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FOURIER DECAY OF MORAN MEASURE WITH QUASI PERIODIC SEQUENCE

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Abstract. In this paper, we introduce a class of Moran measures generated by quasi periodic sequences, and consider power decay of the Fourier transforms of this kind of measures.

1. Introduction

Let μ be a finite positive Borel measure on $\mathbb{R}$, its Fourier transform is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi x} d\mu(x).$$

The measure μ is called a *Rajchman measure* if $\lim_{|\xi|\to\infty} \hat{\mu}(\xi) = 0$. The Riemann-Lebesgue Lemma tells us that μ is Rajchman if it is absolutely continuous. It is difficulty to determine whether or not μ is a Rajchman measure if it is singular. Further information about the rate of decay $\hat{\mu}(\xi)$ is needed for many applications. For example, Davenport, Erdős and LeVeque in [8] established a method finding normal numbers by using the fast rate of decay for $\hat{\mu}(\xi)$. Especially, we are interested in the power Fourier decay of $\hat{\mu}(\xi)$.

Definition 1.1. For $\alpha > 0$, let

$$\mathcal{D}(\alpha) = \left\{\mu : \mu \text{ is a finite positive measure on } \mathbb{R} \text{ with } |\hat{\mu}(\xi)| = O(\tfrac{1}{|\xi|^{\alpha}}), |\xi| \to \infty\right\}$$

and denote $\mathcal{D} = \bigcup_{\alpha>0} \mathcal{D}(\alpha)$. We say that μ has *power Fourier decay* if $\mu \in \mathcal{D}$.

Fractal measure is a hot research field and self-similar measure is an important kind of fractal measures, which were originated from Hutchinson [16].

Definition 1.2. Let $\{f_i : f_i(x) = r_i(x + d_i), r_i \in (0, 1), d_i \in \mathbb{R}, 1 \leq i \leq m\}$ be an *iterated function system* (*IFS*), there exists a unique Borel probability

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measure μ supported on $T \subset \mathbb{R}$ such that

$$T = \bigcup_{i=1}^{m} f_i(T), \qquad \mu = \sum_{i=1}^{m} p_i(\mu \circ f_i^{-1}),$$

where $\{p_i\}_{i=1}^m$ is a probability weights, that is, $p_i > 0$ and $\sum_{i=1}^m p_i = 1$. μ is called a *self-similar measure* and T a *self-similar set*.

There are a lot of papers devoted to the Fourier decay for self-similar measures. For the homogeneous case, that is, all contraction ratios are equal. The best-known result is the Fourier decay for classical Bernoulli convolutions μ_r, which is generated by the IFS $\{rx, rx+1\}$ with $r \in (0,1)$ and probabilities $\{\frac{1}{2}, \frac{1}{2}\}$. Erdős [10,11], Salem [23], Kahane [18] and many mathematicians have done pioneering work on this problem. Whereas very few specific r are known, for which μ_r has power Fourier decay, found by Dai, Feng and Wang [6]. The non-homogeneous case is difficult, since contraction ratios are different and the self-similar measure is not a convolutions. Li and Sahlsten [19], Brémont [4], Varjú and Yu [25], respectively, studied the attenuation of Fourier transform of non-homogeneous self-similar measures under different conditions. Recently, Solomyak [24] showed that after removing a zero Hausdorff dimension exceptional set of parameters, all self-similar measures on the line have power decay of the Fourier transforms.

More generally, Moran measure is supported on a Moran set, which is a generalization of self-similar set and can be generated by Moran IFS.

Definition 1.3. For $k, m_k \in \mathbb{Z}^+$, let $R_k = (r_1^{(k)}, \ldots, r_{m_k}^{(k)}) \in (0,1)^{m_k}$ be a contraction vector, and let

$$0 \in D_k = \left\{ d_1^{(k)}, \ldots, d_{m_k}^{(k)} \right\} \subset \mathbb{R}$$

be a finite digit set. We call the function system

$$\left\{ f_{i_k}^{(k)} : f_{i_k}^{(k)}(x) = r_{i_k}^{(k)}(x + d_{i_k}^{(k)}), 1 \le i_k \le m_k, k \ge 1 \right\}$$

a *Moran IFS*.

Let $P_k = \{p_1^{(k)}, \ldots, p_{m_k}^{(k)}\}$ be a probability weights with respect to D_k, we define

$$\mu_k = \sum_{1 \le i_1 \le m_1, \ldots, 1 \le i_k \le m_k} p_{i_1}^{(1)} \cdots p_{i_k}^{(k)} (\delta_0 \circ (f_{i_1}^{(1)} \circ \cdots \circ f_{i_k}^{(k)})^{-1}), \tag{1.1}$$

where δ_0 is the Dirac measure at 0. For the completeness, we introduce the following theorem without proof. One can refer the proof in [17].

Theorem 1.4. *Suppose that μ_k is given by* (1.1) *and*

$$\sup \left\{ x : x = r_{i_k}^{(k)} d_{i_k}^{(k)}, d_{i_k}^{(k)} \in D_k, 1 \le i_k \le m_k, k \ge 1 \right\} < \infty.$$

Then μ_k converges weakly to a Borel probability measure $\mu = \mu_{\{R_k\},\{D_k\},\{P_k\}}$ with compact support

$$T(\{R_k\},\{D_k\}) = \left\{ x : x = \sum_{k=1}^{\infty} r_{i_1}^{(1)} r_{i_2}^{(2)} \cdots r_{i_k}^{(k)} d_{i_k}^{(k)},\ d_{i_k}^{(k)} \in D_k, 1 \le i_k \le m_k \right\}.$$

The measure μ in Theorem 1.4 is called a Moran measure generated by a sequence of triples $\{(R_k, D_k, P_k)\}_{k=1}^{\infty}$. Moran measures are very important fractal measures. They generalize the self-similar measures and attract much attention of a large number of researchers. There are a lot of researches on the fractal properties of Moran measures and Moran sets [5, 6, 12, 14, 15, 22]. Recently, the study of spectrality of Moran measures is blooming in the fractal community [2,3,7,9,13,20,21]. As the study progressed, researchers found that the attenuation of Fourier transforms plays a important role in determining the spectral properties of Moran measures. Motivated by these findings, we are intend to generalize the power decay of Fourier transform to Moran measures. In this paper, we consider a special class of Moran measures:

Definition 1.5. Let $m_k \equiv m \in \mathbb{Z}^+$, $R_k \equiv R = (r_1, \ldots, r_m) \in (0,1)^m$. If there are finite distinct tuples (D_k, P_k) and there exist a sequence $\{k_j\}_{j=1}^{\infty}$ and $l \in \mathbb{Z}^+$ such that $k_{j+1} - k_j = l$ (or $1 \le k_{j+1} - k_j \le l$) and $(D_{k_j}, P_{k_j}) = (D_{k_{j'}}, P_{k_{j'}})$ for $j \neq j'$, we call $\{(D_k, P_k)\}_{k=1}^{\infty}$ a *quasi periodic sequence with period l (or less than l)*, and say that the Moran measure $\mu_{R,\{D_k\},\{P_k\}}$ is generated by a quasi periodic sequence with period l (or less than l).

The goal of this paper is to prove that the Moran measure generated by quasi periodic sequence enjoys power Fourier decay. For this purpose, we firstly investigate Moran measure with equal contraction components. In this case, since the digit sets varies, Moran measure can not expressed as infinite Bernoulli convolutions. We obtain the power Fourier decay by using the quasi-periodicity of digit sets and improving the Erdős-Kahane method. The conclusion is as follows.

Theorem 1.6. *For $m \ge 1$, there exists an exceptional set $E \subset (0,1)^m$ of zero Hausdorff dimension such that for all $R = (r, r, \ldots, r) \in (0,1)^m \setminus E$, and for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}$.*

For the Moran measure with unequal contraction components and finite quasi-period sequence, we iterate the Moran IFS several times to get a new one, while preserving the measure. Inspired by the method in [24], we introduce the theory of submartingale to calculate the probability of infinite words associated with the exceptional contraction vectors. However, we directly calculate the probability of infinite words instead of considering $\mathbb{Z}^+$ as the vertex set of a directed graph. We get following results.

Theorem 1.7. *For $m \geq 1$, there exists an exceptional set $F \subset (0,1)^m$ of zero Hausdorff dimension such that for all $R = (r_1, r_2, \ldots, r_m) \in (0,1)^m \setminus F$, and for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with finite period, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}$.*

For the organization of the paper, we first introduce some notations in Section 2, and give a corollary, which is needed in the proof of power Fourier decay. We prove Theorem 1.6 and Theorem 1.7 in Section 3 and Section 4, respectively.

2. Preliminaries

For the sake of convenience, we introduce some notations from symbolic dynamical system. Denote $\mathcal{I}_k = \{1, 2, \ldots, m_k\}$, $\mathcal{I}_0 = \{\vartheta\}$ and

$$\mathcal{I}_0 \times \cdots \times \mathcal{I}_k = \{I : I = i_1 \cdots i_n \cdots i_k,\ i_n \in \mathcal{I}_n, 1 \leq n \leq k\},$$

$$\mathcal{I}^* = \bigcup_{k=0}^{\infty} (\mathcal{I}_0 \times \cdots \times \mathcal{I}_k).$$

For any $I = i_1 \cdots i_k \in \mathcal{I}_0 \times \cdots \times \mathcal{I}_k$, we write

$$f_I = f_{i_1}^{(1)} \circ \cdots \circ f_{i_k}^{(k)}, \quad \mathrm{P}_I = p_{i_1}^{(1)} \cdots p_{i_k}^{(k)}.$$

Then (1.1) can be written as

$$\mu_k = \sum_{I \in \mathcal{I}_0 \times \cdots \times \mathcal{I}_k} \mathrm{P}_I (\delta_0 \circ f_I^{-1}). \tag{2.1}$$

Let $m_k \equiv m$, $R_k \equiv R$ and $\{(D_k, P_k)\}_{k=1}^{\infty}$ be a quasi periodic sequence. Then

$$\mathcal{I}_k \equiv \mathcal{I} = \{1, 2, \ldots, m\}, \quad \mathcal{I}_0 \times \cdots \times \mathcal{I}_k = \mathcal{I}^k, \quad \mathcal{I}^* = \bigcup \mathcal{I}^k.$$

Further, (2.1) reduces to

$$\mu_k = \sum_{I \in \mathcal{I}^k} \mathrm{P}_I (\delta_0 \circ f_I^{-1}). \tag{2.2}$$

Given a sequence $\{k_j\}_{j=1}^{\infty}$, for each $j \geq 1$, we divide $\mathcal{I}^{k_j}$ into j parts: $\mathcal{I}^{k_j} = \mathcal{I}^{k_1} \times \mathcal{I}^{k_2 - k_1} \times \cdots \times \mathcal{I}^{k_j - k_{j-1}}$, it means that for each $I \in \mathcal{I}^{k_j}$, we depart it as

$$I = \overbrace{i_1 \cdots i_{k_1}} \cdots \overbrace{i_{k_{s-1}+1} \cdots i_{k_s}} \cdots \overbrace{i_{k_{j-1}+1} \cdots i_{k_j}} := \mathbf{I}_1 \cdots \mathbf{I}_s \cdots \mathbf{I}_j,$$

where $\mathbf{I}_s = i_{k_{s-1}+1} \cdots i_{k_s} \in \mathcal{I}^{k_s - k_{s-1}}$, $1 \leq s \leq j$, $k_0 = 0$. For each $\mathbf{I}_s$, the corresponding function

$$\begin{aligned} \mathbf{f}_{\mathbf{I}_s}^{(s)}(x) = f_{i_{k_{s-1}+1}}^{(k_{s-1}+1)} \circ \cdots \circ f_{i_{k_s}}^{(k_s)}(x) = r_{i_{k_{s-1}+1}} \cdots r_{i_{k_s}} \Big(x + d_{i_{k_s}}^{(k_s)} + r_{i_{k_s}}^{-1} d_{i_{k_s-1}}^{(k_s-1)} \\ + \cdots + (r_{i_{k_{s-1}+2}} \cdots r_{i_{k_s}})^{-1} d_{i_{k_{s-1}+1}}^{(k_{s-1}+1)} \Big) \\ := R^{\mathbf{I}_s} (x + d_{\mathbf{I}_s}^{(s)}), \end{aligned}$$

where

$$R^{\mathbf{I}_s} = r_{i_{k_{s-1}+1}} \cdots r_{i_{k_s}},$$
$$d^{(s)}_{\mathbf{I}_s} = d^{(k_s)}_{i_{k_s}} + r^{-1}_{i_{k_s}} d^{(k_s-1)}_{i_{k_s-1}} + \cdots + (r_{i_{k_{s-1}+2}} \cdots r_{i_{k_s}})^{-1} d^{(k_{s-1}+1)}_{i_{k_{s-1}+1}},$$

and the associated probability weight is $\mathrm{P}^{(s)}_{\mathbf{I}_s} := p^{(k_s-1)}_{i_{k_s-1}} \cdots p^{(k_s)}_{i_{k_s}}$. In this way, we obtain a new Moran IFS $\{\mathbf{f}^{(s)}_{\mathbf{I}_s} : \mathbf{I}_s \in \mathcal{I}^{k_s-k_{s-1}}, s \geq 1\}$. For each $s \geq 1$, the triple $(\mathbf{R}^{(s)}, \mathbf{D}^{(s)}, \mathbf{P}^{(s)})$ is define by

$$\begin{cases} \mathbf{R}^{(s)} = (R^{1\cdots 1}, \ldots, R^{\mathbf{I}_s}, \ldots, R^{m\cdots m}), \\ \mathbf{D}^{(s)} = \{d^{(s)}_{1\cdots 1}, \ldots, d^{(s)}_{\mathbf{I}_s}, \ldots, d^{(s)}_{m\cdots m}\}, \\ \mathbf{P}^{(s)} = \{\mathrm{P}^{(s)}_{1\cdots 1}, \ldots, \mathrm{P}^{(s)}_{\mathbf{I}_s}, \ldots, \mathrm{P}^{(s)}_{m\cdots m}\}. \end{cases} \tag{2.3}$$

For Moran IFS $\{\mathbf{f}^{(s)}_{\mathbf{I}_s} : \mathbf{I}_s \in \mathcal{I}^{k_s-k_{s-1}},\ s \geq 1\}$, we can also define measure sequence

$$\nu_j = \sum_{\mathbf{I}_1 \in \mathcal{I}^{k_1}, \ldots, \mathbf{I}_j \in \mathcal{I}^{k_j-k_{j-1}}} \mathrm{P}_{\mathbf{I}_1 \cdots \mathbf{I}_j} (\delta_0 \circ (\mathbf{f}^{(1)}_{\mathbf{I}_1} \circ \cdots \circ \mathbf{f}^{(j)}_{\mathbf{I}_j})^{-1}).$$

It is easy to see $\nu_j = \mu_{k_j}$. Hence, we have the following corollary.

Corollary 2.1. *For any sequence $\{k_j\}_{j=1}^{\infty}$, $\{\nu_j\}_{j=1}^{\infty}$ converges weakly to $\mu = \mu_{R,\{D_k\},\{P_k\}}$.*

3. The case with equal contraction components

In this section, we consider the power Fourier decay of Moran measures generated by a quasi periodic sequence with equal contraction components. In order to prove Theorem 1.6, we need to prove following proposition.

Proposition 3.1. *Let $1 < a < b < \infty$, $0 < \varpi, \varepsilon$, and $l \in \mathbb{Z}^+$. Then there exist $\alpha > 0$ and $\mathcal{E} \subset [b^{-1}, a^{-1}]$ such that $\dim_H \mathcal{E} < \varpi$ and for all $R = (r, r, \ldots, r)$ with $r \in [b^{-1}, a^{-1}] \setminus \mathcal{E}$, and for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with period less than l and $\inf_{k \geq 1}\{x : x \in P_k\} \geq \varepsilon$, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}(\alpha)$.*

Proof. From (2.2), for $k, K \geq 1$, we have

$$\hat{\mu}_k(\xi) = \sum_{i_1 \cdots i_k \in \mathcal{I}^k} p^{(1)}_{i_1} \cdots p^{(k)}_{i_k} e^{2\pi i (r d^{(1)}_{i_1} + \cdots + r^k d^{(k)}_{i_k})\xi} = \prod_{s=1}^{k} \left(\sum_{i=1}^{m} p^{(s)}_i e^{2\pi i r^s d^{(s)}_i \xi} \right)$$

and

$$\hat{\mu}(\xi) = \prod_{s=1}^{\infty} \left(\sum_{i=1}^{m} p^{(s)}_i e^{2\pi i r^s d^{(s)}_i \xi} \right), \ |\hat{\mu}(r^{-K}\xi)| \leq \prod_{s=1}^{K} \left| \sum_{i=1}^{m} p^{(s)}_i e^{2\pi i r^{s-K} d^{(s)}_i \xi} \right|. \tag{3.1}$$

Note that $\{(D_k, P_k)\}_{k=1}^{\infty}$ is a quasi periodic sequence with period less than l, then there exists a sequence $\{k_j\}$ such that $1 \leq k_{j+1} - k_j \leq l$ and $(D_{k_j}, P_{k_j}) =$

$(D_{k_{j'}}, P_{k_{j'}})$ for $j \neq j'$. Without loss of generality, we can assume $d_1^{(k)} = 0$ and $d_2^{(k_j)} = 1$ for $k \geq 1$, $j \geq 1$. Then for $j \geq 1$,

$$\Big|\sum_{i=1}^{m} p_i^{(k_j)} e^{2\pi i r^{k_j - K} d_i^{(k_j)} \xi}\Big| \leq \sum_{i=3}^{m} p_i^{(k_j)} + |p_1^{(k_j)} + p_2^{(k_j)} e^{2\pi i r^{k_j-K}\xi}| \\ \leq 1 - 2\pi\varepsilon \|r^{k_j-K}\xi\|^2,$$

where $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer. Indeed, we can assume that $p_1^{(k_j)} > p_2^{(k_j)}$, otherwise, write

$$|p_1^{(k_j)} + p_2^{(k_j)} e^{2\pi i r^{k_j-K}\xi}| = |p_1^{(k_j)} e^{-2\pi i r^{k_j-K}\xi} + p_2^{(k_j)}|$$

and repeat the argument. Hence

$$\begin{aligned}|p_1^{(k_j)} + p_2^{(k_j)} e^{2\pi i r^{k_j-K}\xi}| &\leq p_1^{(k_j)} - p_2^{(k_j)} + p_2^{(k_j)}|1 + e^{-2\pi i r^{k_j-K}\xi}| \\ &\leq p_1^{(k_j)} - p_2^{(k_j)} + 2p_2^{(k_j)}|\cos(\pi r^{k_j-K}\xi)| \\ &\leq p_1^{(k_j)} - p_2^{(k_j)} + 2p_2^{(k_j)}(1 - \pi\|r^{k_j-K}\xi\|^2) \\ &\leq p_1^{(k_j)} + p_2^{(k_j)} - 2\pi\varepsilon\|r^{k_j-K}\xi\|^2.\end{aligned}$$

Let

$$r^{k_j-K}\xi = c_{K-k_j} + \epsilon_{K-k_j}, \quad c_{K-k_j} \in \mathbb{N}, \quad \epsilon_{K-k_j} \in [-\frac{1}{2}, \frac{1}{2}),$$

where $\xi > 1$, c_{K-k_j} is the nearest integer to $r^{k_j-K}\xi$ and $|\epsilon_{K-k_j}| = \|r^{k_j-K}\xi\|$. Let $\rho = \frac{1}{2(4+16b^l)^{l+1}l}$, for $K \geq 1$ and $3 \leq n_1 \in \mathbb{Z}^+$, define $\tilde{\mathcal{E}}(n_1, K)$ be the set

$$\Big\{r^{-1} \in [a,b] : \text{there exists } \xi \in [1, r^{-1}) \text{ such that} \\ \text{Card}\,(\{k_j : 1 \leq k_j \leq K, |\epsilon_{K-k_j}| \geq \rho\}) \leq \frac{K}{n_1 l}\Big\}$$

and $\tilde{\mathcal{E}} = \limsup_{K\to\infty} \tilde{\mathcal{E}}(n_1, K)$. If $r^{-1} \notin \tilde{\mathcal{E}}$, then $r^{-1} \notin \tilde{\mathcal{E}}(n_1, K)$ for all K sufficiently large. Fix such a K. From (3.1), it follows that for all $\xi \in [1, r^{-1})$,

$$|\hat{\mu}(r^{-K}\xi)| \leq \prod_{k_j \leq K}\Big(1 - 2\pi\varepsilon\|r^{k_j-K}\xi\|^2\Big) \leq (1 - 2\pi\varepsilon\rho^2)^{\frac{K}{n_1 l}}.$$

Let $\zeta = r^{-K}\xi$, we get $|\hat{\mu}(\zeta)| \leq (1-2\pi\varepsilon\rho^2)^{\frac{\ln\zeta + \ln r}{n_1 l \ln b}} \leq \zeta^{\frac{\ln(1-2\pi\varepsilon\rho^2)}{n_1 l \ln b}}(1-2\pi\varepsilon\rho^2)^{\frac{\ln d - \ln b}{n_1 l \ln b}}$. Since K is arbitrary, sufficiently large, this implies $\mu \in \mathcal{D}(\alpha)$, $\alpha = \frac{\ln(1-2\pi\varepsilon\rho^2)}{-n_1 l \ln b}$.

We now estimate $\dim_H \tilde{\mathcal{E}}$. It is easy to check that for $1 \leq k_j \leq k_{j+1} \leq K$,

$$\tag{3.2} \Big|r^{-(k_{j+1}-k_j)} - \frac{c_{K-k_j}}{c_{K-k_{j+1}}}\Big| = \Big|\frac{r^{-(k_{j+1}-k_j)}\epsilon_{K-k_{j+1}} - \epsilon_{K-k_j}}{c_{K-k_{j+1}}}\Big| \\ \leq \frac{|r^{-(k_{j+1}-k_j)}\epsilon_{K-k_{j+1}}| + |\epsilon_{K-k_j}|}{a^{K-k_{j+1}}} \leq \frac{b^l+1}{2a^{K-k_{j+1}}}.$$

Note

$$1 \leq \frac{c_{K-k_j}}{c_{K-k_{j+1}}} = \frac{r^{k_j-K}\xi - \epsilon_{K-k_j}}{r^{k_{j+1}-K}\xi - \epsilon_{K-k_{j+1}}} \leq \frac{r^{k_j-K}\xi + \frac{1}{2}}{r^{k_{j+1}-K}\xi - \frac{1}{2}} \leq 4r^{k_j-k_{j+1}} \leq 4b^l,$$

and $r^{-1} > 1$. Combining (3.2), we have

$$\left|r^{-1} - \Big(\frac{c_{K-k_j}}{c_{K-k_{j+1}}}\Big)^{\frac{1}{k_{j+1}-k_j}}\right| \leq \left|r^{-(k_{j+1}-k_j)} - \frac{c_{K-k_j}}{c_{K-k_{j+1}}}\right| \leq \frac{b^l+1}{2a^{K-k_{j+1}}}. \tag{3.3}$$

We will cover $\tilde{\mathcal{E}}(n_1, K)$ by intervals of size $\sim a^{-K}$ centered at $\Big(\frac{c_{K-k_1}}{c_{K-k_2}}\Big)^{\frac{1}{k_2-k_1}}$. To this end, we need to estimate the number of possible pairs (c_{K-k_2}, c_{K-k_1}) associated to r^{-1} and ξ. Let J_K be the largest j such that $k_j \leq K$. It is easy to see that $\frac{K}{l} \leq J_K \leq K$. Indeed, we will estimate the number of sequences $c_{K-k_{J_K}}, c_{K-k_{J_K-1}}, \dots, c_{K-k_2}, c_{K-k_1}$.

Lemma 3.2. *Let* $\rho = \frac{1}{2(4+16b^l)^{l+1}l}$ *and* $A = (4+16b^l)^{l+1}l$. *The following results hold for* $0 \leq j \leq J_K - 3$.

(1) *Given* $c_{K-k_{j+2}}, c_{K-k_{j+1}}$, *there are at most* A *possibilities for* c_{K-k_j}, *independent of* $r^{-1} \in [a,b]$ *and* $\xi \in [1, r^{-1})$;

(2) *Given* $c_{K-k_{j+2}}, c_{K-k_{j+1}}$, *if* $\max\{|\epsilon_{K-k_j}|, |\epsilon_{K-k_{j+1}}|, |\epsilon_{K-k_{j+2}}|\} < \rho$, *then* c_{K-k_j} *is uniquely determined, independent of* $r^{-1} \in [a,b]$ *and* $\xi \in [1, r^{-1})$.

Proof. (3.2) and (3.3) tell us

$$\begin{aligned}
&\left|\Big(\frac{c_{K-k_j}}{c_{K-k_{j+1}}}\Big)^{\frac{1}{k_{j+1}-k_j}} - \Big(\frac{c_{K-k_{j+1}}}{c_{K-k_{j+2}}}\Big)^{\frac{1}{k_{j+2}-k_{j+1}}}\right| \\
\leq &\left|\Big(\frac{c_{K-k_j}}{c_{K-k_{j+1}}}\Big)^{\frac{1}{k_{j+1}-k_j}} - r^{-1}\right| + \left|r^{-1} - \Big(\frac{c_{K-k_{j+1}}}{c_{K-k_{j+2}}}\Big)^{\frac{1}{k_{j+2}-k_{j+1}}}\right| \\
\leq &\frac{|r^{-(k_{j+1}-k_j)}\epsilon_{K-k_{j+1}}| + |\epsilon_{K-k_j}|}{c_{K-k_{j+1}}} + \frac{|r^{-(k_{j+2}-k_{j+1})}\epsilon_{K-k_{j+2}}| + |\epsilon_{K-k_{j+1}}|}{c_{K-k_{j+2}}}.
\end{aligned}$$

Denote

$$\begin{aligned}
v = &\frac{(c_{K-k_{j+1}})^{\frac{1}{k_{j+1}-k_j}}\left(|r^{-(k_{j+1}-k_j)}\epsilon_{K-k_{j+1}}| + |\epsilon_{K-k_j}|\right)}{c_{K-k_{j+1}}} \\
&+ \frac{(c_{K-k_{j+1}})^{\frac{1}{k_{j+1}-k_j}}\left(|r^{-(k_{j+2}-k_{j+1})}\epsilon_{K-k_{j+2}}| + |\epsilon_{K-k_{j+1}}|\right)}{c_{K-k_{j+2}}}
\end{aligned}$$

and

$$u = (c_{K-k_{j+1}})^{\left(\frac{1}{k_{j+2}-k_{j+1}} + \frac{1}{k_{j+1}-k_j}\right)} / (c_{K-k_{j+2}})^{\frac{1}{k_{j+2}-k_{j+1}}},$$

it follows $0 < (u-v)^{k_{j+1}-k_j} \leq c_{K-k_j} \leq (u+v)^{k_{j+1}-k_j}$. Since

$$\begin{aligned}
&(u+v)^{k_{j+1}-k_j} - (u-v)^{k_{j+1}-k_j} \\
\leq &\, 2^l v(u^{k_{j+1}-k_j-1} + u^{k_{j+1}-k_j-2}v + \cdots + v^{k_{j+1}-k_j-1}),
\end{aligned}$$

and for any $1 \le s \le k_{j+1} - k_j$,

$$
\begin{aligned}
& u^{k_{j+1}-k_j-s} v^s \\
&= \Big[\Big(\frac{c_{K-k_{j+1}}}{c_{K-k_{j+2}}} \Big)^{\frac{1}{k_{j+2}-k_{j+1}}} (c_{K-k_{j+1}})^{\frac{1}{k_{j+1}-k_j}} \Big]^{k_{j+1}-k_j-s} (c_{K-k_{j+1}})^{\frac{s}{k_{j+1}-k_j}} \\
&\quad \times \Big[\frac{|r^{-(k_{j+1}-k_j)} \epsilon_{K-k_{j+1}}| + |\epsilon_{K-k_j}|}{c_{K-k_{j+1}}} + \frac{|r^{-(k_{j+2}-k_{j+1})} \epsilon_{K-k_{j+2}}| + |\epsilon_{K-k_{j+1}}|}{c_{K-k_{j+2}}} \Big]^s \\
&\le 2^s (4b^l)^{l-s} (1+4b^l)^s \frac{c_{K-k_{j+1}}}{(c_{K-k_{j+2}})^s} \max\{|\epsilon_{K-k_j}|, |\epsilon_{K-k_{j+1}}|, |\epsilon_{K-k_{j+2}}|\} \\
&< 2^l (1+4b^l)^{l+1} \max\{|\epsilon_{K-k_j}|, |\epsilon_{K-k_{j+1}}|, |\epsilon_{K-k_{j+2}}|\},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& (u+v)^{k_{j+1}-k_j} - (u-v)^{k_{j+1}-k_j} \\
&< (4+16b^l)^{l+1} l \max\{|\epsilon_{K-k_j}|, |\epsilon_{K-k_{j+1}}|, |\epsilon_{K-k_{j+2}}|\}.
\end{aligned}
$$

The lemma follows easily as $c_{K-k_j} \in \mathbb{N}$. □

Let us return to estimate the number of sequences

$$c_{K-k_{J_K}}, c_{K-k_{J_K-1}}, \dots, c_{K-k_2}, c_{K-k_1}.$$

Fix $\mathcal{K} \subset \{1, 2, \dots, J_K\}$, we consider those r^{-1} for which there exists $\xi \in [1, r^{-1})$ such that $|\epsilon_{k_j}| < \rho$, $j \in \{1, 2, \dots, J_K\} \setminus \mathcal{K}$. From Lemma 3.2, it follows that the number of sequence $c_{K-k_{J_K}}, c_{K-k_{J_K-1}}, \dots, c_{K-k_2}, c_{K-k_1}$ associated to those r^{-1} is bounded above by $(\lfloor b \rfloor + 1)^3 A^{3\,\mathrm{Card}\,\mathcal{K}}$, where $\lfloor b \rfloor$ is the largest integer less than b. Therefore, the number of sequences $c_{K-k_{J_K}}$, $c_{K-k_{J_K-1}}$, …, c_{K-k_2}, c_{K-k_1} associated to $\tilde{\mathcal{E}}(n_1, K)$ is at most

$$
\begin{aligned}
(\lfloor b \rfloor + 1)^3 \sum_{s=1}^{\lfloor \frac{K}{n_1 l} \rfloor} C_{J_K}^s A^{3s} &< (\lfloor b \rfloor + 1)^3 K C_{J_K}^{\lfloor K/(n_1 l) \rfloor} A^{3\lfloor K/(k_1 l) \rfloor} \\
&< (\lfloor b \rfloor + 1)^3 K e^{A^3 K \frac{\ln(n_1 l)}{n_1 l}},
\end{aligned}
$$

where the last inequality is deduced from Stirling's formula. Thus, we obtain that $\tilde{\mathcal{E}}(n_1, K)$ may be covered by $(\lfloor b \rfloor + 1)^3 K l e^{A^3 K \frac{\ln(n_1 l)}{n_1 l}}$ intervals of size $\sim a^{-K}$ since $1 \le k_2 - k_1 \le l$. It follows

$$\dim_H \tilde{\mathcal{E}} \le \lim_{K \to \infty} \frac{\ln((\lfloor b \rfloor + 1)^3 K l e^{A^3 K \frac{\ln(n_1 l)}{n_1 l}})}{-\ln a^{-K}} = \frac{A^3 \ln(n_1 l)}{n_1 l \ln a}.$$

Note that $n_1 \ge 3$ is arbitrary. We can take n_1 such that $\frac{A^3 \ln(n_1 l)}{n_1 l \ln a} < \varpi$. Hence $\dim_H \tilde{\mathcal{E}} < \varpi$. Let $\mathcal{E} = \{r : r^{-1} \in \tilde{\mathcal{E}}\}$. Since $\frac{1}{x}$ is bi-Lipschitz on $[a, b]$, we get $\dim_H \mathcal{E} = \dim_H \tilde{\mathcal{E}}$. It follows Proposition 3.1. □

Proof of Theorem 1.6. Theorem 1.6 is a easy consequence of Proposition 3.1. In fact, we can let $G = \{R : R = (r, r, \dots, r) \in (0,1)^m,\ r \in (0,1)\}$ and $(0,1) = \cup_{n=2}^{\infty}[\frac{1}{n}, 1 - \frac{1}{n+1}]$. For integers $n, p, q \geq 2$, $l \geq 1$, by Proposition 3.1, there exist $\alpha_{n,p,q,l}$ and $\mathcal{E}_{n,p,q,l}$ such that $\dim_H \mathcal{E}_{n,p,q,l} < \frac{1}{p}$ and for all $R = (r, \dots, r) \in (0,1)^m$ with $r \in [\frac{1}{n}, 1 - \frac{1}{n+1}] \setminus \mathcal{E}_{n,p,q,l}$, for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with period less than l and $\inf_{k \geq 1}\{x : x \in P_k\} \geq \frac{1}{q}$, the associated Moran measure $\mu \in \mathcal{D}(\alpha_{n,p,q,l})$. Let

$$\tilde{E} = \bigcup_{l=1}^{\infty} \bigcup_{n=2}^{\infty} \bigcap_{p=2}^{\infty} \bigcup_{q=2}^{\infty} \mathcal{E}_{n,p,q,l}, \quad E = \{R : R = (r, r, \dots, r) \in (0,1)^m, r \in \tilde{E}\}.$$

It is easy to see $\dim_H \tilde{E} = 0$. Note that $\tilde{E}$ is a projection of E and the projection is bi-Lipschitz. Then $\dim_H E = 0$ and for all $R \in G \setminus E$ and any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}$. $\square$

4. The case with unequal contraction components

In this section, we consider the power Fourier decay of Moran measure generated by a quasi periodic sequence with unequal contraction components. In order to prove Theorem 1.7, we need following proposition.

Proposition 4.1. *Let $m \geq 2$, $1 < a < b < \infty$, $\varepsilon > 0$, $\varpi > 0$ and $l \in \mathbb{Z}^+$. Then there exist $\alpha > 0$ and $\mathcal{F} \subset (0,1)^m$ such that $\dim_H \mathcal{F} < \varpi$ and for all $R = (r_1, \dots, r_m) \in (0,1)^m \setminus \mathcal{F}$ with $b^{-1} \leq \min_{1 \leq i \leq m} r_i < \max_{1 \leq i \leq m} r_i \leq a^{-1}$, and for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with period l and $\inf_{k \geq 1}\{x : x \in P_k\} \geq \varepsilon$, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}(\alpha)$.*

Proof. Since $\{(D_k, P_k)\}_{k=1}^{\infty}$ is a quasi periodic sequence with period l, there exists a sequence $\{k_j\}$ such that $k_{j+1} - k_j = l$ and $(D_{k_j}, P_{k_j}) = (D_{k_{j'}}, P_{k_{j'}})$ for $j \neq j'$. We can choose a subsequence of $\{k_j\}$, which still denoted by $\{k_j\}$ and satisfies $k_{j+1} - k_j = nl$, $n \in \mathbb{Z}^+$. Further, we can choose n such that

$$nl \geq k_1, \quad a^{nl\varpi} > \tilde{m} := C_{nl+m-1}^{m-1},$$

where $\tilde{m} = C_{nl+m-1}^{m-1}$ is the number of ways to write nl as a sum of m non-negative integers. From Corollary 2.1, we can define measure sequence

$$\begin{aligned} \nu_j &= \sum_{\mathbf{I}_1 \in \mathcal{I}^{k_1}, \dots, \mathbf{I}_j \in \mathcal{I}^{k_j - k_{j-1}}} \mathrm{P}_{\mathbf{I}_1 \cdots \mathbf{I}_j}(\delta_0 \circ (\mathbf{f}_{\mathbf{I}_1}^{(1)} \circ \cdots \circ \mathbf{f}_{\mathbf{I}_j}^{(j)})^{-1}) \\ &= \sum_{\mathbf{I}_1 \in \mathcal{I}^{k_1}, \mathbf{I}_2 \in \mathcal{I}^{nl} \dots, \mathbf{I}_j \in \mathcal{I}^{nl}} \mathrm{P}_{\mathbf{I}_1 \cdots \mathbf{I}_j}(\delta_0 \circ (\mathbf{f}_{\mathbf{I}_1}^{(1)} \circ \cdots \circ \mathbf{f}_{\mathbf{I}_j}^{(j)})^{-1}). \end{aligned}$$

Moreover, $\{\nu_j\}$ converges weakly to μ. Then $\lim_{j\to\infty}\hat{\nu}_j=\hat{\mu}$ and for $\xi\in\mathbb{R}$

(4.1)
$$\hat{\nu}_j(\xi)=\sum_{\mathbf{I}_1\in\mathcal{I}^{k_1},\mathbf{I}_2\in\mathcal{I}^{nl}\dots,\mathbf{I}_j\in\mathcal{I}^{nl}}\mathrm{P}_{\mathbf{I}_1\cdots\mathbf{I}_j}e^{2\pi i(R^{\mathbf{I}_1}d^{(1)}_{\mathbf{I}_1}+R^{\mathbf{I}_1}R^{\mathbf{I}_2}d^{(2)}_{\mathbf{I}_2}+\cdots+R^{\mathbf{I}_1}R^{\mathbf{I}_2}\cdots R^{\mathbf{I}_j}d^{(j)}_{\mathbf{I}_j})\xi},$$

where $R^{\mathbf{I}_s},d^{(s)}_{\mathbf{I}_s}\in\mathbf{D}^{(s)}$, $\mathrm{P}^{(s)}_{\mathbf{I}_s}\in\mathbf{P}^{(s)}$ for $1\le s\le j$ are defined by (2.3). Note $0\in D_k$ for $k\ge1$ and $D_{k_j}=D_{k_j-l}$ for $j\ge2$. Without loss of generality, we can assume

$$d^{(k)}_1=0,\quad r_1^{-l+1}d_2^{(k_j-l)}-d_2^{(k_j)}=(r_1^{-l+1}-1)d_2^{(k_j)}=1,\quad k\ge1,\ \ j\ge2.$$

Denote $\bar{\mathbf{I}}=\overbrace{1\cdots1}^{nl-1}2$, $\tilde{\mathbf{I}}=\overbrace{1\cdots1}^{(n-1)l}2\overbrace{1\cdots1}^{l-1}$. One can verify

$$R^{\bar{\mathbf{I}}}=r_1^{nl-1}r_2=R^{\tilde{\mathbf{I}}},\quad d^{(j)}_{\bar{\mathbf{I}}}=d_2^{(k_j)},\quad d^{(j)}_{\tilde{\mathbf{I}}}=r_1^{-l+1}d_2^{(k_j-l)}.$$

Then (4.1) implies

(4.2)
$$\begin{aligned}|\hat{\nu}_j(\xi)|\le&\sum_{\mathbf{I}_1\in\mathcal{I}^{k_1}}\mathrm{P}^{(1)}_{\mathbf{I}_1}\Big(\sum_{\mathbf{I}_2\in\mathcal{I}^{nl},\mathbf{I}_2\neq\bar{\mathbf{I}},\tilde{\mathbf{I}}}\mathrm{P}^{(2)}_{\mathbf{I}_2}\Big|\sum_{\mathbf{I}_3\in\mathcal{I}^{nl}}\mathrm{P}^{(3)}_{\mathbf{I}_3}e^{2\pi iR^{\mathbf{I}_1}R^{\mathbf{I}_2}R^{\mathbf{I}_3}d^{(3)}_{\mathbf{I}_3}\xi}\\&\cdots\Big(\sum_{\mathbf{I}_j\in\mathcal{I}^{nl}}\mathrm{P}^{(j)}_{\mathbf{I}_j}e^{2\pi iR^{\mathbf{I}_1}R^{\mathbf{I}_2}\cdots R^{\mathbf{I}_j}d^{(j)}_{\mathbf{I}_j}\xi}\Big)\Big|+\mathrm{P}^{(2)}_{\bar{\mathbf{I}}}(1-2\pi\varepsilon^{nl}\|R^{\mathbf{I}_1\bar{\mathbf{I}}}\xi\|^2)\\&\times\Big|\sum_{\mathbf{I}_3\in\mathcal{I}^{nl}}\mathrm{P}^{(3)}_{\mathbf{I}_3}e^{2\pi iR^{\mathbf{I}_1}R^{\bar{\mathbf{I}}}R^{\mathbf{I}_3}d^{(3)}_{\mathbf{I}_3}\xi}\cdots\Big(\sum_{\mathbf{I}_j\in\mathcal{I}^{nl}}\mathrm{P}^{(j)}_{\mathbf{I}_j}e^{2\pi iR^{\mathbf{I}_1}R^{\bar{\mathbf{I}}}\cdots R^{\mathbf{I}_j}d^{(j)}_{\mathbf{I}_j}\xi}\Big)\Big|\\&+\mathrm{P}^{(2)}_{\tilde{\mathbf{I}}}(1-2\pi\varepsilon^{nl}\|R^{\mathbf{I}_1\tilde{\mathbf{I}}_2}\xi\|^2)\Big|\sum_{\mathbf{I}_3\in\mathcal{I}^{nl}}\mathrm{P}^{(3)}_{\mathbf{I}_3}e^{2\pi iR^{\mathbf{I}_1}R^{\tilde{\mathbf{I}}}R^{\mathbf{I}_3}d^{(3)}_{\mathbf{I}_3}\xi}\\&\cdots\Big(\sum_{\mathbf{I}_j\in\mathcal{I}^{nl}}\mathrm{P}^{(j)}_{\mathbf{I}_j}e^{2\pi iR^{\mathbf{I}_1}R^{\tilde{\mathbf{I}}}\cdots R^{\mathbf{I}_j}d^{(j)}_{\mathbf{I}_j}\xi}\Big)\Big|\Big),\end{aligned}$$

where we have used the inequality $|1+e^{2\pi ix}|\le2\Big(1-\pi\|x\|^2\Big)$ and $\|x\|$ denotes the distance from $x\in\mathbb{R}$ to the nearest integer, $\mathrm{P}^{(2)}_{\bar{\mathbf{I}}},\mathrm{P}^{(2)}_{\tilde{\mathbf{I}}}\ge\varepsilon^{nl}$.

Recall $\mathcal{I}=\{1,2,\dots,m\}$ and let ϑ be the empty word. For $1\le s\le j$ and a word $I=\mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_j\in\mathcal{I}^{k_j}$, let $I|_0=\vartheta$, $I|_s=\mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_s$ be the prefix of I. We write

$$R^{I|_0}=1,\quad R^{I|_s}=R^{\mathbf{I}_1}R^{\mathbf{I}_2}\cdots R^{\mathbf{I}_s},\quad P_{I|_s}=\mathrm{P}^{(1)}_{\mathbf{I}_1}\mathrm{P}^{(2)}_{\mathbf{I}_2}\cdots\mathrm{P}^{(s)}_{\mathbf{I}_s}.$$

By iterating the inequality (4.2), we obtain

$$|\hat{\nu}_j(\xi)|\le\sum_{I\in\mathcal{I}^{k_j}}P_I\prod_{s:\mathbf{I}_s=\bar{\mathbf{I}},\tilde{\mathbf{I}}}\Big(1-2\pi\varepsilon^{nl}\|R^{I|_s}\xi\|^2\Big),\quad j\ge2.\tag{4.3}$$

Given $\omega = \mathbf{I}'_{s+1}\mathbf{I}'_{s+2}\cdots\mathbf{I}'_{s'}$, $s' \geq s+1$, let

$$I|_s\omega = \mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_s\mathbf{I}'_{s+1}\mathbf{I}'_{s+2}\cdots\mathbf{I}'_{s'}$$

be the concatenation of $I|_s$ and ω. Denote $I|_s \to I'$ if $I' = I|_s\mathbf{I}'_{s+1}$ and call $I|_s \to I'$ an edge. It is easy to see that there are at most $\tilde{m}$ different elements in $\{R^{\mathbf{I}} : \mathbf{I} \in \mathcal{I}^{nl}\}$. Denote $I|_s \rightsquigarrow I'$ if $I' = I|_s$ or $I' = I|_s\omega$, where $\omega = \mathbf{I}'_{s+1}\mathbf{I}'_{s+2}\cdots\mathbf{I}'_{s'}$ satisfies $s+1 \leq s' \leq s+\tilde{m}$ and that the elements of $\{R^{\mathbf{I}'_{s+1}}, R^{\mathbf{I}'_{s+2}}, \cdots, R^{\mathbf{I}'_{s'}}\}$ are distinct. We identify $I|_s$ with the path $I|_0 \to I|_1 \to \cdots \to I|_s$ and denote this path by $\Gamma(I|_s)$. Further, we can define a path from $I|_s$ to $I|_{s'}$ $(0 \leq s < s' \leq j)$ by $I|_s \to I|_{s+1} \to \cdots \to I|_{s'}$, which is denoted by $\Gamma(I|_s, I|_{s'})$. In this case, we call $I|_{s'}$ an $(s'-s)$-level descendant of $I|_s$. It is easy to see that if $I|_s \rightsquigarrow I'$, then $I' = I|_s$, or I' is a descendant of $I|_s$ of level $\leq \tilde{m}$.

Definition 4.2. Let $\rho = \frac{1}{2(1+b^{nl})(1+2b^{nl})}$, $\xi > 0$, $R \in [b^{-1}, a^{-1}]^m$, and let $I \in \mathcal{I}^{k_j}, j \geq 1$. I is called (R,ξ,ρ)-*good* if $R^I\xi \geq 1$ and $\|R^I\xi\| \geq \rho$. We call that I is on an (R,ξ,ρ)-*good track* if there exist $I' \in \mathcal{I}^{k_{j'}}$, $j' \geq j$ such that I' is (R,ξ,ρ)-good and $I \rightsquigarrow I'$.

Definition 4.3. Fix $3 \leq n_1 \in \mathbb{N}$ and $\rho = \frac{1}{2(1+b^{nl})(1+2b^{nl})}$, and let $\mathcal{F}_j = \mathcal{F}_j(n_1,\rho)$ be the set of $R \in [b^{-1}, a^{-1}]^m$ such that there exist $\xi \in [a^{k_j-nl}, a^{k_j}]$, $I \in \mathcal{I}^{k_j}$ satisfying

$$\mathrm{Card}\left(\left\{\tilde{m}+1 \leq s \leq j-\tilde{m}-1 : I|_s \text{ is on an } (R,\xi,\rho)\text{-good track}\right\}\right) \leq \frac{j}{n_1},$$

and let $\mathcal{F} = \limsup_{j\to\infty} \mathcal{F}_j(n_1,\rho)$.

For $R \notin \mathcal{F}$, we now prove the power Fourier decay of μ. From the definition of $\mathcal{F}$, it follows that $R \notin \mathcal{F}_j(n_1,\rho)$ for all j sufficiently large. Fix such a j, the definition of $\mathcal{F}_j(n_1,\rho)$ tells us that for every $\xi \in [a^{k_j-nl}, a^{k_j}]$, and for every $I \in \mathcal{I}^{k_j}$, the number of words on an (R,ξ,ρ)-good track on the path $\Gamma(I|_{\tilde{m}+1}, I|_{j-\tilde{m}-1})$ is greater than $\frac{j}{n_1}$. For a fixed ξ, we will omit (R,ξ,ρ) when talking about words that are good or on a good track.

To prove the power Fourier decay, we consider $\mathcal{I}^{k_j}$ as a probability space, provided a measure $\mathbb{P} : \mathbb{P}(I) = P_I$ for $I \in \mathcal{I}^{k_j}$. Further, for $1 \leq s \leq j$, $0 \leq i \leq \tilde{m}$, we define following random variables: $X_s^{(i)}$ is the number of words on the path $\Gamma(I|_s)$ having a good word among its i-level descendants; $X_s = X_s^{(0)}$ is the number of good words on the path $\Gamma(I|_s)$. Note that if $I|_s$ belongs to the path $\Gamma(I|_{\tilde{m}+1}, I|_{j-\tilde{m}-1})$ and it is on a good track, then there is a good word among its $\tilde{m}$-level descendants. Moreover, the correspondence is at most $(\tilde{m}+1)$-to-1. From $R \notin \mathcal{F}_j(n_1,\rho)$ and the definition of $\mathcal{F}_j(n_1,\rho)$, it follows that for every $I \in \mathcal{I}^{k_j}$,

$$X_j^{(\tilde{m})}(I) \geq \frac{j}{(\tilde{m}+1)n_1}. \tag{4.4}$$

We give some lemmas that are needed in the proof of the Fourier decay.

Lemma 4.4. *For $0 \le i \le \tilde{m}$, there exist $\alpha_i, \beta_i \in \mathbb{R}^+$ and $\gamma_i \in \mathbb{R}$ such that $\mathbb{P}(X_j^{(i)} \le \alpha_i j) \le \gamma_i e^{-\beta_i j}$ for j large enough.*

Proof. We prove the lemma by induction. For $i = \tilde{m}$, (4.4) implies $\mathbb{P}(X_j^{(\tilde{m})} \le \frac{j}{2(\tilde{m}+1)n_1}) = 0$. Hence, the lemma holds in this case. Suppose $0 \le i \le \tilde{m} - 1$ and the lemma holds for $i + 1$. We define a sequence of random variables $\left\{Z_s^{(i)} : Z_s^{(i)} = \frac{X_{s+1}^{(i)}}{2\varepsilon^{nl}} - X_s^{(i+1)}\right\}_{s=1}^{j-1}$. We make a claim.

Claim 4.5. $\{Z_s^{(i)}\}_{s=1}^{j-1}$ is a submartingale.

Proof. To prove the claim, we only need show that

$$\mathbb{E}(Z_s^{(i)} | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \ge Z_{s-1}^{(i)}.$$

We firstly consider the relationship between $X_s^{(i+1)}$ and $X_{s-1}^{(i+1)}$. For $I \in \mathcal{I}^{k_j}$, if $I|_s$ has no good descendant of level $i + 1$, then $X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I)$; if $I|_s$ has a good descendant of level $i + 1$, then $X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I) + 1$. In conclusion, we have either $X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I)$ or $X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I) + 1$. In the former case, it implies that $I|_{s+1}$ has no good descendant of level i, it follows $X_s^{(i)}(I) = X_{s+1}^{(i)}(I)$ and $Z_s^{(i)}(I) = Z_{s-1}^{(i)}(I)$. In the later case, we obtain that $I|_{s+1}$ has a good descendant of level i with probability $\ge 2\varepsilon^{nl}$. It means that $X_s^{(i)}(I) = X_{s+1}^{(i)}(I)$ or $X_{s+1}^{(i)}(I) = X_s^{(i)}(I) + 1$. It follows that $Z_s^{(i)}(I) = Z_{s-1}^{(i)}(I) - 1$ or $Z_s^{(i)}(I) = Z_{s-1}^{(i)}(I) + \frac{1}{2\varepsilon^{nl}} - 1$. According to these relationships, we can give a sequence of partitions of $\mathcal{I}^{k_j}$. Let

$$\Omega_s = \{I : X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I)\}, \quad \Omega_s^c = \{I : X_s^{(i+1)}(I) = X_{s-1}^{(i+1)}(I) + 1\};$$
$$\Psi_s = \{I : X_{s+1}^{(i)}(I) = X_s^{(i)}(I)\}, \quad \Psi_s^c = \{I : X_{s+1}^{(i)}(I) = X_s^{(i)}(I) + 1\}.$$

Then $\mathcal{I}^{k_j} = \Omega_2 \cup \Omega_2^c = \cdots = \Omega_{k_j} \cup \Omega_{k_j}^c = \Psi_1 \cup \Psi_1^c = \cdots = \Psi_{k_j-1} \cup \Psi_{k_j-1}^c$. We get

$$\begin{aligned}
&\mathbb{E}(Z_s^{(i)} | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \\
= &Z_{s-1}^{(i)} \mathbb{P}(\Omega_s | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) + (Z_{s-1}^{(i)} - 1)\mathbb{P}(\Omega_s^c \cap \Psi_s | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \\
&+ (Z_{s-1}^{(i)} + \frac{1}{2\varepsilon^{nl}} - 1)\mathbb{P}(\Omega_s^c \cap \Psi_s^c | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}).
\end{aligned}$$

Note $\mathcal{I}^{k_j} = \Omega_s \cup (\Omega_s^c \cap \Psi_s) \cup (\Omega_s^c \cap \Psi_s^c)$, we have

$$\begin{aligned}
&\mathbb{E}(Z_s^{(i)} | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \\
= &Z_{s-1}^{(i)} - \mathbb{P}(\Omega_s^c \cap \Psi_s | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) + \mathbb{P}(\Omega_s^c \cap \Psi_s^c | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \\
&\times (\frac{1}{2\varepsilon^{nl}} - 1) \\
= &Z_{s-1}^{(i)} + \frac{1}{2\varepsilon^{nl}} \mathbb{P}(\Omega_s^c \cap \Psi_s^c | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) - \mathbb{P}(\Omega_s^c | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}).
\end{aligned}$$

From $\mathbb{P}(\Omega_s^c \cap \Psi_s^c | Z_{s-1}^{(i)}, Z_{s-2}^{(i)}, \ldots, Z_1^{(i)}) \geq 2\varepsilon^{nl}$, it follows the claim. $\square$

We will prove the lemma by using the Azuma-Hoeffding inequality [1], which says that, if $\{u_i\}_{i=1}^k$ is a submartingale (or martingale) and satisfies $|u_{i+1} - u_i| \leq a_i$, then $\mathbb{P}(u_k - u_1 \leq -v) \leq e^{\frac{-v^2}{2\sum_{i=2}^{k-1} a_i^2}}$. From the proof of Claim 4.5, we obtain $|Z_s^{(i)} - Z_{s-1}^{(i)}| \leq \frac{1}{2\varepsilon^{nl}}$. Let $v = \frac{\alpha_{i+1} j}{4}$, by Azuma-Hoeffding inequality, we have

$$\mathbb{P}\Big(Z_{j-1}^{(i)} - Z_1^{(i)} \leq -\frac{\alpha_{i+1} j}{4}\Big) \leq e^{-\frac{\varepsilon^{2nl}\alpha_{i+1}^2 j^2}{8(j-2)}} \leq e^{-\frac{\varepsilon^{2nl}\alpha_{i+1}^2 j}{8}}.$$

Note that $Z_1^{(i)}$ is bounded. Then for j large enough,

$$\mathbb{P}\Big(Z_{j-1}^{(i)} \leq -\frac{\alpha_{i+1} j}{2}\Big) \leq \mathbb{P}\Big(Z_{j-1}^{(i)} - Z_1^{(i)} \leq -\frac{\alpha_{i+1} j}{4}\Big) \leq e^{-\frac{\varepsilon^{2nl}\alpha_{i+1}^2 j}{8}}.$$

Recall that $X_j^{(i)} = 2\varepsilon^{2nl}(Z_{j-1}^{(i)} + X_{j-1}^{(i+1)})$ and

$$\mathbb{P}\Big(X_j^{(i+1)} \leq \alpha_{i+1} j\Big) \leq \gamma_{i+1} e^{-\beta_{i+1} j}$$

by the inductive assumption. Hence, for j large enough,

$$\begin{aligned}\mathbb{P}\Big(X_j^{(i)} \leq 8\varepsilon^{2nl}\alpha_{i+1} j\Big) &\leq \mathbb{P}\Big(X_{j-1}^{(i+1)} \leq \frac{3\alpha_{i+1} j}{4}\Big) + \mathbb{P}\Big(Z_{j-1}^{(i)} \leq -\frac{\alpha_{i+1} j}{2}\Big)\\ &\leq \mathbb{P}\Big(X_j^{(i+1)} \leq \alpha_{i+1} j\Big) + \mathbb{P}\Big(Z_{j-1}^{(i)} \leq -\frac{\alpha_{i+1} j}{2}\Big)\\ &\leq \gamma_{i+1} e^{-\beta_{i+1} j} + e^{-\frac{\varepsilon^{2nl}\alpha_{i+1}^2 j}{8}}.\end{aligned}$$

It follows the lemma. $\square$

Let's return to the proof of the Fourier decay for the case $R \notin \mathcal{F}$ and $\xi \in [a^{k_j - nl}, a^{k_j}]$. For the inequality (4.3), we split it into two cases: (i) $X_j(I) < \alpha_0 j$, (ii) $X_j(I) \geq \alpha_0 j$, where α_0 is given by Lemma 4.4. Then (4.3) becomes

$$\begin{aligned}|\hat{\nu}_j(\xi)| &\leq \sum_{I \in \mathcal{I}^{k_j}} P_I \prod_{s:\mathbf{I}_s = \bar{\mathbf{I}}, \tilde{\mathbf{I}}} \Big(1 - 2\pi\varepsilon^{nl}\|R^{I|_s}\xi\|^2\Big)\\ &= \sum_{I: X_j(I) < \alpha_0 j} P_I \prod_{s:\mathbf{I}_s = \bar{\mathbf{I}}, \tilde{\mathbf{I}}} \Big(1 - 2\pi\varepsilon^{nl}\|R^{I|_s}\xi\|^2\Big)\\ &\quad + \sum_{I: X_j(I) \geq \alpha_0 j} P_I \prod_{s:\mathbf{I}_s = \bar{\mathbf{I}}, \tilde{\mathbf{I}}} \Big(1 - 2\pi\varepsilon^{nl}\|R^{I|_s}\xi\|^2\Big)\\ &\leq \sum_{I: X_j(I) < \alpha_0 j} P_I + \sum_{I: X_j(I) \geq \alpha_0 j} \prod_{s:\mathbf{I}_s = \bar{\mathbf{I}}, \tilde{\mathbf{I}}} \Big(1 - 2\pi\varepsilon^{nl}\|R^{I|_s}\xi\|^2\Big).\end{aligned}$$

From Lemma 4.4 and the definition of X_j, we have

$$|\hat{\nu}_j(\xi)| \leq \mathbb{P}(X_j \leq \alpha_0 j) + (1 - 2\pi\varepsilon^{nl}\rho^2)^{\alpha_0 j} \leq \gamma_0 e^{-\beta_0 j} + (1 - 2\pi\varepsilon^{nl}\rho^2)^{\alpha_0 j}.$$

Note $\xi \in [a^{k_j-nl}, a^{k_j}] = [a^{k_1+(j-1)nl}, a^{k_1+jnl}]$. We have

$$|\hat{\nu}_j(\xi)| \leq \gamma_0 \xi^{\frac{-1}{2nl\ln b}} + \xi^{\frac{\alpha_0 \ln(1-2\pi\varepsilon^{nl}\rho^2)}{2nl\ln b}}$$

for j large enough. Note that $\hat{\nu}_j(\xi)$ converges uniformly to $\hat{\mu}(\xi)$ on each compact set and that j,ξ are arbitrary. Then $\mu \in \mathcal{D}(\alpha)$, $\alpha = \min\{\frac{1}{2nl\ln b}, \frac{\alpha_0\ln(1-2\pi\varepsilon^{nl}\rho^2)}{-2nl\ln b}\}$.

It remains to estimate $\dim_H \mathcal{F}$. Fix $R \in \mathcal{F} = \limsup_{j\to\infty} \mathcal{F}_j(n_1,\rho)$. Then $R \in \mathcal{F}_j(n_1,\rho)$ for infinitely many j. Fix such an enough large j, there exist $\xi \in [a^{k_j-nl}, a^{k_j}]$, $I \in \mathcal{I}^{k_j}$ such that $\mathrm{Card}\big(\{\tilde{m}+1 \leq s \leq j-\tilde{m}-1 : I|_s$ is on an (R,ξ,ρ)-good track$\}\big) \leq \frac{j}{n_1}$. Let $R^{I|_s}\xi = c(I|_s) + \epsilon(I|_s)$, where $c(I|_s)$ is the nearest integer to $R^{I|_s}$ and $|\epsilon(I|_s)| = \|R^{I|_s}(\xi)\|$. We are going to give a cover for $\mathcal{F}_j(n_1,\rho)$. To this end, we introduce the following lemmas.

Lemma 4.6. *Let* $\rho = \frac{1}{2(1+b^{nl})(1+2b^{nl})}$, $B = (1+b^{nl})(1+2b^{nl})+1$. *For* $I|_{s+2}, I|_{s+1}, I|_s$, $1 \leq s \leq j-2$, *if* $I|_{s+2} = I|_{s+1}\mathbf{I}_{s+2} = I|_s\mathbf{I}_{s+1}\mathbf{I}_{s+2}, \mathbf{I}_{s+1} = \mathbf{I}_{s+2} \in \mathcal{I}^{nl}$ *and* $c(I|_{s+2}) \geq 1$, *then following statements hold*:

(1) *Given* $c(I|_{s+2})$ *and* $c(I|_{s+1})$, *there are at most* B *choices for* $c(I|_s)$;

(2) *Given* $c(I|_{s+2})$ *and* $c(I|_{s+1})$, *if* $\max\{|\epsilon(I|_s)|, |\epsilon(I|_{s+1})|, |\epsilon(I|_{s+2})|\} < \rho$, *then* $c(I|_s)$ *is uniquely determined.*

Proof. From the assumption, we have

$$R^{I|_s}\xi = c(I|_s)+\epsilon(I|_s),\ R^{I|_{s+1}}\xi = c(I|_{s+1})+\epsilon(I|_{s+1}),\ R^{I|_{s+2}}\xi = c(I|_{s+2})+\epsilon(I|_{s+2}).$$

It implies

$$\begin{aligned}
\frac{c(I|_{s+1})}{c(I|_{s+2})} &= \frac{R^{I|_{s+1}}\xi - \epsilon(I|_{s+1})}{R^{I|_{s+2}}\xi - \epsilon(I|_{s+2})} \leq \frac{R^{I|_{s+1}}\xi + \frac{1}{2}}{R^{I|_{s+2}}\xi} \leq \frac{2R^{I|_{s+1}}\xi}{R^{I|_{s+2}}\xi} \\
&= 2(R^{\mathbf{I}_{s+2}})^{-1} \leq 2b^{nl}, \\
\Big|(R^{\mathbf{I}_{s+1}})^{-1} - \frac{c(I|_s)}{c(I|_{s+1})}\Big| &= \frac{|\epsilon(I|_s) - (R^{\mathbf{I}_{s+1}})^{-1}\epsilon(I|_{s+1})|}{c(I|_{s+1})} \\
&\leq \frac{|\epsilon(I|_s)| + b^{nl}|\epsilon(I|_{s+1})|}{c(I|_{s+1})}, \\
\Big|(R^{\mathbf{I}_{s+2}})^{-1} - \frac{c(I|_{s+1})}{c(I|_{s+2})}\Big| &= \frac{|\epsilon(I|_{s+1}) - (R^{\mathbf{I}_{s+2}})^{-1}\epsilon(I|_{s+2})|}{c(I|_{s+2})} \\
&\leq \frac{|\epsilon(I|_{s+1})| + b^{nl}|\epsilon(I|_{s+2})|}{c(I|_{s+2})}.
\end{aligned}$$

From above inequalities, it follows

$$\begin{aligned}
&\Big|c(I|_s) - \frac{c^2(I|_{s+1})}{c(I|_{s+2})}\Big| \\
&\leq c(I|_{s+1})\Big(\Big|\frac{c(I|_s)}{c(I|_{s+1})} - (R^{\mathbf{I}_{s+1}})^{-1}\Big| + \Big|(R^{\mathbf{I}_{s+1}})^{-1} - \frac{c(I|_{s+1})}{c(I|_{s+2})}\Big|\Big)
\end{aligned}$$

$$\leq c(I|_{s+1})\Big(\Big|\frac{c(I|_s)}{c(I|_{s+1})}-(R^{\mathbf{I}_{s+1}})^{-1}\Big|\Big)+\frac{c(I|_{s+1})}{c(I|_{s+2})}c(I|_{s+2})$$
$$\times\Big(\Big|(R^{\mathbf{I}_{s+1}})^{-1}-\frac{c(I|_{s+1})}{c(I|_{s+2})}\Big|\Big)$$
$$\leq |\epsilon(I|_s)|+b^{nl}|\epsilon(I|_{s+1})|+2b^{nl}(|\epsilon(I|_{s+1})|+b^{nl}|\epsilon(I|_{s+2})|)$$
$$\leq (1+b^{nl})(1+2b^{nl})\max\{|\epsilon(I|_s)|,|\epsilon(I|_{s+1})|,|\epsilon(I|_{s+2})|\}.$$

Now both parts of the lemma are immediate since $c(I|_s)\in\mathbb{N}$. □

Lemma 4.7. *Let ρ, B be given by Lemma 4.6. If $I|_s\to I'$ and we are given $c(\omega')$ for all ω' such that $I'\rightsquigarrow\omega'$ and $c(\omega')\geq 1$. Then following statements hold*:

(1) *for $I|_s\rightsquigarrow\omega$, there are at most B choices for $c(\omega)$;*

(2) *for $I|_s\rightsquigarrow\omega$, if neither ω nor ω' is (R,ξ,ρ)-good, then $c(\omega)$ is uniquely determined.*

Proof. Since $I|_s\to I'$, there exists $\mathbf{I}'_{s+1}\in\mathcal{I}^{nl}$ such that $I'=I|_s\mathbf{I}'_{s+1}$. Fix a ω such that $I|_s\rightsquigarrow\omega$. Then $\omega=I|_s\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}$, where $\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}$ satisfies $s+1\leq s'\leq s+m$ and that the elements of $\{R^{\tilde{\mathbf{I}}_{s+1}},\dots,R^{\tilde{\mathbf{I}}_{s'}}\}$ are distinct.

If $R^{\mathbf{I}'_{s+1}}\in\{R^{\tilde{\mathbf{I}}_{s+1}},\dots,R^{\tilde{\mathbf{I}}_{s'}}\}$, without loss of generality, we assume $R^{\mathbf{I}'_{s+1}}=R^{\tilde{\mathbf{I}}_{s+1}}$. Let

$$\omega'=I|_s\mathbf{I}'_{s+1}\tilde{\mathbf{I}}_{s+2}\cdots\tilde{\mathbf{I}}_{s'}=I'\tilde{\mathbf{I}}_{s+2}\cdots\tilde{\mathbf{I}}_{s'}.$$

One can verify $R^{\omega'}\xi=R^{\omega}\xi$, hence $c(\omega)=c(\omega')$. Since $I'\rightsquigarrow\omega'$, we know that $c(\omega)=c(\omega')$ is already known.

If $R^{\mathbf{I}'_{s+1}}\notin\{R^{\tilde{\mathbf{I}}_{s+1}},\dots,R^{\tilde{\mathbf{I}}_{s'}}\}$, then $s'<s+\tilde{m}$. Let

$$\omega'=I'\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}=I|_s\mathbf{I}'_{s+1}\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'},$$
$$\omega''=I'\mathbf{I}'_{s+1}\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}=I|_s\mathbf{I}'_{s+1}\mathbf{I}'_{s+1}\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}.$$

It is easy to see $I'\rightsquigarrow\omega'$, $I'\rightsquigarrow\omega''$. Hence, $c(\omega'),c(\omega'')$ are already known. Let

$$\omega^*=I|_s\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}\mathbf{I}'_{s+1}=\omega\mathbf{I}'_{s+1},\quad \omega^{**}=I|_s\tilde{\mathbf{I}}_{s+1}\cdots\tilde{\mathbf{I}}_{s'}\mathbf{I}'_{s+1}\mathbf{I}'_{s+1}=\omega\mathbf{I}'_{s+1}\mathbf{I}'_{s+1}.$$

Then $\omega^{**}=\omega^*\mathbf{I}'_{s+1}=\omega\mathbf{I}'_{s+1}\mathbf{I}'_{s+1}$. It is easy to check $R^{\omega^*}\xi=R^{\omega'}\xi$, $R^{\omega^{**}}\xi=R^{\omega''}\xi$ and $c(\omega^*)=c(\omega')$, $c(\omega^{**})=c(\omega'')$. From Lemma 4.6, we can get the desired result. □

We continue the estimate of $\dim_H\mathcal{F}$. Let

$$\tilde{\mathcal{F}}_j(n_1,\rho)=\{(r_1^{-1},r_2^{-1},\dots,r_m^{-1}):R=(r_1,r_2,\dots,r_m)\in\mathcal{F}_j(n_1,\rho)\}.$$

Since the function $f((r_1^{-1},r_2^{-1},\dots,r_m^{-1}))=(r_1,r_2,\dots,r_m)$ is bi-Lipschitz on $[a,b]^m$, we have $\dim_H\mathcal{F}_j(n_1,\rho)=\dim_H\tilde{\mathcal{F}}_j(n_1,\rho)$. Note for any $I\in\mathcal{I}^{k_j}$, $\mathbf{I}\in\mathcal{I}^{nl}$,

$$\Big|(R^{\mathbf{I}})^{-1}-\frac{c(I|_1)}{c(I|_1\mathbf{I})}\Big|\leq\frac{1+b^{nl}}{2c(I|_1\mathbf{I})}\leq\frac{1+b^{nl}}{2(R^{I|_1\mathbf{I}}\xi-\frac{1}{2})}$$

$$\le \frac{1+b^{nl}}{\frac{2a^{k_j-nl}}{b^{k_1+nl}}-1} = \frac{b^{k_1+nl}(1+b^{nl})}{2a^{k_1+(j-2)nl}-b^{k_1+nl}}.$$

Especially, if $\mathbf{I} = \overbrace{ii\cdots i}^{nl}$, $1 \le i \le m$, then

$$\Big|r_i^{-1} - \sqrt[nl]{\frac{c(I|_1)}{c(I|_1\mathbf{I})}}\Big| \le \Big|(R^{\mathbf{I}})^{-1} - \frac{c(I|_1)}{c(I|_1\mathbf{I})}\Big|$$
$$\le \frac{b^{k_1+nl}(1+b^{nl})}{2a^{k_1+(j-2)nl}-b^{k_1+nl}}.$$

Hence $(r_1^{-1}, r_2^{-1}, \ldots, r_m^{-1})$ is in the ball centered at

$$\Big(\sqrt[nl]{\frac{c(I|_1)}{c(I|_1 1\cdots 1)}}, \sqrt[nl]{\frac{c(I|_1)}{c(I|_1 2\cdots 2)}}, \ldots, \sqrt[nl]{\frac{c(I|_1)}{c(I|_1 m\cdots m)}}\Big)$$

with diameter $\sim a^{-nlj}$. To estimate $\dim_H \tilde{\mathcal{F}}_j(n_1,\rho)$, we need to estimate the number of possible pairs $(c(I|_1), c(I|_1\mathbf{I}))$ for $\mathbf{I} \in \mathcal{I}^{nl}$, $I \in \mathcal{I}^{k_j}$. For fixed j, $I = \mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_j \in \mathcal{I}^{k_j}$, $\xi \in [a^{k_j-nl}, a^{k_j}]$, let $q \in \mathbb{N}$ be maximal, such that

$$R^{I|_q}\xi \ge b^{\tilde{m}nl}. \tag{4.5}$$

It is easy to see $q \le j - \tilde{m}$ and that $q > \tilde{m}$ when j is sufficiently large. For any $I' \in \mathcal{I}^{k_j}$ satisfying $I|_q \rightsquigarrow I'$, from (4.5), we get $R^{I'}\xi \ge 1$, $c(I') \ge 1$. Note $I|_1 \rightsquigarrow I|_1\mathbf{I}$ for any $\mathbf{I} \in \mathcal{I}^{nl}$. It is sufficient to estimate the number of all possible different configurations of integers $c(\omega)$, where $I|_s \rightsquigarrow \omega$ for each s, $1 \le s \le q$. Note that for any $I|_s$, the number of different words ω with $I|_s \rightsquigarrow \omega$ is at most $1 + \tilde{m} + \tilde{m}(\tilde{m}-1) + \cdots + \tilde{m}! < (\tilde{m}+1)!$. From the choice of q, it follows $c(I|_q) \le b^{(\tilde{m}+1)nl} + 1$. Since $R^{\omega}\xi \le R^{I|_q}\xi$ for $I|_q \rightsquigarrow \omega$, we have $c(\omega) \in [1, b^{(\tilde{m}+1)nl}+1]$. It implies that number of possible configurations of integers $c(\omega)$ for all ω such that $I|_q \rightsquigarrow \omega$, is at most $(b^{(\tilde{m}+1)nl}+1)^{(\tilde{m}+1)!}$. We obtain the estimate following the path $I|_q, I|_{q-1}, \ldots, I|_1$ backwards. For $s \le q$, Lemma 4.7 tells us that for any ω with $I|_s \rightsquigarrow \omega$, there are at most B choices for $c(\omega)$, if all $c(\omega')$ are determined with $I|_{s-1} \rightsquigarrow \omega'$. Moreover, if neither ω nor ω' is (R,ξ,ρ)-good, then $c(\omega)$ is uniquely determined. Recall that the number of words on a good (R,ξ,ρ)-track on the path $\Gamma(I|_{\tilde{m}+1}, I|_{j-\tilde{m}})$ does not exceed $\frac{j}{n_1}$. For each set $\mathcal{Q} \subset \{1,2,\ldots,q\}$ corresponding to the words on a good (R,ξ,ρ)-track on the path $\Gamma(I|_{\tilde{m}+1}, I|_{j-\tilde{m}})$, we will obtain a subset $\mathcal{Q}' \subset \{1,2,\ldots,q\}$ with cardinality less than $\frac{2j}{n_1}$. For each $s \in \mathcal{Q}'$, there are at most $B^{(\tilde{m}+1)!}$ new configurations of $c(\omega)$ with $I|_s \rightsquigarrow \omega$ if all configurations of $c(\omega')$ with $I|_{s+1} \rightsquigarrow \omega'$ are given. Hence, this $\mathcal{Q}$ yields at most $(b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!}B^{(\tilde{m}+1)!(2j/n_1+2(\tilde{m}+1))}$ total configurations of integers $c(\omega)$, where $I|_s \rightsquigarrow \omega$ for each s, $1 \le s \le q$. Considering all the subset $\mathcal{Q}$ and all possible $q \le j$, we get that the total number of configurations is at most

$$
\begin{aligned}
& j \sum_{s=1}^{\lfloor j/n_1 \rfloor} C_q^s (b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!} B^{(\tilde{m}+1)!(2j/n_1+2(\tilde{m}+1))} \\
& < (b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!} B^{2(\tilde{m}+1)(\tilde{m}+1)!} j^2 C_j^{\lfloor j/n_1 \rfloor} B^{2(\tilde{m}+1)! j/n_1} \\
& < (b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!} B^{2(\tilde{m}+1)(\tilde{m}+1)!} j^2 e^{B^{2(\tilde{m}+1)!(\ln j)/n_1}},
\end{aligned}
$$

which is independent of ξ. Note that for each ξ, we just need calculate the words $I \in \mathcal{I}^{k_j}$ such that $\{R^{I|_s} : 1 \le s \le q\}$ distinct. It is easy to see that the number of this kind of words is less than $\tilde{m}^j$, we obtain that the set $\tilde{\mathcal{F}}_j(n_1, \rho)$ may be covered by

$$
(b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!} B^{2(\tilde{m}+1)(\tilde{m}+1)!} j^2 e^{B^{2(\tilde{m}+1)!(\ln j)/n_1}} \tilde{m}^j
$$

balls of diameter $\sim a^{-jnl}$. It follows

$$
\begin{aligned}
\dim_H \mathcal{F} &= \limsup_{j\to\infty} \dim_H \mathcal{F}_j(n_1, \rho) = \limsup_{j\to\infty} \dim_H \tilde{\mathcal{F}}_j(n_1, \rho) \\
&\le \lim_{j\to\infty} \frac{\ln((b^{(\tilde{m}+1)nl} + 1)^{(\tilde{m}+1)!} B^{2(\tilde{m}+1)(\tilde{m}+1)!} j^2 e^{B^{2(\tilde{m}+1)!(\ln j)/n_1}} \tilde{m}^j)}{-\ln a^{-jnl}} \\
&= \frac{\ln \tilde{m}}{nl \ln a} < \varpi.
\end{aligned}
$$

We finish the proof of Proposition 4.1. □

Proof of Theorem 1.7. Now we prove Theorem 1.7 by using Proposition 4.1. Let $G = \{R : R = (r, r, \ldots, r) \in (0,1)^m, r \in (0,1)\}$ and $(0,1)^m = \cup_{n=2}^{\infty} [\frac{1}{n}, 1 - \frac{1}{n+1}]^m$. From Theorem 1.6, we know that there exists $E \subset (0,1)^m$ such that $\dim_H E = 0$ and for all $R \in G \setminus E$, for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}$. It remains to consider $R \in (0,1)^m \setminus G$. For integers $n, p, q \ge 2$, $l \ge 1$, by Proposition 4.1, there exist $\alpha_{n,p,q,l}$ and $\mathcal{F}_{n,p,q,l}$ such that $\dim_H \mathcal{F}_{n,p,q,l} < \frac{1}{p}$ and for all $R = (r_1, \ldots, r_m) \in ((0,1)^m \setminus G) \setminus \mathcal{F}_{n,p,q,l}$ with $\frac{1}{n} \le \min_{1\le i\le m} r_i < \max_{1\le i\le m} r_i \le (1 - \frac{1}{n+1})$, for any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with $\inf_{k\ge1}\{x : x \in P_k\} \ge \frac{1}{q}$ and period l, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}(\alpha_{n,p,q,l})$. Let

$$
F_1 = \bigcup_{l=1}^{\infty} \bigcup_{n=2}^{\infty} \bigcap_{p=2}^{\infty} \bigcup_{q=2}^{\infty} \mathcal{F}_{n,p,q,l}, \quad F = E \cup F_1.
$$

It is easy to see $\dim_H F = 0$, and for all $R \in (0,1)^m \setminus F$, any quasi periodic sequence $\{(D_k, P_k)\}_{k=1}^{\infty}$ with a finite quasi period, the associated Moran measure $\mu = \mu_{R,\{D_k\},\{P_k\}} \in \mathcal{D}$. □

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