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CERTAIN DIFFERENTIAL IDENTITIES IN PRIME RINGS WITH ANTI-AUTOMORPHISMS

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Abstract. The objective of this paper is to study some central identities involving generalized derivations and anti-automorphisms in prime rings. Using the tools of the theory of functional identities, several known results have been generalized as well as improved.

1. Introduction

Throughout this paper, we tacitly assume that $\mathcal{A}$ is a prime ring with center $\mathcal{Z}(\mathcal{A})$. A ring $\mathcal{A}$ is called prime if for any $x, y \in \mathcal{A}$, whenever $x\mathcal{A}y = \{0\}$ implies that either $x = 0$ or $y = 0$. A ring $\mathcal{A}$ is n-torsion free if for any $x \in \mathcal{A}$, $nx = 0$ implies $x = 0$. We denote the maximal left (resp. right) ring of quotients of $\mathcal{A}$ by $\mathcal{Q}_{ml}(\mathcal{A})$ (resp. $\mathcal{Q}_{mr}(\mathcal{A})$), and the maximal symmetric ring of quotients of $\mathcal{A}$ by $\mathcal{Q}_{ms}(\mathcal{A})$. It is well known that $\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A}) \subseteq \mathcal{Q}_{ml}(\mathcal{A})$. The super rings $\mathcal{Q}_{ms}(\mathcal{A})$ and $\mathcal{Q}_{ml}(\mathcal{A})$ are also prime, and have the same centre $\mathcal{C}$, known as the extended centroid of $\mathcal{A}$. Moreover $\mathcal{C} = \{\lambda \in \mathcal{Q}_{ms}(\mathcal{A}) \mid \lambda a = a\lambda \text{ for all } a \in \mathcal{A}\}$ and $\mathcal{A}$ is prime if and only if $\mathcal{C}$ is a field. For $x \in \mathcal{A}$, we write $\deg(x) = n$ if x is algebraic of minimal degree n over $\mathcal{C}$ and $\deg(x) = \infty$ otherwise. For a nonempty subset $\mathcal{M}$ of $\mathcal{A}$, we define $\deg(\mathcal{M}) = \sup\{\deg(y) \mid y \in \mathcal{M}\}$. For details one may refer to [7].

An additive map '$*$': $\mathcal{A} \to \mathcal{A}$ is called an involution if $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in \mathcal{A}$, that is, an involution '$*$' on $\mathcal{A}$ is an anti-automorphism of period 1 or 2. An involution '$*$' on $\mathcal{A}$ is called symplectic if $a + a^* \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. For example in the ring of real quaternions the conjugation map is a symplectic involution. A ring $\mathcal{A}$ is said to be a $*$-ring if $\mathcal{A}$ admits an involution '$*$'. The set $\mathcal{K}(\mathcal{A}) = \{a \in \mathcal{A} \mid a^* = -a\}$ is known as the set of skew-symmetric elements of $\mathcal{A}$. For details on involution one may refer to [18]. For $x, y \in \mathcal{A}$, we denote the commutator $xy - yx$ by $[x,\ y]$, anti-commutator $xy + yx$ by $x \circ y$ and $xy - yx^\tau$ by $[x, y]_\tau$, where τ is an anti-automorphism of $\mathcal{A}$. For a

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positive integer n and $x, y \in \mathcal{A}$, $[x,\ y]_n = [[x,\ y],\ y]_{n-1}$, where $[x,\ y]_0 = x$ and $[x,\ y]_1 = xy - yx$.

It is well known that any anti-automorphism of $\mathcal{A}$ can be uniquely extended to an anti-automorphism of $\mathcal{Q}_{ms}(\mathcal{A})$ and hence can also be viewed as an anti-automorphism of $\mathcal{C}$. An anti-automorphism τ of $\mathcal{A}$ is said to be of the first kind if it induces the identity map on $\mathcal{C}$ and of the second kind otherwise. Also, τ is said to be of the first kind on $\mathcal{Z}(\mathcal{A})$ if $\alpha^\tau = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$ otherwise of the second kind on $\mathcal{Z}(\mathcal{A})$. Note that if $\mathcal{A}$ is a $*$-ring and $\alpha^* \neq \alpha \in \mathcal{Z}(\mathcal{A})$, then $0 \neq \beta = \alpha^* - \alpha \in \mathcal{K}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A})$. Therefore, if $\mathcal{A}$ is a 2-torsion free $*$-ring, then '$*$' is of the second kind on $\mathcal{Z}(\mathcal{A})$ if and only if $\mathcal{K}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A}) \neq \{0\}$.

An additive map $f : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called a derivation if $f(ab) = af(b) + f(a)b$ for all $a, b \in \mathcal{A}$. An additive map $\mathcal{F} : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called a generalized derivation if there exists a derivation $f : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{F}(ab) = af(b) + \mathcal{F}(a)b$ for all $a, b \in \mathcal{A}$. Note that if $\mathcal{A}$ is a prime ring and $\mathcal{F} : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is a generalized derivation, then there exists a unique derivation $f : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ associated with $\mathcal{F}$. Moreover a map $\phi : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called centralizing (resp. commuting) on $\mathcal{B} \subseteq \mathcal{A}$ if $[\phi(a), a] \in \mathcal{C}$ (resp. $[\phi(a), a] = 0$) for every $a \in \mathcal{B}$.

Many results in the literature indicate how the structure of the ring $\mathcal{A}$ and of the mappings defined on $\mathcal{A}$ are intimately related to the algebraic identities satisfied by appropriate subsets of $\mathcal{A}$. The most remarkable result in this direction was obtained by Posner [35], who proved that the existence of the nonzero centralizing derivation on a prime ring $\mathcal{A}$ forces $\mathcal{A}$ to be commutative. This result was extended by Lanski to Lie ideals [22, Theorem 2]. Starting from this result, several authors studied the relationship between the structure of a (semi)prime ring $\mathcal{A}$ and the behaviour of the additive maps defined on $\mathcal{A}$ satisfying some identities. For example, Brešar [9, Theorem 4.1] proved that a prime ring $\mathcal{A}$ is commutative if there exist derivations $f, g : \mathcal{A} \to \mathcal{A}$ such that $f(a)a - ag(a) \in \mathcal{Z}(\mathcal{A})$ holds for every $a \in \mathcal{K}$, where $\mathcal{K}$ is a nonzero left ideal of $\mathcal{A}$ and $g \neq 0$. Herstein [19] proved that if $\mathcal{A}$ is a 2-torsion free prime ring and $f : \mathcal{A} \to \mathcal{A}$ is a derivation such that $[f(a), f(b)] = 0$ for all $a, b \in \mathcal{A}$, then $\mathcal{A}$ is commutative. Fošner et al. [17, Theorem 2.7] proved that if $\mathcal{A}$ is a prime ring of characteristic different from two and $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{A}$ are generalized derivations satisfying the relation $\mathcal{F}(a)\mathcal{G}(a) - \mathcal{G}(a)\mathcal{F}(a) = 0$ for all $a \in \mathcal{A}$, then either $\mathcal{F} = 0$ or $\mathcal{G} = 0$. For other results see [5, 9, 11, 14, 15, 17, 20, 27, 38, 39] and the references therein.

On the other hand several authors studied derivations and generalized derivations in the setting of prime $*$-rings. For instance Ali et al. [1, Main Theorem], proved that if $\mathcal{A}$ is a 2-torsion free prime ring equipped with an involution '$*$' and $d : \mathcal{A} \to \mathcal{A}$ is a derivation such that $[d(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$ and $d(\mathcal{K}(\mathcal{A}) \cap \mathcal{Z}(\mathcal{A})) \neq \{0\}$, then $\mathcal{A}$ is commutative. Nejjar et al. [30, Theorem 3.7] obtained that if $\mathcal{A}$ is a 2-torsion free prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $d : \mathcal{A} \to \mathcal{A}$ is a nonzero derivation such that

$[d(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$ or $d(a) \circ a^* \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is commutative. In an attempt to generalize this result, Mamouni et al. [29, Theorems 2.1 and 2.2] proved that if $\mathcal{A}$ is a 2-torsion free noncommutative prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $f, g : \mathcal{A} \to \mathcal{A}$ are derivations such that $f(a)a^* - a^*g(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$ or $f(a^*)a - a^*g(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $f = g = 0$. Ali et al. [3] proved that a 2-torsion free prime ring $\mathcal{A}$ equipped with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ is commutative if there exists a nonzero derivation $f : \mathcal{A} \to \mathcal{A}$ such that either of the following conditions holds: $f([a, a^*]) = 0$ for all $a \in \mathcal{A}$; $f(a \circ a^*) = 0$ for all $a \in \mathcal{A}$; $f(aa^*) \pm aa^* = 0$ for all $a \in \mathcal{A}$; $f(aa^*) \pm a^*a = 0$ for all $a \in \mathcal{A}$; $f(a)f(a^*) - aa^* = 0$ for all $a \in \mathcal{A}$; $f(a)f(a^*) - a^*a = 0$ for all $a \in \mathcal{A}$. Zemzami et al. [41, Theorem 2(1)] proved that if $\mathcal{A}$ is a 2-torsion free prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a nonzero generalized derivation such that $[\mathcal{F}(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is commutative. In 2021, Oukhtite and Zemzami [34, Theorem 2.4(1)] proved that if $\mathcal{A}$ is a 2-torsion free prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a nonzero generalized derivation with $f : \mathcal{A} \to \mathcal{A}$ as associated derivation such that $\mathcal{F}([a, a^*]) - [f(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is commutative. Recently, Ali et al. [4, Theorem 4] obtained that if $\mathcal{A}$ is a 2-torsion free prime ring with involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is a generalized derivation such that $[a, \mathcal{F}(a^*)]_* \pm [a, a^*]_* \in \mathcal{C}$ for all $a \in \mathcal{A}$, then either $\mathcal{A}$ is commutative or $\mathcal{F}(a) = \mp a$ for all $a \in \mathcal{A}$. For other results see [1,2,4,8,12,13,21,28–33] and the references therein. Note that in all these cited results, involution '$*$' is assumed to be of the second kind on $\mathcal{Z}(\mathcal{A})$.

The main purpose of the paper is to characterize generalized derivations in prime rings with anti-automorphisms satisfying some algebraic identities. More precisely, we characterize generalized derivations $(\mathcal{F}, f), (\mathcal{G}, g) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$, where $\mathcal{A}$ is a noncommutative prime ring with an anti-automorphism τ of the second kind, satisfying any one among the following conditions:

(i) $\mathcal{F}(a)a^\tau - a\mathcal{G}(a^\tau) \in \mathcal{C}$ for all $a \in \mathcal{A}$.
(ii) $\mathcal{F}(a)a^\tau - a^\tau\mathcal{G}(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$.
(iii) $\mathcal{F}([a, a^\tau]) - [f(a), a^\tau] \in \mathcal{C}$ for all $a \in \mathcal{A}$.
(iv) $\mathcal{F}(aa^\tau) - a^\tau a \in \mathcal{C}$ for all $a \in \mathcal{A}$.
(v) $\mathcal{F}(aa^\tau) - aa^\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$.
(vi) $[a, \mathcal{F}(a^\tau)]_\tau \pm [a, a^\tau]_\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$.

In fact our results generalize as well as improve [3, Theorems 2.4-2.5], [4, Theorem 4], [1, Main Theorem], [28, Theorem 1(1) and (2)], [29, Theorems 2.1 and 2.2], [30, Theorem 3.7], [34, Theorem 2.4(1)] and [41, Theorem 2(1)] in the following directions.

(i) We prove our results without any restriction on the characteristic of ring.

(ii) We prove our results for any anti-automorphism τ of the second kind instead of involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$.
(iii) We will take generalized derivations from $\mathcal{A}$ to $\mathcal{Q}_{ml}(\mathcal{A})$ instead of $\mathcal{A}$ to $\mathcal{A}$.

2. Preliminary results

For the establishment of our results we fix some notations and recall the definition of a d-free subring (see [10, Definition 3.1]). Let Q be a unital ring with center C and A be a subring of Q. For a fixed positive integer p, we let $\bar{a}_p = (a_1, a_2, \ldots, a_p) \in A^p$,

$$\bar{a}_p^i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p) \in A^{p-1}$$

and for $1 \leq i < j \leq p$,

$$\bar{a}_p^{ij} = \bar{a}_p^{ji} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_p) \in A^{p-2}.$$

Let $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, p\}$ and $E_i, F_j : A^{p-1} \to Q$ be arbitrary maps, where $i \in \mathcal{I}$, $j \in \mathcal{J}$. Consider the following functional identity

$$\sum_{i\in\mathcal{I}} E_i(\bar{a}_p^i)a_i + \sum_{j\in\mathcal{J}} a_j F_j(\bar{a}_p^j) \in \mathcal{V} \tag{2.1}$$

for all $\bar{a}_p \in A^p$, where $\mathcal{V} \in \{0, C\}$ and the following standard solutions

$$\begin{aligned} E_i(\bar{a}_p^i) &= \sum_{j\in\mathcal{J}, j\neq i} a_j f_{ij}(\bar{a}_p^{ij}) + \lambda_i(\bar{a}_p^i), \\ F_j(\bar{a}_p^j) &= -\sum_{i\in\mathcal{I}, i\neq j} f_{ij}(\bar{a}_p^{ij})a_i - \lambda_j(\bar{a}_p^j), \\ \lambda_k &= 0 \text{ if } k \notin \mathcal{I}\cap\mathcal{J}, \end{aligned} \tag{2.2}$$

where $f_{ij} : A^{p-2} \to Q$ and $\lambda_i : A^{p-1} \to C$.

Definition 2.1. A ring A is called a d-free subring of Q, where d is a positive integer, if for all $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, p\}$ and $p \geq 1$ the following two conditions are satisfied:

(i) If $\mathcal{V} = 0$ and $\max\{|\mathcal{I}|,\ |\mathcal{J}|\} \leq d$, then (2.1) implies (2.2).
(ii) If $\mathcal{V} = C$ and $\max\{|\mathcal{I}|,\ |\mathcal{J}|\} \leq d-1$, then (2.1) implies (2.2).

Note that by [10, Lemma 3.2(vii)] if all E_i's and F_j's are $(p-1)$-additive, then all f_{ij}'s are $(p-2)$-additive and all the λ_i's are $(p-1)$-additive.

The following lemmas play a pivotal role in the proof of our main results.

Lemma 2.1 ([10, Corollary 5.12])**.** *Let $\mathcal{A}$ be a prime ring, and let d be a positive integer. Then $\mathcal{A}$ is a d-free subring of $\mathcal{Q}_{ml}(\mathcal{A})$ if and only if* $\deg(\mathcal{A}) \geq d$.

Lemma 2.2 ([10, Corollary 5.13])**.** *A prime ring $\mathcal{A}$ is a d-free subring of $\mathcal{Q}_{ml}(\mathcal{A})$ for every positive integer d if and only if* $\deg(\mathcal{A}) = \infty$*, that is, $\mathcal{A}$ is not a PI-ring.*

Lemma 2.3 ([26, Theorem 2.1])**.** *Let $\mathcal{A}$ be a prime ring with an anti-automorphism τ of the second kind. Suppose that $E_{ik}, F_{j1} : \mathcal{A}^{p-1} \to Q_{ml}(\mathcal{A})$ are $(p-1)$-additive maps such that*

$$\sum_{i=1}^{p} E_{i1}(\bar{a}_p^i)a_i + \sum_{i=1}^{p} E_{i2}(\bar{a}_p^i)a_i^\tau + \sum_{j=1}^{p} a_j F_{j1}(\bar{a}_p^j) \in \mathcal{C}$$

for all $\bar{a}_p \in \mathcal{A}^p$, where $1 \le i, j \le p$ and $k = 1, 2$. If $\mathcal{A}$ is not a PI-ring, then there exist a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$, $(p-2)$-additive maps $h_{ikl1} : \mathcal{J}^{p-2} \to \mathcal{Q}_{ml}(\mathcal{A})$ and $(p-1)$-additive maps $\mu_{i1} : \mathcal{J}^{p-1} \to \mathcal{C}$ such that

$$E_{i1}(\bar{a}_p^i) = \sum_{\substack{1\le j\le p\\ j\ne i}}^{p} a_j h_{i1j1}(\bar{a}_p^{ij}) + \mu_{i1}(\bar{a}_p^i),$$

$$E_{i2}(\bar{a}_p^i) = \sum_{\substack{1\le j\le p\\ j\ne i}} a_j h_{i2j1}(\bar{a}_p^{ij}),$$

$$F_{j1}(\bar{a}_p^j) = -\sum_{\substack{1\le i\le p\\ i\ne j}} h_{i1j1}(\bar{a}_p^{ij})a_i - \sum_{\substack{1\le i\le p\\ i\ne j}} h_{i2j1}(\bar{a}_p^{ij})a_i^\tau - \mu_{j1}(\bar{a}_p^j)$$

for all $\bar{a}_p \in \mathcal{J}^p$, where $1 \le i, j \le p$ and $k = 1, 2$. Moreover, if $E_{i1} = 0$ for all $1 \le i \le p$, then $h_{i1j1} = 0$ and $\mu_{i1} = 0$ for $1 \le i, j \le p$.

Lemma 2.4 ([37, Theorem 2.1])**.** *Let $\mathcal{A}$ be a $(d+1)$-free prime $*$-ring and $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L} \subseteq \{1, 2, \dots, p\}$. Let $E_i, F_j, G_k, H_l : A^{p-1} \to Q_{ml}(\mathcal{A})$ be arbitrary maps, where $i \in \mathcal{I}$, $j \in \mathcal{J}$, $k \in \mathcal{K}$ and $l \in \mathcal{L}$. Suppose that max $\{|\mathcal{I}| + |\mathcal{K}| + 1, |\mathcal{J}| + |\mathcal{L}|\} \le d$ and*

$$\sum_{i\in\mathcal{I}} E_i(\bar{a}_p^i)a_i + \sum_{j\in\mathcal{J}} a_j F_j(\bar{a}_p^j) + \sum_{k\in\mathcal{K}} G_k(\bar{a}_p^k)a_k^* + \sum_{l\in\mathcal{L}} a_l^* H_l(\bar{a}_p^l) \in \mathcal{C}$$

for all $\bar{a}_p \in A^p$. Then there exist unique maps $f_{ij}, g_{il}, h_{kj}, r_{kl} : A^{p-2} \to Q_{ml}(\mathcal{A})$ and $\lambda_i, \mu_k : A^{p-1} \to C$ such that

$$E_i(\bar{a}_p^i) = \sum_{j\in\mathcal{J}, j\ne i} a_j f_{ij}(\bar{a}_p^{ij}) + \sum_{l\in\mathcal{L}, l\ne i} a_l^* g_{il}(\bar{a}_p^{il}) + \lambda_i(\bar{a}_p^i),$$

$$F_j(\bar{a}_p^j) = -\sum_{i\in\mathcal{I}, i\ne j} f_{ij}(\bar{a}_p^{ij})a_i - \sum_{k\in\mathcal{K}, k\ne j} h_{kj}(\bar{a}_p^{kj})a_k^* - \lambda_j(\bar{a}_p^j),$$

$$G_k(\bar{a}_p^k) = \sum_{j\in\mathcal{J}, j\ne k} a_j h_{kj}(\bar{a}_p^{kj}) + \sum_{l\in\mathcal{L}, l\ne k} a_l^* r_{kl}(\bar{a}_p^{kl}) + \mu_k(\bar{a}_p^k),$$

$$H_l(\bar{a}_p^l) = -\sum_{i\in\mathcal{I}, i\ne l} g_{il}(\bar{a}_p^{il})a_i - \sum_{k\in\mathcal{K}, k\ne l} r_{kl}(\bar{a}_p^{kl})a_k^* - \mu_l(\bar{a}_p^l),$$

$$\lambda_k = 0 \text{ if } k \notin \mathcal{I} \cap \mathcal{J} \text{ and } \mu_k = 0 \text{ if } k \notin \mathcal{K} \cap \mathcal{L}.$$

If all E_i's, F_j's, G_k's and H_l's are $(p-1)$-additive, then all f_{ij}'s, g_{il}'s, h_{kj}'s, r_{kl}'s are $(p-2)$-additive and all the λ_i's, μ_k's are $(p-1)$-additive.

Lemma 2.5. *Let $\mathcal{A}$ be a prime PI-ring with an anti-automorphism τ. Then τ is of the first kind if and only if τ is of the first kind on $\mathcal{Z}(\mathcal{A})$.*

Proof. By [10, Theorem C.1], $\dim_{\mathcal{C}}\mathcal{AC} < \infty$. Therefore $\mathcal{AC} = \mathcal{Q}_{ml}(\mathcal{A})$, $\mathcal{Z}(\mathcal{A}) \neq \{0\}$ and any element in $\mathcal{AC}$ is of the form $\frac{a}{\alpha}$, for some $a \in \mathcal{A}$ and some nonzero $\alpha \in \mathcal{Z}(\mathcal{A})$ (see [36, Corollary 1]). Now if τ is of the first kind on $\mathcal{Z}(\mathcal{A})$, then clearly τ can be uniquely extended to an anti-automorphism of $\mathcal{AC}$, denoted by τ also, by defining $(\frac{a}{\alpha})^\tau = \frac{a^\tau}{\alpha}$ for $a \in \mathcal{A}$ and $0 \neq \alpha \in \mathcal{Z}(\mathcal{A})$. Therefore τ is of the first kind. The converse part holds trivially. □

Lemma 2.6 ([24, Corollary 1.2])**.** *Let $\mathcal{A}$ be a semiprime ring, and let τ be a surjective anti-homomorphism of $\mathcal{A}$. Then the following are equivalent*:

(i) $[a^\tau, a] = 0$ *for all* $a \in \mathcal{A}$.
(ii) $aa^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.
(iii) $a + a^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.

Lemma 2.7. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ. Then τ is of the first kind if any one of the following holds*:

(i) $[a^\tau, a] \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.
(ii) $aa^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.
(iii) $a + a^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.

Proof. If $[a^\tau, a] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $[a^\tau, a]_2 = 0$ for all $a \in \mathcal{A}$. In view of [16, Theorem 1.1] it follows that τ is a commuting anti-automorphism of $\mathcal{A}$. Therefore by Lemma 2.6, (i), (ii) and (iii) are equivalent. Also for any $a \in \mathcal{A}$, we have $a^2 - (a+a^\tau)a + aa^\tau = 0$. By [40, Lemma 2.1], it follows that $\mathcal{A}$ satisfies a polynomial identity with coefficients ± 1. Thus $\mathcal{A}$ is a PI-ring. Hence in view of Lemma 2.5, it suffices to prove that if τ is commuting, then τ is of the first kind on $\mathcal{Z}(\mathcal{A})$. Now by [23, Lemma 2.8], τ is an involution of $\mathcal{A}$. Hence by [30 Lemma 2.1], τ is of the first kind on $\mathcal{Z}(\mathcal{A})$. □

The following result characterizes the elements of $\mathcal{C}$ if $\mathcal{A}$ is a noncommutative prime ring of characteristic different from 2 and admits an anti-automorphism τ.

Corollary 2.1. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ satisfying any one of the following conditions*:

(i) $[a^\tau, a] \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.
(ii) $aa^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.
(iii) $a + a^\tau \in \mathcal{Z}(\mathcal{A})$ *for all* $a \in \mathcal{A}$.

If $\mathrm{char}(\mathcal{A}) \neq 2$, *then* $\alpha \in \mathcal{C}$ *if and only if* $\alpha^\tau = \alpha$.

Proof. The direct part follows from Lemma 2.7. For the converse part first note that by Lemma 2.6 and [16, Theorem 1.1], (i), (ii) and (iii) are equivalent. Hence in each case, $[a + a^\tau, b] = 0$ for all $a, b \in \mathcal{A}$. Now applying [7, Theorem 6.4.6], we find that $a + a^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{Q}_s(\mathcal{A})$. Therefore if $\alpha^\tau = \alpha$, then from the previous relation, we infer that $\alpha \in \mathcal{C}$. □

The following example demonstrates that the above corollary does not hold if $\mathrm{char}(\mathcal{A}) = 2$.

Example 2.1. Consider the ring $\mathcal{M}_2(\mathbb{F})$ of all 2×2 matrices over any field $\mathbb{F}$ of characteristic 2 with an anti-automorphism τ given by $\left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right]^\tau = \left[\begin{smallmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{smallmatrix}\right]$. Then the elements of the form $\left[\begin{smallmatrix} 0 & \alpha_2 \\ \alpha_3 & 0 \end{smallmatrix}\right]$, $\alpha_2 \neq 0$, are noncentral which are fixed by τ.

Lemma 2.8. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ, and let $\mathcal{J}$ be a nonzero ideal of $\mathcal{A}$. Suppose that $q, q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $q_1 a^\tau = aq$ for all $a \in \mathcal{J}$ (or $a^\tau q_1 = qa$ for all $a \in \mathcal{J}$). Then $q = q_1 = 0$.*

Proof. First assume that $qa^\tau = aq$ for all $a \in \mathcal{J}$. Then $baq = bqa^\tau = q(ab)^\tau = abq$ that is, $[a, b]q = 0$ for all $a, b \in \mathcal{J}$, from which it can be easily deduced that $q = 0$. Now suppose that $q_1 a^\tau = aq$ for all $a \in \mathcal{J}$. Then $(bq)a^\tau = q_1(ab)^\tau = a(bq)$ for all $a, b \in \mathcal{J}$. By above $bq = 0$ for all $b \in \mathcal{J}$. Hence $q = 0$ which further gives us $q_1 = 0$. Using similar techniques it can be shown that if $a^\tau q_1 = qa$ for all $a \in \mathcal{J}$, then $q = q_1 = 0$. □

3. Main results

In [30, Theorem 3.7(1)], Nejjar et al. improved [1, Main Theorem] and showed that if $\mathcal{A}$ is a 2-torsion free noncommutative prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $f : \mathcal{A} \to \mathcal{A}$ is a derivation such that $[f(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $f = 0$. In the following result we shall improve this result by showing that the torsion restriction and the condition "$*$ is of the second kind on $\mathcal{Z}(\mathcal{A})$" are superfluous.

Proposition 3.1. *Let $\mathcal{A}$ be a noncommutative prime ring and suppose that $f : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is a derivation. Then $f = 0$ if any one of the following holds:*

(i) *$[f(a), a] \in \mathcal{C}$ for all $a \in \mathcal{A}$.*
(ii) *$\mathcal{A}$ admits an involution '$*$' such that $[f(a), a^*] \in \mathcal{C}$ for all $a \in \mathcal{A}$.*

Proof. (i) By [27, Theorem 1.1] there exist $\lambda \in \mathcal{C}$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $f(a) = \lambda a + \mu(a)$ for all $a \in \mathcal{A}$. Therefore $\lambda ab + \mu(ab) = f(ab) = f(a)b + af(b) = \lambda ab + \mu(a)b + \lambda ab + \mu(b)a$ for all $a, b \in \mathcal{A}$. Hence

$$\lambda ab + \mu(a)b + \mu(b)a \in \mathcal{C} \tag{3.1}$$

for all $a, b \in \mathcal{A}$. In particular, we have $\lambda a^2 + 2\mu(a)a \in \mathcal{C}$ for all $a \in \mathcal{A}$. Linearizing this, we get

$$\lambda(ab + ba) + 2\mu(a)b + 2\mu(b)a \in \mathcal{C} \tag{3.2}$$

for all $a, b \in \mathcal{A}$. From (3.1) and (3.2), we see that $\lambda[a, b] \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Therefore $\lambda = 0$. Hence $f(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$. This gives us $f(a)b + af(b) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Thus $f(b)[a, b] = 0$ for all $a, b \in \mathcal{A}$. By the primeness of $\mathcal{A}$, we conclude that for each $b \in \mathcal{A}$, either $f(b) = 0$ or $b \in \mathcal{Z}(\mathcal{A})$. The sets $\{b \in \mathcal{A} \mid f(b) = 0\}$ and $\{b \in \mathcal{A} \mid b \in \mathcal{Z}(\mathcal{A})\}$ form additive subgroups of $\mathcal{A}$ whose union is $\mathcal{A}$. But a group can not be a set theoretic union of its two proper subgroups. Therefore $f = 0$.

(ii) Suppose $[f(a), a^*] \in \mathcal{C}$ for all $a \in \mathcal{A}$. Then $[f(a^*), a] \in \mathcal{C}$ for all $a \in \mathcal{A}$. Now in view of [27, Theorem 1.1], it follows that there exist $\lambda \in \mathcal{C}$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $f(a) = \lambda a^* + \mu(a^*)$ for all $a \in \mathcal{A}$. Therefore $f(ab) = \lambda b^* a^* + \mu(b^* a^*)$ for all $a, b \in \mathcal{A}$. On the other hand $f(ab) = f(a)b + af(b) = \lambda a^* b + \mu(a^*)b + \lambda ab^* + \mu(b^*)a$ for all $a, b \in \mathcal{A}$. Thus,

$$a(\lambda b^* + \mu(b^*)) + (\lambda a^* + \mu(a^*))b - (\lambda b^*)a^* \in \mathcal{C} \tag{3.3}$$

for all $a, b \in \mathcal{A}$. Now if $\deg(\mathcal{A}) > 3$, then by Lemma 2.1, $\mathcal{A}$ is 4-free. Therefore, by Lemma 2.4, there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and an additive map $\tau : \mathcal{A} \to \mathcal{C}$ such that $\lambda b^* - qb = \tau(b) - \mu(b^*)$ for all $b \in \mathcal{A}$. Applying [6, Corollary 3.4], we get $\tau(b) = \mu(b^*)$ for all $b \in \mathcal{A}$. Hence $\lambda b^* = qb$ for all $b \in \mathcal{A}$. Invoking Lemma 2.8, we have $\lambda = 0$ and hence $f(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$. By (i), we conclude that $f = 0$.

Next suppose that $\deg(\mathcal{A}) \leq 3$. Then $\mathcal{A}$ is a PI-ring. From (3.3), we have

$$(\lambda b^* + \mu(b^*))a + b(\lambda a^* + \mu(a^*)) - (\lambda a^*)b^* \in \mathcal{C} \tag{3.4}$$

for all $a, b \in \mathcal{A}$. Combining (3.3) and (3.4), we arrive at $\lambda[a, b] + \lambda[a + a^*, b^*] \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. We claim that $\lambda = 0$, otherwise we have $[a, b] + [a + a^*, b^*] \in \mathcal{Z}(\mathcal{A})$ for all $a, b \in \mathcal{A}$. Now if $\alpha^* \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$, then replacing b by αb in the last relation and using it again, we see that $[a, b] \in \mathcal{Z}(\mathcal{A})$ for all $a, b \in \mathcal{A}$; which leads to a contradiction that $\mathcal{A}$ is commutative. Therefore in view of Lemma 2.5, we assume that '$*$' is of the first kind. Clearly '$*$' can be uniquely extended to an involution of $\mathcal{AC}$, denoted by '$*$' also, by defining $(\frac{a}{\alpha})^* = \frac{a^*}{\alpha}$ for $a \in \mathcal{A}$ and $0 \neq \alpha \in \mathcal{Z}(\mathcal{A})$.

Now let $\mathbb{F}$ be the algebraic closure of $\mathcal{C}$. Then, '$*$' can be extended uniquely to an involution on $\mathcal{R} = \mathcal{AC} \otimes_{\mathcal{C}} \mathbb{F} \cong M_k(\mathbb{F})$, where $k = \deg(\mathcal{A}) > 1$, denoted by '$*$' also, by defining

$$\Big(\sum_i a_i \otimes \alpha_i\Big)^* = \sum_i a_i^* \otimes \alpha_i$$

for $a_i \in \mathcal{AC}$ and $\alpha_i \in \mathbb{F}$. Now it can be easily verified that

$$[a, b] + [a^*, b + b^*] \in \mathbb{F} \tag{3.5}$$

holds for all $a, b \in \mathcal{R}$. Moreover, '$*$' is either the ordinary transpose or the sympletic involution (see [7, Theorem 4.6.12 and Corollary 4.6.13] and [18] for details). Now if '$*$' is the symplectic involution, then from (3.5), we find that $[a, b] \in \mathbb{F}$ for all $a, b \in \mathcal{R}$; which leads to a contradiction. Also if '$*$' is the transpose involution, then setting $a = e_{11}$ and $b = e_{12}$ in (3.5), we see that $e_{12} - 2e_{21} \in \mathbb{F}$; which is a contradiction. Therefore $\lambda = 0$ and hence $f(a) \in \mathcal{C}$ for all $a \in \mathcal{R}$. By (i), we conclude that $f = 0$. □

Theorem 3.1. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ of the second kind. Suppose that $(\mathcal{F}, f), (\mathcal{G}, g) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ are generalized derivations such that*

$$\mathcal{F}(a)a^\tau - a\mathcal{G}(a^\tau) \in \mathcal{C} \tag{3.6}$$

for all $a \in \mathcal{A}$. Then there exists $q \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{F}(a) = aq$ and $\mathcal{G}(a) = qa$ for all $a \in \mathcal{A}$.

Proof. Linearizing (3.6), we have

$$\mathcal{F}(a)b^\tau + \mathcal{F}(b)a^\tau - a\mathcal{G}(b^\tau) - b\mathcal{G}(a^\tau) \in \mathcal{C} \tag{3.7}$$

for all $a, b \in \mathcal{A}$. Firstly we deal with the case when $\mathcal{A}$ is not a PI-ring. By Lemma 2.3, it follows that there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{F}(a) = aq$ and $\mathcal{G}(a^\tau) = qa^\tau$ for all $a \in \mathcal{J}$. Now let $a \in \mathcal{J}$ and $b \in \mathcal{A}$. Then, we have $abq = \mathcal{F}(ab) = aqb + af(b)$. This gives us $f(b) = [b, q]$ and hence $baq = \mathcal{F}(ba) = \mathcal{F}(b)a + bf(a) = \mathcal{F}(b)a + baq - bqa$. Therefore $\mathcal{F}(b) = bq$ for all $b \in \mathcal{A}$. Also for any $a \in \mathcal{A}$ and $b \in \mathcal{J}$, we have $qb^\tau a^\tau = \mathcal{G}(b^\tau a^\tau) = qb^\tau a^\tau + b^\tau g(a^\tau)$. Therefore $g = 0$ and hence $\mathcal{G}(a) = qa$ for all $a \in \mathcal{A}$.

Next assume that $\mathcal{A}$ is a PI-ring. Then by Lemma 2.5, τ is of the second kind on $\mathcal{Z}(\mathcal{A})$. Let $\alpha \in \mathcal{Z}(\mathcal{A})$ be such that $\alpha^\tau \neq \alpha$. Substituting αb for b in (3.7), we have

$$\alpha^\tau \mathcal{F}(a)b^\tau + \alpha\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau - \alpha^\tau a\mathcal{G}(b^\tau) - g(\alpha^\tau)ab^\tau - \alpha b\mathcal{G}(a^\tau) \in \mathcal{C} \tag{3.8}$$

for all $a, b \in \mathcal{A}$. Multiplying (3.7) by α and subtracting from (3.8), we arrive at

$$\mathcal{F}(a)b^\tau - a\mathcal{G}(b^\tau) + \beta ba^\tau - \gamma ab^\tau \in \mathcal{C} \tag{3.9}$$

for all $a, b \in \mathcal{A}$, where $\beta = (\alpha^\tau - \alpha)^{-1}f(\alpha) \in \mathcal{C}$ and $\gamma = (\alpha^\tau - \alpha)^{-1}g(\alpha^\tau) \in \mathcal{C}$. We claim that $\beta = \gamma = 0$, otherwise we have the following cases:

Case I. When $\beta = 0$ and $\gamma \neq 0$ or $\beta \neq 0$ and $\gamma = 0$. In this situation putting $b = a$ in (3.9), we get $aa^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. By Lemma 2.7, this is a contradiction.

Case II. When $\beta \neq 0$ and $\gamma \neq 0$. Setting $b = a$ in (3.9), we get $(\beta - \gamma)aa^\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$. Hence $\beta = \gamma$. So from (3.9), we have

$$\mathcal{F}(a)b^\tau - a\mathcal{G}(b^\tau) + \beta(ba^\tau - ab^\tau) \in \mathcal{C} \tag{3.10}$$

for all $a, b \in \mathcal{A}$. Replacing b by αb in (3.10) and using it again, we have $g(\alpha^\tau)ab^\tau - \beta(\alpha - \alpha^\tau)ba^\tau \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now replacing b by α here, we find that τ is commuting, which is not possible by Lemma 2.7.

Therefore $\beta = \gamma = 0$ and hence from (3.9), we have

$$\mathcal{F}(a)b - a\mathcal{G}(b) \in \mathcal{C} \tag{3.11}$$

for all $a, b \in \mathcal{A}$. In particular $\mathcal{F}(a)a - a\mathcal{G}(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$. According to [25, Theorem 1.1] there exist $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$, $\beta \in \mathcal{C}$ and additive maps $\zeta, \mu : \mathcal{A} \to \mathcal{C}$

such that $\mathcal{F}(a) = aq_1 + \zeta(a)$ and $\mathcal{G}(a) = (q_1 + \beta)a + \mu(a)$ for all $a \in \mathcal{A}$. Using these relations in (3.11), we arrive at

$$\zeta(a)b - \mu(b)a - \beta ab \in \mathcal{C} \tag{3.12}$$

for all $a, b \in \mathcal{A}$. Therefore, we have

$$(\zeta(a) - \beta a)[b, a] = 0 \tag{3.13}$$

for all $a, b \in \mathcal{A}$. Replacing b by cb in (3.13), we have $(\zeta(a) - \beta a)\mathcal{A}[b, a] = \{0\}$ for all $a, b \in \mathcal{A}$. Therefore for every $a \in \mathcal{A}$ either $\zeta(a) = \beta a$ or $a \in \mathcal{Z}(\mathcal{A})$. Let $\mathcal{M} = \{a \in \mathcal{A} \,|\, \zeta(a) = \beta a\}$ and $\mathcal{N} = \{a \in \mathcal{A} \,|\, a \in \mathcal{Z}(\mathcal{A})\}$. Then $\mathcal{M}$ and $\mathcal{N}$ are additive subgroups of $\mathcal{A}$ whose union is $\mathcal{A}$. But a group can not be a set theoretic union of its two proper subgroups. Hence $\zeta(a) = \beta a$ for all $a \in \mathcal{A}$. Using this in (3.12), we find that $\mu = 0$. Thus $\mathcal{F}(a) = aq$ and $\mathcal{G}(a) = qa$ for all $a \in \mathcal{A}$, where $q = q_1 + \beta$. This completes the proof. □

Corollary 3.1 ([29, Theorem 2]). *Let $\mathcal{A}$ be a 2-torsion free noncommutative prime ring with involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$. Then there exist no nonzero derivations $d_1, d_2 : \mathcal{A} \to \mathcal{A}$ such that $d_1(a^*)a - a^*d_2(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$.*

Now using similar techniques as in the proof of Theorem 3.1, with necessary alterations and applying Lemma 2.4 instead of Lemma 2.3, we can prove the following.

Theorem 3.2. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$' and let $(\mathcal{F}, f), (\mathcal{G}, g) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ be generalized derivations such that $\mathcal{F}(a)a^* - a^*\mathcal{G}(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$. If either* $\deg(\mathcal{A}) > 3$ *or '$*$' is of the second kind, then there exists $q \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{F}(a) = aq$ and $\mathcal{G}(a) = qa$ for all $a \in \mathcal{A}$.*

Theorem 3.3. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ of the second kind. Suppose that $(\mathcal{F}, f), (\mathcal{G}, g) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ are generalized derivations such that*

$$\mathcal{F}(a)a^\tau + a^\tau\mathcal{G}(a) \in \mathcal{C} \tag{3.14}$$

for all $a \in \mathcal{A}$. Then $\mathcal{F} = \mathcal{G} = 0$.

Proof. Let $\theta = \tau^{-1}$. Then from (3.14), we have

$$\mathcal{F}(a^\theta)a + a\mathcal{G}(a^\theta) \in \mathcal{C} \tag{3.15}$$

for all $a \in \mathcal{A}$. Linearizing (3.15), we have

$$\mathcal{F}(a^\theta)b + \mathcal{F}(b^\theta)a + a\mathcal{G}(b^\theta) + b\mathcal{G}(a^\theta) \in \mathcal{C} \tag{3.16}$$

for all $a, b \in \mathcal{A}$. First suppose that $\mathcal{A}$ is not a PI-ring. Then by Lemma 2.2, $\mathcal{A}$ is d-free for every positive integer d. Hence there exist $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $\mathcal{F}(a^\theta) = aq_1 + \mu(a)$. Thus $\mathcal{F}(a^\theta) - aq_1 \in \mathcal{C}$, which further gives us $\mathcal{F}(b)a^\theta + bf(a^\theta) - a(b^\tau q_1) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. By Lemma 2.3, it follows that there exist $q_2 \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$

such that $\mathcal{F}(b) = bq_2$ and $f(b) = -q_2 b^\theta$ for all $b \in \mathcal{J}$. This yields that $q_2 = 0$ and hence $\mathcal{F} = 0$. Therefore from (3.16), we have $a\mathcal{G}(b^\theta) + b\mathcal{G}(a^\theta) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now applying Lemma 2.2, we conclude that $\mathcal{G} = 0$.

Next assume that $\mathcal{A}$ is a PI-ring. In view of Lemma 2.5, it follows that there exists $\alpha \in \mathcal{Z}(\mathcal{A})$ such that $\alpha^\tau \neq \alpha$. Linearizing (3.14), we have

$$\mathcal{F}(a)b^\tau + \mathcal{F}(b)a^\tau + a^\tau\mathcal{G}(b) + b^\tau\mathcal{G}(a) \in \mathcal{C} \tag{3.17}$$

for all $a, b \in \mathcal{A}$. Substituting αb for b in (3.17), we have

$$\alpha^\tau\mathcal{F}(a)b^\tau + \alpha\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau + \alpha a^\tau\mathcal{G}(b) + g(\alpha)a^\tau b + \alpha^\tau b^\tau\mathcal{G}(a) \in \mathcal{C} \tag{3.18}$$

for all $a, b \in \mathcal{A}$. From (3.17) and (3.18), we deduce that

$$\mathcal{F}(a)b^\tau + b^\tau\mathcal{G}(a) + \beta ba^\tau + \gamma a^\tau b \in \mathcal{C} \tag{3.19}$$

for all $a, b \in \mathcal{A}$, where $\beta = (\alpha^\tau - \alpha)^{-1} f(\alpha) \in \mathcal{C}$ and $\gamma = (\alpha^\tau - \alpha)^{-1} g(\alpha) \in \mathcal{C}$.

We claim that $\beta = \gamma = 0$, otherwise we have the following cases:

Case I. When $\beta = 0$ and $\gamma \neq 0$ or $\beta \neq 0$ and $\gamma = 0$. In this situation putting $b = a$ in (3.19), we get $aa^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$; which by Lemma 2.7, leads to a contradiction.

Case II. $\beta \neq 0$ and $\gamma \neq 0$. Setting $b = a$ in (3.19), we get $(\beta + \gamma)aa^\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$. In view of Lemma 2.7, we infer that $\beta = -\gamma$. Thus replacing b by a in (3.19), we see that $[a^\tau, a] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$ which is not possible by Lemma 2.7.

Therefore $\beta = \gamma = 0$ and hence from (3.19), we have

$$\mathcal{F}(a)b + b\mathcal{G}(a) \in \mathcal{C} \tag{3.20}$$

for all $a, b \in \mathcal{A}$. Setting $b = \alpha$ in (3.20), we see that $(\mathcal{F} + \mathcal{G})(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$. By [20, Lemma 3], $\mathcal{F} = -\mathcal{G}$ and hence $[\mathcal{F}(a), b] \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Invoking [27, Theorem 1.1], it follows that there exist $\lambda \in \mathcal{C}$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $\mathcal{F}(a) = \lambda a + \mu(a)$ for all $a \in \mathcal{A}$. Therefore $\lambda[a, b] = 0$ for all $a, b \in \mathcal{A}$. And, hence $\lambda = 0$. Thus $\mathcal{F}(a) \in \mathcal{C}$ for all $a \in \mathcal{A}$, whence by [20, Lemma 3], we conclude that $\mathcal{F} = 0$. This completes the proof. □

Corollary 3.2 ([29, Theorem 1]). *Let $\mathcal{A}$ be a 2-torsion free noncommutative prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$. If $d_1, d_2 : \mathcal{A} \to \mathcal{A}$ are derivations such that $d_1(a)a^* - a^*d_2(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, then $d_1 = d_2 = 0$.*

Now using similar techniques as in the proof of Theorem 3.3 and applying Lemma 2.4 instead of Lemma 2.3, one can prove the following result.

Theorem 3.4. *Let $\mathcal{A}$ be a noncommutative prime ring with involution '$*$' and let $(\mathcal{F}, f), (\mathcal{G}, g) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ be generalized derivations such that $\mathcal{F}(a)a^* - a\mathcal{G}(a^*) \in \mathcal{C}$ for all $a \in \mathcal{A}$. If either $\deg(\mathcal{A}) > 3$ or '$*$' is of the second kind, then $\mathcal{F} = \mathcal{G} = 0$.*

Theorems 3.1 and 3.3 do not hold if τ is of the first kind. For if $\mathcal{A} = \mathbb{H}$ is the ring of real quaternions and anti-automorphism τ is the conjugate map. Then for fixed nonzero $q \in \mathbb{H}$, maps $\mathcal{F}, \mathcal{G} : \mathbb{H} \to \mathbb{H}$ given by $\mathcal{F}(a) = qa$ and $\mathcal{G}(a) = -a\bar{q}$ are generalized derivations such that $\mathcal{F}(a)a^\tau - a\mathcal{G}(a^\tau) \in \mathbb{R}$ and $\mathcal{F}(a)a^\tau - a^\tau\mathcal{G}(a) \in \mathbb{R}$, where $\mathbb{R}$ denotes the field of real numbers. However, $\mathcal{F}$ and $\mathcal{G}$ are not of the forms as described in Theorems 3.1 and 3.3.

Theorem 3.5. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ of the second kind. Suppose that there exists a generalized derivation $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ such that*

$$\mathcal{F}([a, a^\tau]) - [f(a), a^\tau] \in \mathcal{C} \tag{3.21}$$

for all $a \in \mathcal{A}$. Then $\mathcal{F} = 0$.

Proof. From (3.21), we have

$$\mathcal{F}(a)a^\tau + af(a^\tau) - \mathcal{F}(a^\tau)a - f(a)a^\tau \in \mathcal{C} \tag{3.22}$$

for all $a \in \mathcal{A}$. Linearizing (3.22), we have

$$(\mathcal{F}(a)-f(a))b^\tau+(\mathcal{F}(b)-f(b))a^\tau+af(b^\tau)+bf(a^\tau)-\mathcal{F}(a^\tau)b-\mathcal{F}(b^\tau)a \in \mathcal{C} \tag{3.23}$$

for all $a, b \in \mathcal{A}$. First suppose that $\mathcal{A}$ is not a PI-ring. Then applying Lemma 2.3, it follows that there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{F}(a) - f(a) = aq$ for all $a \in \mathcal{J}$. Now it can be easily verified that $\mathcal{F}(a) - f(a) = aq$ for all $a \in \mathcal{A}$. Substituting $\mathcal{F}(a) = aq + f(a)$ in (3.23), we arrive at

$$a(f(b^\tau) + qb^\tau) + b(f(a^\tau) + qa^\tau) - (f(a^\tau) + a^\tau q)b - (f(b^\tau) + b^\tau q)a \in \mathcal{C}$$

for all $a, b \in \mathcal{A}$. Now, by Lemma 2.2, $\mathcal{A}$ is d-free for every positive integer d. Hence there exist $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $qa^\tau + f(a^\tau) = q_1a + \mu(a)$ for all $a \in \mathcal{A}$. Therefore $(f(a)+qa)b+af(b)-(q_1b^\theta)a^\theta \in \mathcal{C}$ for all $a, b \in \mathcal{A}$, where $\theta = \tau^{-1}$. Applying Lemma 2.3, it follows that there exist $q_2 \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $q_1b^\theta = bq_2$ for all $b \in \mathcal{J}$. Thus by Lemma 2.8, we infer that $q_1 = q_2 = 0$. Therefore, $f(a) + qa \in \mathcal{C}$ for all $a \in \mathcal{A}$ which further, in view of [20, Lemma 3], gives us $f(a) = -qa$ for all $a \in \mathcal{A}$. Hence $f = 0$. Therefore from (3.23), we have

$$\mathcal{F}(a)b^\tau + \mathcal{F}(b)a^\tau - \mathcal{F}(a^\tau)b - \mathcal{F}(b^\tau)a \in \mathcal{C}$$

for all $a, b \in \mathcal{A}$. Invoking Lemma 2.3, we conclude that there exist $q_1, q_2 \in \mathcal{Q}_{ml}(\mathcal{A})$, a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ and an additive map $\mu : \mathcal{J} \mapsto \mathcal{C}$ such that $\mathcal{F}(a) = aq_2$ and $\mathcal{F}(a^\tau) = aq_1 + \mu(a)$ for all $a \in \mathcal{J}$. Therefore $aq_2 - a^\theta q_1 \in \mathcal{C}$ for all $a \in \mathcal{J}$, where $\theta = \tau^{-1}$. Hence $abq_2-b^\theta a^\theta q_1 \in \mathcal{C}$, that is, $a(b^\tau q_2)-b(a^\theta q_1) \in \mathcal{C}$ for all $a, b \in \mathcal{J}$. By Lemma 2.2, we infer that $q_2 = 0$. Consequently, $\mathcal{F} = 0$.

Next suppose that $\mathcal{A}$ is a PI-ring. Then by Lemma 2.5, $\alpha^\tau \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$. Replacing b by αb in (3.23), we get

$$\begin{aligned} &\alpha^\tau(\mathcal{F}(a) - f(a))b^\tau + \alpha(\mathcal{F}(b) - f(b))a^\tau + f(\alpha^\tau)ab^\tau + \alpha^\tau af(b^\tau) \\ &\quad + \alpha bf(a^\tau) - \alpha\mathcal{F}(a^\tau)b - \alpha^\tau\mathcal{F}(b^\tau)a - f(\alpha^\tau)b^\tau a \in \mathcal{C} \end{aligned} \tag{3.24}$$

for all $a, b \in \mathcal{A}$. From (3.23) and (3.24), we deduce that

$$(\alpha^\tau - \alpha)(\mathcal{F}(a) - f(a))b^\tau + f(\alpha^\tau)ab^\tau + (\alpha^\tau - \alpha)af(b^\tau) \\ - (\alpha^\tau - \alpha)\mathcal{F}(b^\tau)a - f(\alpha^\tau)b^\tau a \in \mathcal{C} \tag{3.25}$$

for all $a, b \in \mathcal{A}$. If $f(\alpha^\tau) = 0$, then (3.25) gives us

$$(\mathcal{F}(a) - f(a))b + af(b) - \mathcal{F}(b)a \in \mathcal{C} \tag{3.26}$$

for all $a, b \in \mathcal{A}$. Now setting $b = a$ in (3.26), we find that $[f(a), a] \in \mathcal{C}$. Applying Proposition 3.1, we infer that $f = 0$. Therefore $\mathcal{F}([a, b]) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now let x be a noncentral element of $\mathcal{A}$. Then $\mathcal{F}([a, xa]) = \mathcal{F}([a, x])a \in \mathcal{C}$ for all $a \in \mathcal{A}$. Therefore for each $a \in \mathcal{A}$ either $a \in \mathcal{Z}(\mathcal{A})$ or $\mathcal{F}([a, x]) = 0$. By the standard argument, we must have $\mathcal{F}([a, x]) = 0$ for all $a \in \mathcal{A}$ and hence $\mathcal{F}(a)[b, x] = \mathcal{F}([ab, x]) = 0$ for all $a, b \in \mathcal{A}$. Now it can be easily deduced that $\mathcal{F} = 0$. Next if $f(\alpha^\tau) \neq 0$, then substituting αa for a in (3.22) and using it again, we find that $[a, b] = 0$ for all $a, b \in \mathcal{A}$, which is a contradiction. □

Corollary 3.3 ([34, Theorem 2.4(1)])**.** *Let $\mathcal{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$ and $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ be a nonzero generalized derivation associated with a derivation $f : \mathcal{A} \to \mathcal{A}$ such that $\mathcal{F}([a, a^*]) - [f(a), a^*] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Then $\mathcal{A}$ is commutative.*

The following result generalizes as well as improves [3, Theorems 2.4-2.5].

Theorem 3.6. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ of the second kind. Suppose that $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is a generalized derivation such that*

$$\mathcal{F}(aa^\tau) - aa^\tau \in \mathcal{C} \tag{3.27}$$

for all $a \in \mathcal{A}$. Then $\mathcal{F}(a) = a$ for all $a \in \mathcal{A}$. Moreover, there exists no generalized derivation $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ such that

$$\mathcal{F}(aa^\tau) - a^\tau a \in \mathcal{C} \tag{3.28}$$

for all $a \in \mathcal{A}$.

Proof. From (3.27), we have

$$\mathcal{F}(a)a^\tau + af(a^\tau) - aa^\tau \in \mathcal{C} \tag{3.29}$$

for all $a \in \mathcal{A}$. Linearizing (3.29), we have

$$(\mathcal{F}(a) - a)b^\tau + (\mathcal{F}(b) - b)a^\tau + af(b^\tau) + bf(a^\tau) \in \mathcal{C} \tag{3.30}$$

for all $a, b \in \mathcal{A}$. Firstly we deal with the case when $\mathcal{A}$ is not a PI-ring. Applying Lemma 2.3, it follows that there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{F}(a) - a = aq$ for all $a \in \mathcal{J}$. Now it can be easily seen that $\mathcal{F}(a) = a + aq$ for all $a \in \mathcal{A}$. Using this in (3.27), we find that $aa^\tau q \in \mathcal{C}$ for all $a \in \mathcal{A}$. Replacing a by $a + b$ in the last relation, we get $a(b^\tau q) + b(a^\tau q) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Using Lemma 2.2, we conclude that $q = 0$.

Next assume that $\mathcal{A}$ is a PI-ring. Then by Lemma 2.5, $\alpha^\tau \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$. Substituting αb for b in (3.30), we obtain

$$\begin{aligned} \alpha^\tau(\mathcal{F}(a)-a)b^\tau + \alpha\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau - \alpha ba^\tau + \alpha^\tau af(b^\tau) \\ + f(\alpha^\tau)ab^\tau + \alpha bf(a^\tau) \in \mathcal{C} \end{aligned} \tag{3.31}$$

for all $a, b \in \mathcal{A}$. From (3.30) and (3.31), we find that

$$(\alpha-\alpha^\tau)\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau - (\alpha-\alpha^\tau)ba^\tau + f(\alpha^\tau)ab^\tau + (\alpha-\alpha^\tau)bf(a^\tau) \in \mathcal{C} \tag{3.32}$$

for all $a, b \in \mathcal{A}$. Now if $f(\alpha^\tau) = 0$ and $f(\alpha) = 0$, then from (3.32), we find that $\mathcal{F}(ba) - ba \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now setting $a = b = \alpha$ in the last relation, we find that $\mathcal{F}(\alpha) \in \mathcal{C}$. Hence $\mathcal{F}(\alpha a) - \alpha a \in \mathcal{C}$ gives us $[f(a), a] = 0$ for all $a \in \mathcal{A}$. Applying Proposition 3.1, it follows that $f = 0$. Therefore $(\mathcal{F}(b) - b)a \in \mathcal{C}$ for all $a \in \mathcal{A}$, which further gives us $\mathcal{F}(a) = a$ for all $a \in \mathcal{A}$.

Next if $f(\alpha^\tau) \neq 0$ and $f(\alpha) = 0$ or $f(\alpha^\tau) = 0$ and $f(\alpha) \neq 0$, then taking $a = b$ in (3.32), we get $aa^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, which is not possible by Lemma 2.7. Finally if $f(\alpha^\tau) \neq 0$ and $f(\alpha) \neq 0$, then from (3.32), we have $(f(\alpha) + f(\alpha^\tau))aa^\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$. Therefore $f(\alpha) = -f(\alpha^\tau)$. Using this in (3.32), we arrive at

$$\mathcal{F}(b)a^\tau + \lambda(ba^\tau - ab^\tau) - ba^\tau + bf(a^\tau) \in \mathcal{C} \tag{3.33}$$

for all $a, b \in \mathcal{A}$, where $\lambda = (\alpha - \alpha^\tau)^{-1} f(\alpha) \in \mathcal{C}$. Replacing b by αb in (3.33) and using it again, we get $ba^\tau - ab^\tau \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now setting $b = \alpha$ in the last relation, we see that $[a^\tau, a] = 0$ for all $a \in \mathcal{A}$, which is a contradiction.

Now suppose on the contrary that there exists a generalized derivation $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ such that (3.28) holds. Then, we have

$$\mathcal{F}(a)a^\tau + af(a^\tau) - a^\tau a \in \mathcal{C} \tag{3.34}$$

for all $a \in \mathcal{A}$. Linearizing (3.34), we have

$$\mathcal{F}(a)b^\tau + \mathcal{F}(b)a^\tau + af(b^\tau) + bf(a^\tau) - a^\tau b - b^\tau a \in \mathcal{C} \tag{3.35}$$

for all $a, b \in \mathcal{A}$. First suppose that $\mathcal{A}$ is not a PI-ring. Then applying Lemma 2.3, it follows that there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{F}(a) = aq$ for all $a \in \mathcal{J}$. Therefore $\mathcal{F}(a) = aq$ and $f(a) = [a, q]$ for all $a \in \mathcal{A}$. Using this in (3.35), we see that

$$b(a^\tau q) + a(b^\tau q) - a^\tau b - b^\tau a \in \mathcal{C}$$

for all $a, b \in \mathcal{A}$. Now by Lemma 2.2, we infer that there exist $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ and an additive map $\mu : \mathcal{A} \to \mathcal{C}$ such that $a^\tau q = q_1 a + \mu(a)$ for all $a \in \mathcal{A}$. This yields, $q_1 a^\theta = aq - \mu(a^\theta)$ and $abq - q_1 b^\theta a^\theta \in \mathcal{C}$ for all $a, b \in \mathcal{A}$, where $\theta = \tau^{-1}$. Hence $a(bq) + b(-qa^\theta) + \mu(b^\theta)a^\theta + \zeta(a)b^\theta \in \mathcal{C}$ for all $a, b \in \mathcal{A}$, where $\zeta = 0$. Applying Lemma 2.3, we infer that there exist $p \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $bq = pb^\theta$ for all $b \in \mathcal{J}$. By Lemma 2.8, $q = p = 0$ and so $\mathcal{F} = 0$. Thus from (3.28), we have $aa^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Now by Lemma 2.6, $a + a^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Thus for any $a \in \mathcal{A}$, we have

$a^2 - (a + a^\tau)a + aa^\tau = 0$. In view of [40, Lemma 2.1], it follows that $\mathcal{A}$ is a PI-ring, which is a contradiction.

Next assume that $\mathcal{A}$ is a PI-ring. Then by Lemma 2.5, $\alpha^\tau \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$. Substituting αb for b in (3.35), we obtain

$$\begin{aligned}\alpha^\tau \mathcal{F}(a)b^\tau + \alpha\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau + \alpha^\tau a f(b^\tau) \\ + f(\alpha^\tau)ab^\tau + \alpha b f(a^\tau) - \alpha a^\tau b - \alpha^\tau b^\tau a \in \mathcal{C}\end{aligned} \tag{3.36}$$

for all $a, b \in \mathcal{A}$. From (3.36) and (3.35), we find that

$$(\alpha - \alpha^\tau)\mathcal{F}(b)a^\tau + f(\alpha)ba^\tau + f(\alpha^\tau)ab^\tau + (\alpha - \alpha^\tau)bf(a^\tau) - (\alpha - \alpha^\tau)a^\tau b \in \mathcal{C} \tag{3.37}$$

for all $a, b \in \mathcal{A}$. If $f(\alpha^\tau) = f(\alpha) = 0$, then from (3.37), we have $\mathcal{F}(b)a + bf(a) - ab \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Also it can be easily seen that $\mathcal{F}(\alpha) \in \mathcal{C}$. Therefore setting $b = \alpha$ in the last relation, we have $[f(a), a] = 0$ for all $a \in \mathcal{A}$. Hence by Proposition 3.1, $f = 0$. Thus $\mathcal{F}(b)a - ab \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Putting $a = \alpha$ here, we see that $\mathcal{F}(a) - a \in \mathcal{C}$ for all $a \in \mathcal{A}$. Therefore by [20, Lemma 3], $\mathcal{F}(a) = a$ for all $a \in \mathcal{A}$. Hence $[a, b] \in \mathcal{Z}(\mathcal{A})$ for all $a, b \in \mathcal{A}$, which is a contradiction.

Now if $f(\alpha^\tau) \neq 0$ and $f(\alpha) \neq 0$, then putting $b = a$ in (3.37), we find that $(f(\alpha^\tau) + f(\alpha))aa^\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$. Thus $f(\alpha^\tau) = -f(\alpha)$. Hence setting αa at a in (3.37) and using it again, we find that $ab^\tau + ba^\tau \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now replacing b by α in the last relation, we have $[a^\tau, a] = 0$ for all $a \in \mathcal{A}$, which is a contradiction. Finally if $f(\alpha^\tau) \neq 0$ and $f(\alpha) = 0$ or $f(\alpha^\tau) = 0$ and $f(\alpha) \neq 0$, then putting $b = a$ in (3.37), we find that $aa^\tau \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$, which is a contradiction. □

Corollary 3.4 ([28, Theorem 1(1) and (2)]). *Let $\mathcal{A}$ be a 2-torsion free prime ring with an involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$. Let $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{A}$ be a generalized derivation such that either $\mathcal{F}(aa^*) - aa^* \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$ or $\mathcal{F}(aa^*) - a^*a \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Then $\mathcal{A}$ is commutative.*

Theorem 3.7. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ of the second kind. Suppose that $(\mathcal{F}, f) : \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is a generalized derivation such that*

$$[a, \mathcal{F}(a^\tau)]_\tau \pm [a, a^\tau]_\tau \in \mathcal{C} \tag{3.38}$$

for all $a \in \mathcal{A}$. Then $\mathcal{F}(a) = \mp a$ for all $a \in \mathcal{A}$.

Proof. Linearizing (3.38), we have

$$a\mathcal{F}(b^\tau) + b\mathcal{F}(a^\tau) + (a - a^\tau - \mathcal{F}(a^\tau))b^\tau + (b - b^\tau - \mathcal{F}(b^\tau))a^\tau \in \mathcal{C} \tag{3.39}$$

for all $a, b \in \mathcal{A}$. Firstly we deal with the case when $\mathcal{A}$ is not a PI-ring. Applying Lemma 2.3, it follows that there exist $q \in \mathcal{Q}_{ml}(\mathcal{A})$ and a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\mathcal{F}(a^\tau) = qa^\tau$ for all $a \in \mathcal{J}$. Therefore for every $a \in \mathcal{A}$ and $b \in \mathcal{J}$, we have $qb^\tau a^\tau + b^\tau f(a^\tau) = \mathcal{F}((ab)^\tau) = qb^\tau a^\tau$. Thus $f = 0$ and hence $\mathcal{F}(a) = qa$ for all $a \in \mathcal{A}$. Using this in (3.39), we arrive at

$$(a^\theta q - qa + a^\theta - a)b + (b^\theta q - qb + b^\theta - b)a \in \mathcal{C} \tag{3.40}$$

for all $a, b \in \mathcal{A}$, where $\theta = \tau^{-1}$. Now $\mathcal{A}$ is not a PI-ring. Therefore by Lemma 2.2, $\mathcal{A}$ is d-free for every positive integer d. Consequently, $(q+1)a = a^\tau(q+1)$ for all $a \in \mathcal{A}$. By Lemma 2.8, $q = -1$.

Next assume that $\mathcal{A}$ is a PI-ring. In view of Lemma 2.5, it follows that there exists $\alpha \in \mathcal{Z}(\mathcal{A})$ such that $\alpha^\tau \neq \alpha$. Substituting αb for b in (3.39) and using it again, we get

$$f(\alpha^\tau)ab^\tau + (\alpha - \alpha^\tau)b\mathcal{F}(a^\tau) + (\alpha - \alpha^\tau)ba^\tau - f(\alpha^\tau)b^\tau a^\tau \in \mathcal{C} \tag{3.41}$$

for all $a, b \in \mathcal{A}$. If $f(\alpha^\tau) = 0$, then from (3.41), we have $b(\mathcal{F}(a) + a) \in \mathcal{C}$ for all $a, b \in \mathcal{A}$. Now putting $b = \alpha$ in the previous relation, we find that $\mathcal{F}(a) + a \in \mathcal{C}$ for all $a \in \mathcal{A}$. Therefore, by [20, Lemma 3], we get $\mathcal{F}(a) = -a$ for all $a \in \mathcal{A}$. Also if $f(\alpha^\tau) \neq 0$, then replacing a by αa in (3.38) and using it again, we have

$$a\mathcal{F}(a^\tau) + \lambda_1 aa^\tau - \lambda_2(a^\tau)^2 + aa^\tau \in \mathcal{C} \tag{3.42}$$

for all $a \in \mathcal{A}$, where $\lambda_1 = (\alpha^\tau(\alpha - \alpha^\tau))^{-1}\alpha f(\alpha^\tau)$ and $\lambda_2 = (\alpha - \alpha^\tau)^{-1}f(\alpha^\tau)$. Linearizing this, we get

$$a\mathcal{F}(b^\tau) + b\mathcal{F}(a^\tau) + (\lambda_1 + 1)ab^\tau + (\lambda_1 + 1)ba^\tau - \lambda_2 a^\tau b^\tau - \lambda_2 b^\tau a^\tau \in \mathcal{C} \tag{3.43}$$

for all $a, b \in \mathcal{A}$. Substituting αb for b in (3.43) and using it again, we have

$$(\alpha - \alpha^\tau)b\mathcal{F}(a^\tau) + (\lambda_1 + 1)(\alpha - \alpha^\tau)ba^\tau + f(\alpha^\tau)ab^\tau \in \mathcal{C} \tag{3.44}$$

for all $a \in \mathcal{A}$. Now it can be easily seen that $\mathcal{F}(\alpha^\tau) \in \mathcal{C}$. Therefore setting $a = \alpha$ in (3.44), we see that $[b^\tau, b] = 0$ for all $b \in \mathcal{A}$, which is a contradiction. Similarly it can be shown that if $[a, \mathcal{F}(a^\tau)]_\tau - [a, a^\tau]_\tau \in \mathcal{C}$ for all $a \in \mathcal{A}$, then $\mathcal{F}(a) = a$ for all $a \in \mathcal{A}$. □

Corollary 3.5 ([4, Theorem 4]). *Let $\mathcal{A}$ be a 2-torsion free prime ring with involution '$*$' of the second kind on $\mathcal{Z}(\mathcal{A})$. If $\mathcal{A}$ admits a generalized derivation $(\mathcal{F}, f)$ such that $[a, \mathcal{F}(a^*)]_* \pm [a, a^*]_* \in \mathcal{C}$ for all $a \in \mathcal{A}$, then either $\mathcal{A}$ is commutative or $\mathcal{F}(a) = \mp a$ for all $a \in \mathcal{A}$.*

We conclude this article with the following example which shows that Proposition 3.1 and Theorems 3.1-3.6 do not hold for semiprime rings and hence the condition of primeness is essential.

Example 3.1. Let $\mathcal{A}_1$ be a noncommutative prime ring with commuting anti-automorphism τ. Also let $\mathcal{A}_2$ be a commutative integral domain with nonidentity automorphism σ and let $\delta : \mathcal{A}_2 \to \mathcal{Q}_{ml}(\mathcal{A}_2)$ be any nonzero derivation. Then the map $(a, b) \to (a^\tau, b^\sigma)$ is an anti-automorphism on $\mathcal{A}_1 \times \mathcal{A}_2$ and the map $d : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{Q}_{ml}(\mathcal{A}_1 \times \mathcal{A}_2)$ given by $d(a, b) = (0, \delta(b))$ is a derivation. Here all the hypotheses, except primeness, of Proposition 3.1 and Theorems 3.1-3.6 are satisfied but conclusions of lemma and theorems do not hold.

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