{-\gamma}\tan\theta \left|\nabla^{D^{\alpha,\gamma}}(V^\gamma \phi_{\mathbf{n}})\right|^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right)$$
$$+ \left(H^{D^{\alpha,\gamma}} V^{-\alpha} - (n-1)\tan\theta\right) V^{2\gamma-\alpha}\phi_{\mathbf{n}}^2 \Big] \,\mathrm{d}A \cdot \left(\int_M V^\tau \phi^2 \,\mathrm{d}\Omega\right)^{-1}.$$

If $\theta \neq 0$, it follows from the boundary condition that

$$\phi\cot\theta + V^\gamma \phi_{\mathbf{n}} = 0.$$

By using an analogous argument, we obtain the following inequality:

$$(\tau_1^2 - n\tau_1)\int_M V^\tau \phi^2 \,\mathrm{d}\Omega \tag{4.6}$$
$$\geq \frac{n}{n-1}\int_{\partial M} V^\tau \Big[H^{D^{\alpha,\gamma}} V^{-2\alpha}\phi^2\cot^2\theta + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right)$$
$$- [(n-1)\cot\theta] V^{-\alpha}\phi^2 + 2V^{-\gamma}\cot\theta \left|\nabla^{D^{\alpha,\gamma}}\phi\right|^2 \Big] \,\mathrm{d}A.$$

Hence, we have

$$(\tau_1^2 - n\tau_1) \tag{4.7}$$
$$\geq \frac{n}{n-1}\int_{\partial M} V^\tau \Big[2V^{-\gamma}\cot\theta \left|\nabla^{D^{\alpha,\gamma}}\phi\right|^2 + II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right)$$
$$+ \left(H^{D^{\alpha,\gamma}} V^{-\alpha}\cot^2\theta - (n-1)\cot\theta\right) V^{-\alpha}\phi^2 \Big] \,\mathrm{d}A \cdot \left(\int_M V^\tau \phi^2 \,\mathrm{d}\Omega\right)^{-1}.$$

This completes the proof of Theorem 1.3. □

Next we prove Theorem 1.5.

Proof of Theorem 1.5. (i) By the assumption we have

$$V^{-\gamma}\tan\theta \left|\nabla^{D^{\alpha,\gamma}}(V^\gamma \phi_{\mathbf{n}})\right|^2 \geq 0,$$
$$II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) \geq 0$$

and

$$\left(H^{D^{\alpha,\gamma}} V^{-\alpha} - (n-1)\tan\theta\right) V^{2\gamma-\alpha}\phi_{\mathbf{n}}^2 \geq 0.$$

Therefore $C_1(M,f,\phi,\theta) \geq 0$. This together with Theorem 1.3 implies that $\tau_1^2 - n\tau_1 \geq 0$. So $\tau_1 \geq n$.

(ii) By the assumption we have

$$V^{-\gamma}\cot\theta\left|\nabla^{D^{\alpha,\gamma}}\phi\right|^2 = 0,\ II^{D^{\alpha,\gamma}}\left(\nabla^{D^{\alpha,\gamma}}\phi, \nabla^{D^{\alpha,\gamma}}\phi\right) = 0$$

and

$$\left(H^{D^{\alpha,\gamma}}V^{-\alpha}\cot^2\theta - (n-1)\cot\theta\right)V^{-\alpha}\phi^2 \geq 0.$$

Therefore $C_2(M,f,\phi,\theta) \geq 0$. This together with Theorem 1.3 implies that $\tau_1^2 - n\tau_1 \geq 0$. So $\tau_1 \geq n$. This completes the proof of Theorem 1.5. □

References

[1] R. Avalos and A. Freitas, *The Pohozaev-Schoen identity on asymptotically Euclidean manifolds: conservation laws and their applications*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **38** (2021), no. 6, 1703–1724. https://doi.org/10.1016/j.anihpc.2021.01.002

[2] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math., 1123, Springer, Berlin, 1985. https://doi.org/10.1007/BFb0075847

[3] E. R. Barbosa, *A note on the almost-Schur lemma on 4-dimensional Riemannian closed manifolds*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4319–4322. https://doi.org/10.1090/S0002-9939-2012-11255-9

[4] Y. Chen, Y. Hu, and H. Li, *Geometric inequalities for free boundary hypersurfaces in a ball*, Ann. Global Anal. Geom. **62** (2022), no. 1, 33–45. https://doi.org/10.1007/s10455-022-09836-2

[5] J.-T. R. Chen, T. Saotome, and C.-T. Wu, *The CR almost Schur lemma and Lee conjecture*, Kyoto J. Math. **52** (2012), no. 1, 89–98. https://doi.org/10.1215/21562261-1503763

[6] X. Cheng, *A generalization of almost-Schur lemma for closed Riemannian manifolds*, Ann. Global Anal. Geom. **43** (2013), no. 2, 153–160. https://doi.org/10.1007/s10455-012-9339-8

[7] X. Cheng, *An almost-Schur type lemma for symmetric* $(2,0)$ *tensors and applications*, Pacific J. Math. **267** (2014), no. 2, 325–340. https://doi.org/10.2140/pjm.2014.267.325

[8] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci Flow*, Lectures in Contemporary Mathematics 3, Science Press, American Mathematical Society, 2006.

[9] F. Cruz Jr., A. Freitas, and M. Santos, *De Lellis–Topping inequalities on weighted manifolds with boundary*, Results Math. **77** (2022), no. 1, Paper No. 16, 14 pp. https://doi.org/10.1007/s00025-021-01549-5

[10] C. De Lellis and P. M. Topping, *Almost-Schur lemma*, Calc. Var. Partial Differential Equations **43** (2012), no. 3-4, 347–354. https://doi.org/10.1007/s00526-011-0413-z

[11] A. Freitas and M. Santos, *Some almost-Schur type inequalities for k-Bakry-Emery Ricci tensor*, Differential Geom. Appl. **66** (2019), 82–92. https://doi.org/10.1016/j.difgeo.2019.05.009

[12] Y. Ge, G. Wang, and C. Xia, *On problems related to an inequality of Andrews, De Lellis, and Topping*, Int. Math. Res. Not. IMRN **2013** (2013), no. 20, 4798–4818. https://doi.org/10.1093/imrn/rns196

[13] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 2015.

[14] P. T. Ho, *Almost Schur lemma for manifolds with boundary*, Differential Geom. Appl. **32** (2014), 97–112. https://doi.org/10.1016/j.difgeo.2013.11.006

[15] G. Huang, B. Ma, and M. Zhu, *A Reilly type integral formula and its applications*, arXiv:2201.09439.
[16] G. Huang, B. Ma, and M. Zhu, *Colesanti type inequalities for affine connections*, Anal. Math. Phys. **13** (2023), no. 1, Paper No. 12, 15 pp. https://doi.org/10.1007/s13324-022-00773-8
[17] Q. Huang and Q. H. Ruan, *Applications of some elliptic equations in Riemannian manifolds*, J. Math. Anal. Appl. **409** (2014), no. 1, 189–196. https://doi.org/10.1016/j.jmaa.2013.07.004
[18] G. Huang and F. Zeng, *De Lellis-Topping type inequalities for f-Laplacians*, Studia Math. **232** (2016), no. 3, 189–199. https://doi.org/10.4064/sm8236-4-2016
[19] G. Huang and M. Zhu, *Some geometric inequalities on Riemannian manifolds associated with the generalized modified Ricci curvature*, J. Math. Phys. **63** (2022), no. 11, Paper No. 111508, 12 pp. https://doi.org/10.1063/5.0116994
[20] A. V. Kolesnikov and E. Milman, *Brascamp-Lieb-type inequalities on weighted Riemannian manifolds with boundary*, J. Geom. Anal. **27** (2017), no. 2, 1680–1702. https://doi.org/10.1007/s12220-016-9736-5
[21] A. V. Kolesnikov and E. Milman, *Poincaré and Brunn-Minkowski inequalities on the boundary of weighted Riemannian manifolds*, Amer. J. Math. **140** (2018), no. 5, 1147–1185. https://doi.org/10.1353/ajm.2018.0027
[22] H. Li and Y. Wei, *f-minimal surface and manifold with positive m-Bakry-Émery Ricci curvature*, J. Geom. Anal. **25** (2015), no. 1, 421–435. https://doi.org/10.1007/s12220-013-9434-5
[23] J. Li and C. Xia, *An integral formula for affine connections*, J. Geom. Anal. **27** (2017), no. 3, 2539–2556. https://doi.org/10.1007/s12220-017-9771-x
[24] J. Li and C. Xia, *An integral formula and its applications on sub-static manifolds*, J. Differential Geom. **113** (2019), no. 3, 493–518. https://doi.org/10.4310/jdg/1573786972
[25] L. Ma and S.-H. Du, *Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 21-22, 1203–1206. https://doi.org/10.1016/j.crma.2010.10.003
[26] M. Meng and S. Zhang, *De Lellis-Topping type inequalities on smooth metric measure spaces*, Front. Math. China **13** (2018), no. 1, 147–160. https://doi.org/10.1007/s11464-017-0670-z
[27] G. Qiu and C. Xia, *A generalization of Reilly's formula and its applications to a new Heintze-Karcher type inequality*, Int. Math. Res. Not. IMRN **2015** (2015), no. 17, 7608–7619. https://doi.org/10.1093/imrn/rnu184
[28] R. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), no. 3, 459–472. https://doi.org/10.1512/iumj.1977.26.26036
[29] X.-A. Ren and H. W. Xu, *A lower bound for first eigenvalue with mixed boundary condition*, Appl. Math. J. Chinese Univ. Ser. B **19** (2004), no. 2, 223–228. https://doi.org/10.1007/s11766-004-0057-2
[30] J. Y. Wu, *De Lellis-Topping type inequalities for smooth metric measure spaces*, Geom. Dedicata **169** (2014), 273–281. https://doi.org/10.1007/s10711-013-9855-0
[31] F. Zeng, *Some almost-Schur type inequalities and applications on sub-static manifolds*, Electron. Res. Arch. **30** (2022), no. 8, 2860–2870. https://doi.org/10.3934/era.2022145

Huiting Chang
School of Mathematics and Statistics
Xinyang Normal University
Xinyang 464000, P. R. China
Email address: changhuiting163@163.com

Fanqi Zeng
School of Mathematics and Statistics
Xinyang Normal University
Xinyang 464000, P. R. China
Email address: zengfq@xynu.edu.cn