Bull. Korean Math. Soc. **61** (2024), No. 2, pp. 317–333
https://doi.org/10.4134/BKMS.b230018
pISSN: 1015-8634 / eISSN: 2234-3016

SHARP BOUNDS OF FIFTH COEFFICIENT AND HERMITIAN-TOEPLITZ DETERMINANTS FOR SAKAGUCHI CLASSES

Surya Giri and S. Sivaprasad Kumar

Abstract. For the classes of analytic functions f defined on the unit disk satisfying

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \quad \text{and} \quad \frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \varphi(z),$$

denoted by $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$, respectively, the sharp bound of the n^{th} Taylor coefficients are known for $n = 2$, 3 and 4. In this paper, we obtain the sharp bound of the fifth coefficient. Additionally, the sharp lower and upper estimates of the third order Hermitian Toeplitz determinant for the functions belonging to these classes are determined. The applications of our results lead to the establishment of certain new and previously known results.

1. Introduction

Let $\mathcal{H}$ be the class of holomorphic functions in the unit disk $\mathbb{D}$ and $A \subset \mathcal{H}$ represent the class of functions f satisfying $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent functions. A function $f \in \mathcal{H}$ is said to be starlike with respect to symmetric point if for r less than and sufficiently close to 1 and every z_0 on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. Sakaguchi [20] showed that a function $f \in \mathcal{A}$ is starlike with respect to symmetrical point if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)-f(-z)} > 0.$$

The class of all such functions is denoted by $\mathcal{S}_s^*$. It is noted that the class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd functions starlike with respect to the origin

Received January 7, 2023; Revised October 24, 2023; Accepted January 8, 2024.
2020 *Mathematics Subject Classification.* Primary 30C45, 30C50, 30C80.
Key words and phrases. Univalent functions, starlike functions with respect to symmetric points, fifth coefficient, Hermitian-Toeplitz determinants.

[20]. Afterwards, Das and Singh [5] introduced the class $\mathcal{K}_s$ of $f \in \mathcal{A}$, known as convex functions with respect to symmetric points, which satisfy

$$\operatorname{Re} \frac{(2zf'(z))'}{(f(z)-f(-z))'} > 0.$$

The functions in the class are convex and Das and Singh proved that the n^{th} coefficient of functions in $\mathcal{K}_s$ is bounded by $1/n$, $n \geq 2$.

Incorporating the notion of subordination, Ravichandran [19] generalized these classes as

$$\mathcal{S}_s^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \varphi(z) \right\},$$
$$\mathcal{C}_s(\varphi) = \left\{ f \in \mathcal{A} : \frac{(2zf'(z))'}{(f(z)-f(-z))'} \prec \varphi(z) \right\},$$

where $\varphi(z)$ is an analytic univalent function in $\mathbb{D}$ satisfying (i) $\varphi(\mathbb{D})$ is symmetric about the real axis, (ii) $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0)=1$ (iii) $\varphi'(0)>0$ and (iv) $\operatorname{Re}\varphi(z)>0$ for all $z \in \mathbb{D}$. Let us take

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0. \tag{1.1}$$

They obtained certain convolution conditions and growth and distortion estimates for functions belonging to these classes. Later, Shanmugam et al. [21] found the sharp bound of Feketo-Szegö functional, $|a_3 - \mu a_2^2|$ for the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$, which easily provides the bound for initial coefficients $|a_2|$ and $|a_3|$. Further, the sharp bound of $|a_4|$ was determined by Khatter et al. [9] and for certain important choices of φ such as

$$\begin{cases} \mathcal{S}_{s,e}^* := \mathcal{S}_s^*(e^z), \quad \mathcal{S}_{s,L}^* := \mathcal{S}_s^*(\sqrt{1+z}) \quad \text{and} \\ \mathcal{S}_{s,RL}^* := \mathcal{S}_s^*\big(\sqrt{2} - (\sqrt{2}-1)\sqrt{(1-z)/(1+2(\sqrt{2}-1)z)}\big), \end{cases} \tag{1.2}$$

the sharp bound of $|a_5|$ was also established. The sharp bound of $|a_5|$ for functions belonging to the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$ was still unknown. We get this bound in Section 2. Recently, Gangania and Kumar [6] studied generalized Bohr Rogosinski type inequalities for the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$. Kumar and Kumar [11] obtained the sharp bound of second and third order Hermitian-Toeplitz determinant for Sakaguchi functions and the classes defined in (1.2).

For $f \in \mathcal{A}$ and $m, n \in \mathbb{N}$, the Hermitian-Toeplitz determinant of order m is given by

$$T_m(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_n & \cdots & a_{n+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \cdots & a_n \end{vmatrix}. \tag{1.3}$$

It can be easily seen that the determinant of $T_{m,1}(f)$ is rotationally invariant that is determinant of $T_{m,1}(f)$ and $T_{m,1}(f_\theta)$ are same, where $f_\theta = e^{-i\theta} f(e^{i\theta} z)$

and $\theta \in \mathbb{R}$. Since for $n = 1$ and $f \in \mathcal{A}$, $a_1 = 1$. Thus, the third order Hermitian-Toepilitz determinant is

$$T_{3,1}(f) = 1 - 2|a_2|^2 + 2\,\mathrm{Re}\left(a_2^2\bar{a}_3\right) - |a_3|^2. \tag{1.4}$$

Ye and Lim [23] proved that any $n \times n$ matrix over $\mathbb{C}$ generically can be written as the product of some Toeplitz matrices or Hankel matrices. The applications of Toeplitz matrices and Toeplitz determinants can be seen in the field of pure as well as applied mathematics. They arise in algebraic geometry, numerical integration, numerical integral equations and queueing networks. For more applications, we refer to [23] and the references cited therein.

Numerous papers have recently focused on finding the sharp upper and lower bounds of the Hermitian Toeplitz determinants for functions in $\mathcal{A}$. Cudna et al. [4] initiated this work by determining the sharp lower and upper estimates for $T_{2,1}(f)$ and $T_{3,1}(f)$ for the class of starlike and convex functions of order α, $0 \le \alpha < 1$. The bounds of $T_{2,1}(f)$ and $T_{3,1}(f)$ for the class $\mathcal{S}$ and its certain subclasses were derived by Obradović and Tuneski [18]. For more recent work on this topic, we refer to [1, 10, 12–14] and the references cited therein.

The aim of this paper is to derive the bound of $|a_5|$ and third order Hermitian Toeplitz determinant for f belonging to the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$.

2. Fifth coefficient bound

Let $\mathcal{P}$ be the class of Carathéodory functions $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ satisfying $\mathrm{Re}\, p(z) > 0$ $(z \in \mathbb{D})$. The subsequent lemmas are used in order to prove the bound of $|a_5|$.

Lemma 2.1 ([17])**.** *If the functions $1 + \sum_{n=1}^{\infty} p_n z^n$ and $1 + \sum_{n=1}^{\infty} q_n z^n$ are members of $\mathcal{P}$, then the same is true of the function*

$$1 + \sum_{n=1}^{\infty} \frac{p_n q_n}{2} z^n.$$

Lemma 2.2 ([17])**.** *Let $h(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$ and $1 + H(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be functions in $\mathcal{P}$, and set*

$$\gamma_n = \frac{1}{2^n}\left[1 + \frac{1}{2}\sum_{\nu=1}^{n}\binom{n}{\nu}\beta_\nu\right], \quad \gamma_0 = 1.$$

If A_n is defined by

$$\sum_{n=1}^{\infty}(-1)^{n+1}\gamma_{n-1}H^n(z) = \sum_{n=1}^{\infty} A_n z^n,$$

then $|A_n| \le 2$.

It is worth recalling the Möbius function Ψ_ξ, which maps the unit disk $\mathbb{D}$ onto itself and given by

$$\Psi_\xi(z) = \frac{z - \xi}{1 - \bar{\xi} z}, \quad \xi \in \mathbb{D}. \tag{2.1}$$

Lemma 2.3 ([3]). *If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$, then for some $\xi_1, \xi_2, \xi_3 \in \overline{\mathbb{D}}$,*

$$\begin{cases} p_1 = 2\xi_1, \quad p_2 = 2\xi_1^2 + 2(1-|\xi_1|^2)\xi_2, \\ p_3 = 2\xi_1^3 + 4(1-|\xi_1|^2)\xi_1\xi_2 - 2(1-|\xi_1|^2)\overline{\xi_1}\xi_2^2 \\ \qquad\qquad + 2(1-|\xi_1|^2)(1-|\xi_2|^2)\xi_3. \end{cases} \tag{2.2}$$

Further, for $\xi_1, \xi_2 \in \mathbb{D}$ and $\xi_3 \in \mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p(z) = (1+\omega(z))/(1-\omega(z)) \in \mathcal{P}$ with p_1, p_2 and p_3 as in (2.2)*, where*

$$\omega(z) = z\Psi_{-\xi_1}(z\Psi_{-\xi_2}(\xi_3 z)), \tag{2.3}$$

that is

$$p(z) = \frac{1 + (\overline{\xi_2}\xi_3 + \overline{\xi_1}\xi_2 + \xi_1)z + (\overline{\xi_1}\xi_3 + \xi_1\overline{\xi_2}\xi_3 + \xi_2)z^2 + \xi_3 z^3}{1 + (\overline{\xi_2}\xi_3 + \overline{\xi_1}\xi_2 - \xi_1)z + (\overline{\xi_1}\xi_3 - \xi_1\overline{\xi_2}\xi_3 - \xi_2)z^2 - \xi_3 z^3}.$$

Conversely, for given $\xi_1, \xi_2 \in \mathbb{D}$ and $\xi_3 \in \overline{\mathbb{D}}$, we can construct a (unique) function $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that p_1, p_2 and p_3 satisfy the identities in (2.2). For this, we define

$$\omega(z) = \omega_{\xi_1,\xi_2,\xi_3}(z) = z\Psi_{-\xi_1}(z\Psi_{-\xi_2}(\xi_3 z)). \tag{2.4}$$

Moreover, if we define $p(z) = (1+\omega(z))/(1-\omega(z))$, then p_1, p_2 and p_3 satisfy the identities in (2.2) (see the proof of [3, Lemma 2.4]).

Assumption 2.4. *Let $\varphi(z)$ be given by* (1.1)*. The following conditions on coefficients of φ helps us to prove the result.*

C1 : $|B_1^3 - 2B_1B_2 + 2B_2^2| < |2B_1^2 - B_1^3 - 2B_1B_2|$,

C2 : $|B_1^3 - B_1^2B_2 + 3B_2^2 - 3B_1B_3| < 3|B_1^3 - B_1^2 + B_2^2|$,

C3 : $|B_1^7 - B_1^6(8B_2+3) - 6B_1^4(B_2(3B_2+2B_3+2) - 6B_3 + 9B_4)$
$+ B_1^5(7B_2(B_2+4) - 24B_3 + 18B_4) + 6B_1^3(B_2^3 - 2B_2^2 + 8B_2B_3 - 3B_3^2$
$+ 6(B_2+1)B_4) - 6B_1B_2(3B_2^3 - 6B_3^2 + B_2^2(4B_3-6) + 6B_2(B_4-2B_3))$
$+ 18B_2^2(-2B_3^2 + B_2((B_2-2)B_2 + 2B_4)) + B_1^2B_2(B_2(B_2(5B_2+6)$
$- 24B_3 + 18B_4) - 36(2B_3+B_4))| < 2|((B_1-2)B_1 + 2B_2)$
$(B_1(2B_1+B_2-3) + 3B_3)(4B_1^3 + 6B_2^2 - B_1^2(B_2+3) - 3B_1B_3)|$,

C4 : $0 < (2B_1 - B_1^2 - 2B_2)/(2(B_1 - B_2)) < 1$.

Theorem 2.5. *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_s^*(\varphi)$ and coefficients of $\varphi(z)$ satisfy the conditions* **C1**, **C2**, **C3** *and* **C4**, *then*

$$|a_5| \le \frac{B_1}{4}.$$

The bound is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*(\varphi)$. Then there exists a Schwarz function $\omega(z)$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \varphi(\omega(z)).$$

By the one-to-one correspondence between the class of Schwarz functions and the class $\mathcal{P}$, we obtain

$$\frac{2zf'(z)}{f(z) - f(-z)} = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) \tag{2.5}$$

for some $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$. On the comparison of the same powers of z with the series expansions of functions $f(z)$, $\varphi(z)$ and $p(z)$, the above equation yields

$$a_5 = \frac{B_1}{8}(\Upsilon_1 p_1^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4), \tag{2.6}$$

where

$$\begin{cases} \Upsilon_1 = \dfrac{B_1^2 - 2B_1 + 6B_2 - 2B_1B_2 + B_2^2 - 6B_3 + 2B_4}{16B_1}, \\ \Upsilon_2 = \dfrac{3B_1 - B_1^2 - 6B_2 + B_1B_2 + 3B_3}{4B_1}, \quad \Upsilon_3 = \dfrac{B_2 - B_1}{B_1}, \\ \Upsilon_4 = \dfrac{B_1^2 - 2B_1 + 2B_2}{4B_1}. \end{cases} \tag{2.7}$$

Let us consider that $q(z) = 1 + \sum_{n=1}^{\infty} \kappa_n z^n$ and $h(z) = 1 + \sum_{n=2}^{\infty} \nu_n z^n$ are the members of $\mathcal{P}$, then by Lemma 2.1 for $p \in \mathcal{P}$, we have

$$1 + H(z) := 1 + \sum_{n=1}^{\infty} \frac{p_n \kappa_n}{2} z^n \in \mathcal{P}. \tag{2.8}$$

For $h \in \mathcal{P}$ and the function $1 + H(z)$ given in (2.8), Lemma 2.2 gives

$$A_4 = \frac{1}{2}\gamma_0 \kappa_4 p_4 - \frac{1}{4}\gamma_1 \kappa_2^2 p_2^2 - \frac{1}{2}\gamma_1 \kappa_1 \kappa_3 p_1 p_3 + \frac{3}{8}\gamma_2 \kappa_1^2 \kappa_2 p_1^2 p_2 - \frac{1}{16}\gamma_3 \kappa_1^4 p_1^4, \tag{2.9}$$

where $\gamma_0 = 1$,

$$\gamma_1 = \frac{1}{2}\left(1 + \frac{1}{2}\nu_1\right), \ \gamma_2 = \frac{1}{4}\left(1 + \nu_1 + \frac{1}{2}\nu_2\right), \ \gamma_3 = \frac{1}{8}\left(1 + \frac{3}{2}\nu_1 + \frac{3}{2}\nu_2 + \frac{1}{2}\nu_3\right) \tag{2.10}$$

and

$$|A_4| \leq 2. \tag{2.11}$$

Now, in order to establish the required bound, we construct functions $h(z)$ and $q(z)$ such that

$$A_4 = \Upsilon_1 p_1^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4, \tag{2.12}$$

where Υ's and A_4 are given in (2.7) and (2.9), respectively. For $0<\tau<1$, define

$$q(z)=\frac{1+2\tau z+2\tau^2z^2+2\tau z^3+z^4}{1-z^4},$$

which yields

$$\kappa_1=\kappa_3=2\tau,\quad \kappa_2=2\tau^2\quad\text{and}\quad \kappa_4=2. \tag{2.13}$$

From [2, Theorem 1], we have $q\in\mathcal{P}$. To construct function $h(z)$, using Lemma 2.3, let

$$h(z)=\frac{1+\omega_1(z)}{1-\omega_1(z)}$$

such that

$$\omega_1(z)=z\Psi_{-\varepsilon_1}(z\Psi_{-\varepsilon_2}(\varepsilon_3 z)), \tag{2.14}$$

where $\varepsilon_1,\varepsilon_2\in\mathbb{D}$ and $\varepsilon_3\in\overline{\mathbb{D}}$. Thus, we have

$$\begin{aligned}&\nu_1=2\varepsilon_1,\quad \nu_2=2\varepsilon_1^2+2(1-|\varepsilon_1|^2)\varepsilon_2,\\ &\nu_3=2\varepsilon_1^3+4(1-|\varepsilon_1|^2)\varepsilon_1\varepsilon_2-2(1-|\varepsilon_1|^2)\overline{\varepsilon_1}\varepsilon_2^2+2(1-|\varepsilon_1|^2)(1-|\varepsilon_2|^2)\varepsilon_3.\end{aligned}$$

The above set of equations may be satisfied by many ε's. For our purpose, we impose some restriction on ε's and take all ε's as real numbers. Therefore,

$$\begin{cases}\nu_1=2\varepsilon_1,\quad \nu_2=2\varepsilon_1^2+2(1-\varepsilon_1^2)\varepsilon_2,\\ \nu_3=2\varepsilon_1^3+4(1-\varepsilon_1^2)\varepsilon_1\varepsilon_2-2(1-\varepsilon_1^2)\varepsilon_1\varepsilon_2^2+2(1-\varepsilon_1^2)(1-\varepsilon_2^2)\varepsilon_3.\end{cases} \tag{2.15}$$

In addition, if we define

$$\begin{aligned}\varepsilon_1&=\frac{B_1^3-2B_1B_2+2B_2^2}{2B_1^2-B_1^3-2B_1B_2},\quad \varepsilon_2=\frac{B_1^3-B_1^2B_2+3B_2^2-3B_1B_3}{3(-B_1^2+B_1^3+B_2^2)},\\ \varepsilon_3&=\Big(B_1^7-B_1^6(8B_2+3)-6B_1^4(B_2(3B_2+2B_3+2)-6B_3+9B_4)\\ &\quad+B_1^5(7B_2(B_2+4)-24B_3+18B_4)+6B_1^3(B_2^3-2B_2^2+8B_2B_3-3B_3^2\\ &\quad+6(B_2+1)B_4)-6B_1B_2(3B_2^3-6B_3^2+B_2^2(4B_3-6)-6B_2(2B_3-B_4))\\ &\quad+18B_2^2(-2B_3^2+B_2((B_2-2)B_2+2B_4))+B_1^2B_2(-36(2B_3+B_4)\\ &\quad+B_2(B_2(6+5B_2)-24B_3+18B_4))\Big)\Big/\Big(2((B_1-2)B_1+2B_2)\\ &\quad(B_1(2B_1+B_2-3)+3B_3)(4B_1^3+6B_2^2-B_1^2(3+B_2)-3B_1B_3)\Big)\end{aligned}$$

and

$$\tau=\sqrt{\frac{2B_1-B_1^2-2B_2}{2(B_1-B_2)}},$$

then by Assumption 2.4, we have $|\varepsilon_1| < 1$, $|\varepsilon_2| < 1$, $|\varepsilon_3| < 1$ and $0 < \tau < 1$. Putting these defined ε's in (2.15), we obtain ν_i's, which in turn together with (2.10) yields

$$(2.16)\quad \begin{cases} \gamma_1 = -\dfrac{(B_1-B_2)^2}{B_1(B_1^2-2B_1+2B_2)}, \\ \gamma_2 = -\dfrac{(B_1-B_2)^2(B_1^2+6B_2-B_1(3+B_2)-3B_3)}{3B_1(B_1^2-2B_1+2B_2)^2}, \\ \gamma_3 = -\dfrac{(B_1-B_2)^2(B_1^2+6B_2+B_2^2-2B_1(1+B_2)-6B_3+2B_4)}{4B_1(B_1^2-2B_1+2B_2)^2}. \end{cases}$$

On putting the values of κ_i's and γ_i's from (2.13) and (2.16), respectively, in (2.9), we get (2.12). Using the bound $|A_4| \leq 2$ in (2.12), we get

$$|\Upsilon_1 p_1^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4| \leq 2,$$

which together with (2.6) gives the desired bound of $|a_5|$.

Consider the function $\tilde{f}_5(z) = z + \sum_{n=2}^{\infty} \tilde{a}_n z^n$ in the unit disk satisfying

$$\frac{2z\tilde{f}_5'(z)}{\tilde{f}_5(z) - \tilde{f}_5(-z)} = \varphi(z^4),$$

where $\varphi(z)$ is given by (1.1). Clearly, $\tilde{f}_5 \in \mathcal{S}_s^*(\varphi)$. Equating the coefficients in the above equation, we obtain $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$ and $\tilde{a}_5 = B_1/4$, that demonstrates the sharpness of the bound. □

For $-1 \leq B < A \leq 1$, consider the classes $\mathcal{S}_s^*[A,B] := \mathcal{S}^*((1+Az)/(1+Bz))$ and $\mathcal{S}_{s,SG}^* := \mathcal{S}^*(2/(1+e^{-z}))$. These classes are analogues to the corresponding classes of starlike functions introduced and studied in [7, 8]. Theorem 2.5 directly gives the following result for these classes.

Corollary 2.6. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*[A,B]$ such that A and B satisfy the following conditions*

$$(2.17)\quad \begin{cases} \mathbf{C1}: |(A-B)^2(A+B+2B^2)| < |(A-3B-2)(A-B)^2|, \\ \mathbf{C2}: |(A-B)^3(B+1)| < 3|(A-B)^2(A-1+(B-1)B)|, \\ \mathbf{C3}: |(A-B)^5(B+1)(A^2(7B+1)+B(B(38+(12-17B)B) \\ \qquad +15)+A(B(B(5B-31)-27)-3))| < 2|(A-B)^4(A \\ \qquad -3B-2)(A(B-2)-4B^2+2B+3)(A(B+4)+2B(B \\ \qquad -2)-3)|, \\ \mathbf{C4}: 0 < (3B-A+2)/(2B+2) < 1, \end{cases}$$

then

$$|a_5| \leq (A-B)/4.$$

The bound is sharp.

Example 2.7. For $A = 0$ and $B = -1/2$, all conditions in Corollary 2.6 are satisfied. Thus, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*[0, -1/2]$, then $|a_5| \leq 1/8$.

Corollary 2.8. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_{s,SG}^*$, then $|a_5| \leq 1/8$ and the bound is sharp.*

In case of the classes $\mathcal{S}_{s,L}^*$ and $\mathcal{S}_{s,RL}^*$, the coefficients of corresponding φ satisfy the conditions **C1**, **C2**, **C3** and **C4**. Theorem 2.5 yields the following result for these classes:

Remark 2.9. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_{s,L}^*$, then $|a_5| \leq 1/8$ [9, Theorem 5(a)].

Remark 2.10. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_{s,RL}^*$, then $|a_5| \leq (5 - 3\sqrt{2})/8$ [9, Theorem 5(b)].

Theorem 2.11. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{C}_s(\varphi)$ and coefficients of $\varphi(z)$ satisfy the conditions* **C1**, **C2**, **C3** *and* **C4**, *then*

$$|a_5| \leq \frac{B_1}{20}.$$

The bound is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_s(\varphi)$. Then there exists a Schwarz function $\omega(z)$ such that

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \varphi(\omega(z)).$$

Corresponding to the Schwarz function $\omega(z)$, let there be a function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ satisfying $p(z) = (1 + \omega(z))/(1 - \omega(z))$. Thus, we obtain

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right). \tag{2.18}$$

Comparing the coefficients of the same powers of z after applying the series expansion of $f(z)$, $\varphi(z)$ and $p(z)$ leads to

$$a_5 = \frac{B_1}{20}(\Upsilon_1 p_1^4 + \Upsilon_2 p_1^2 p_2 + \Upsilon_3 p_1 p_3 + \Upsilon_4 p_2^2 + p_4),$$

where Υ_i's are given in (2.7). Since, Υ_i's are the same as in the case of $\mathcal{S}_s^*(\varphi)$, therefore following the same methodology as in Theorem 2.5, we get the bound of $|a_5|$.

To see the sharpness, consider the function $\tilde{g}_5(z) = z + \sum_{n=2}^{\infty} \tilde{a}_n z^n$ in $\mathbb{D}$ such that

$$\frac{(2z\tilde{g}_5'(z))'}{(\tilde{g}_5(z) - \tilde{g}_5(-z))'} = \varphi(z^4).$$

Comparison of coefficients of same powers yields $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$ and $\tilde{a}_5 = B_1/20$, which proves the sharpness of the bound. □

We can define the classes $\mathcal{C}_s[A,B]$, $\mathcal{C}_{s,e}$, $\mathcal{C}_{s,SG}$, $\mathcal{C}_{s,L}$ and $\mathcal{C}_{s,RL}$ in a similar manner as $\mathcal{S}^*_s[A,B]$, $\mathcal{S}^*_{s,e}$, $\mathcal{S}^*_{s,SG}$, $\mathcal{S}^*_{s,L}$ and $\mathcal{S}^*_{s,RL}$, respectively. For these classes, Theorem 2.11 yields the following:

Corollary 2.12. (1) *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_s[A,B]$ such that A and B satisfy the conditions given in Corollary 2.6, then $|a_5|\leq(A-B)/20$.*
(2) *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_{s,e}$, then $|a_5|\leq 1/20$.*
(3) *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_{s,L}$, then $|a_5|\leq 1/40$.*
(4) *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_{s,RL}$, then $|a_5|\leq(5-3\sqrt{2})/40$.*
(5) *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_{s,SG}$, then $|a_5|\leq 1/40$.*
All these bounds are sharp.

3. Hermitian-Toeplitz determinant

Shanmugam et al. [21] obtained the following bounds of $|a_3-\mu a_2^2|$ for $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ belonging to the classes $\mathcal{S}^*_s(\varphi)$ and $\mathcal{C}_s(\varphi)$.

Lemma 3.1 ([21, Theorem 2.1])**.** *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}^*_s(\varphi)$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{1}{2}\left(B_2-\dfrac{\mu}{2}B_1^2\right) & \text{if } \mu\leq\nu_1,\\ \dfrac{B_1}{2} & \text{if } \nu_1\leq\mu\leq\nu_2,\\ -\dfrac{1}{2}\left(B_2-\dfrac{\mu}{2}B_1^2\right) & \text{if } \mu\geq\nu_2,\end{cases}$$

where $\nu_1=(2(B_2-B_1))/B_1^2$ and $\nu_2=(2(B_2+B_1))/B_1^2$. The bound is sharp.

Lemma 3.2 ([21, Corollary 2.4])**.** *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{C}_s(\varphi)$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{1}{6}\left(B_2-\dfrac{3}{8}\mu B_1^2\right) & \text{if } \mu\leq\nu_1,\\ \dfrac{B_1}{6} & \text{if } \nu_1\leq\mu\leq\nu_2,\\ -\dfrac{1}{6}\left(B_2-\dfrac{3}{8}\mu B_1^2\right) & \text{if } \mu\geq\nu_2,\end{cases}$$

where $\nu_1=(8(B_2-B_1))/(3B_1^2)$ and $\nu_2=(8(B_2+B_1))/(3B_1^2)$. The bound is sharp.

For $\mu=0$, the following bounds for $|a_3|$ directly follow, which help us to prove the results:

Lemma 3.3. *If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{S}^*_s(\varphi)$ and $B_1\leq|B_2|$, then*

$$|a_3|\leq\frac{|B_2|}{2}.$$

Lemma 3.4. *If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{C}_s(\varphi)$ and $B_1\leq|B_2|$, then*

$$|a_3|\leq\frac{|B_2|}{6}.$$

Theorem 3.5. *If $f \in \mathcal{S}_s^*(\varphi)$ and $B_1 \leq |B_2|$, then*

$$T_{3,1}(f) \leq 1.$$

The bound is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*(\varphi)$. Then

$$T_{3,1}(f) = 1 - 2|a_2|^2 - |a_3|^2 + 2\,\mathrm{Re}(a_2^2 \bar{a}_3). \tag{3.1}$$

Applying the inequality $2\,\mathrm{Re}(a_2^2\bar{a}_3) \leq 2|a_2^2||a_3|$ in the last equation, we obtain

$$T_{3,1}(f) \leq 1 - 2|a_2|^2 - |a_3|^2 + 2|a_2^2||a_3| =: g(x),$$

where $g(x) = 1 - 2|a_2|^2 - x^2 + 2|a_2^2|x$ with $x = |a_3|$. For $f \in \mathcal{S}_s^*(\varphi)$, we have $|a_2| \leq B_1/2$ and from Lemma 3.3, $|a_3| \leq |B_2|/2$. Thus $|a_2| \in [0,1]$ and $x = |a_3| \in [0,1]$. As $g'(x) = 0$ at $x = |a_2|^2$ and $g''(x) < 0$ for all $x \in [0,1]$. Consequently, we have

$$\begin{aligned} T_{3,1}(f) &\leq \max g(x) \\ &= g(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1. \end{aligned}$$

Since the identity function $f(z) = z$ is a member of the class $\mathcal{S}_s^*(\varphi)$ and for this function, we have $a_2 = 0$, $a_3 = 0$ and $T_{3,1}(f) = 1$, which shows that the bound is sharp. □

Theorem 3.6. *If $f \in \mathcal{C}_c(\varphi)$ and $B_1 \leq |B_2|$, then*

$$T_{3,1}(f) \leq 1.$$

The result is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_s(\varphi)$. Then using the inequality $\mathrm{Re}(a_2^2\bar{a}_3) \leq |a_2|^2|a_3|$ in (1.4) for $f \in \mathcal{C}_s(\varphi)$, we obtain

$$T_{3,1}(f) \leq 1 - 2|a_2|^2 - |a_3|^2 + 2|a_2|^2|a_3| =: g(x),$$

where $g(x) = 1 - 2|a_2|^2 - x^2 + 2|a_2|^2 x$. Since $|a_2| \leq B_1/4$ and from Lemma 3.4, we have $|a_3| \leq |B_2|/6$, therefore $|a_2| \in [0,1/2]$ and $|a_3| \in [0,1/3]$. Also, note that $g(x)$ attains its maximum value at $x = |a_2|^2$. Hence

$$\begin{aligned} T_{3,1}(f) &\leq \max g(x) \\ &= g(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1. \end{aligned}$$

The equality case holds for $f(z) = z$. □

Theorem 3.7. *If $f \in \mathcal{S}_s^*(\varphi)$ such that $B_1^2 > 2B_2$, then the following estimates hold:*

$$T_{3,1}(f) \geq \begin{cases} \min\left\{1 - \dfrac{B_1^2}{4}, 1 - \dfrac{B_1^2}{2} + \dfrac{B_1^2 B_2}{4} - \dfrac{B_2^2}{4}\right\}, & \sigma_1 \notin [0,4], \\ 1 - \dfrac{B_1^2}{2} + \dfrac{B_1^2 B_2}{4} - \dfrac{B_2^2}{4}, & \sigma_1 = 4, \\ 1 - \dfrac{B_1^3(B_1^3 + 4B_1^2 - 4B_1 - 8B_2)}{16(B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)}, & \sigma_1 \in (0,4), \end{cases}$$

where

$$\sigma_1 = \frac{2B_1(B_1^2 - 2B_2)}{(B_1^2 - B_1 - B_2)(B_1 + B_2)}.$$

First two inequalities are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_s^*(\varphi)$. Then from (2.5), we obtain

$$a_2 = \frac{B_1 p_1}{4} \quad \text{and} \quad a_3 = \frac{1}{8}(-B_1 p_1^2 + B_2 p_1^2 + 2B_1 p_2).$$

Since the class $\mathcal{S}_s^*(\varphi)$ and the class $\mathcal{P}$ is rotationally invariant, therefore we can take $p_1 = p \in [0, 2]$. Moreover, Libera et al. [15] showed that $2p_2 = p_1^2 + (4 - p_1^2)\zeta$, $\zeta \in \mathbb{D}$ for $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$. Thus, we have

$$-|a_3|^2 = -\frac{1}{64}\left(B_2^2 p_1^4 + B_1^2(4 - p_1^2)^2|\zeta|^2 + 2B_1 B_2 p_1^2(4 - p_1^2)\operatorname{Re}\bar{\zeta}\right),$$

$$2\operatorname{Re}(a_2^2 \bar{a}_3) = \frac{1}{64} B_1^2 p_1^2 \left((B_2 - B_1)p_1^2 + B_1(p_1^2 + (4 - p_1^2)\operatorname{Re}\bar{\zeta})\right).$$

Taking these into account in (1.4), we get

$$\begin{aligned} T_{3,1}(f) = \frac{1}{64}\Big(&(B_1^2 - B_2)B_2 p_1^4 - B_1^2(4 - p_1^2)^2|\zeta|^2 \\ &+ B_1(B_1^2 - 2B_2)p_1^2(4 - p_1^2)\operatorname{Re}\bar{\zeta}\Big) - \frac{B_1^2 p_1^2}{8} + 1 \\ =:&\ F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}). \end{aligned}$$

It can be seen that $F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}) \geq F(p_1, |\zeta|, -|\zeta|) =: G(x, y)$ by considering $p_1^2 = x$ and $|\zeta| = y$, where

$$\begin{aligned} G(x, y) = &\frac{1}{64}\left((B_1^2 - B_2)B_2 x^2 - B_1^2(4 - x)^2 y^2 - B_1(B_1^2 - 2B_2)x(4 - x)y\right) \\ &- \frac{B_1^2 x}{8} + 1. \end{aligned}$$

Whenever $B_1^2 > 2B_2$, we have

$$\frac{\partial G}{\partial y} = \frac{1}{64}(-2B_1^2(4 - x)^2 y - B_1(B_1^2 - 2B_2)x(4 - x)) \leq 0$$

for $x \in [0, 4]$ and $y \in [0, 1]$, which means that $G(x, y)$ is a decreasing function of y and $G(x, y) \geq G(x, 1) =: I(x)$ with

$$I(x) = \frac{1}{64}(B_1^3 + B_1^2(B_2 - 1) - 2B_1 B_2 - B_2^2)x^2 + \frac{B_1}{16}(2B_2 - B_1^2)x - \frac{B_1^2}{4} + 1.$$

An easy computation yields that $I'(x) = 0$ at

$$x_0 = \frac{2B_1(B_1^2 - 2B_2)}{(B_1^2 - B_1 - B_2)(B_1 + B_2)}$$

and

$$I''(x_0) = \frac{1}{32}(B_1^2 - B_1 - B_2)(B_1 + B_2).$$

Since $B_1^2 > 2B_2$, therefore numerator of x_0 is always positive. Moreover, denominator of x_0 and numerator of $I''(x_0)$ are same, therefore $x_0 < 0$ (or $x_0 > 0$) if and only if $I''(x_0) < 0$ (or $I''(x_0) > 0$). Here we discuss the following cases:

Case I: Whenever $x_0 \in (0, 4)$, then $I''(x_0) > 0$. Thus $I(x)$ attains its minimum value at x_0, which gives

$$\begin{aligned} T_{3,1}(f) &\geq I(x_0) \\ &= 1 - \frac{B_1^3(B_1^3 + 4B_1^2 - 4B_1 - 8B_2)}{16(B_1^3 + B_1^2(B_2 - 1) - 2B_1B_2 - B_2^2)}. \end{aligned}$$

Case II: When $x_0 < 0$ or $x_0 > 4$, which indicates that $I(x)$ does not have any critical point, therefore

$$\begin{aligned} T_{3,1}(f) &\geq \min\{I(0), I(4)\} \\ &= \min\left\{1 - \frac{B_1^2}{4}, 1 - \frac{B_1^2}{2} + \frac{B_1^2B_2}{4} - \frac{B_2^2}{4}\right\}. \end{aligned}$$

For $x_0 = 4$, $T_{3,1}(f) \geq I(4)$.

Functions $\tilde{f}_2 \in \mathcal{S}_s^*(\varphi)$ and $\tilde{f}_3 \in \mathcal{S}_s^*(\varphi)$ given by

$$\frac{2z\tilde{f}_2'(z)}{\tilde{f}_2(z) - \tilde{f}_2(-z)} = \varphi(z), \quad \frac{2z\tilde{f}_3'(z)}{\tilde{f}_3(z) - \tilde{f}_3(-z)} = \varphi(z^2)$$

show that these bounds are sharp as

$$T_{3,1}(\tilde{f}_2) = 1 - \frac{B_1^2}{2} + \frac{B_1^2B_2}{4} - \frac{B_2^2}{4} \quad \text{and} \quad T_{3,1}(\tilde{f}_3) = 1 - \frac{B_1^2}{4},$$

which completes the proof. □

Theorem 3.8. *If $f \in \mathcal{C}_s(\varphi)$ and $3B_1^2 \geq 8B_2$, then the following estimates hold:*

$$T_{3,1}(f) \geq \begin{cases} \min\left\{1 - \dfrac{B_1^2}{36}, 1 - \dfrac{B_1^2}{8} + \dfrac{B_1^2B_2}{48} - \dfrac{B_2^2}{36}\right\}, & \sigma_2 \notin [0, 4], \\ 1 - \dfrac{B_1^2}{8} + \dfrac{B_1^2B_2}{48} - \dfrac{B_2^2}{36}, & \sigma_2 = 4, \\ 1 - \dfrac{B_1^3(B_1^3 + 12B_1^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}, & \sigma_2 \in (0, 4), \end{cases}$$

where

$$\sigma_2 = \frac{2B_1(3B_1^2 + 10B_1 - 8B_2)}{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}.$$

First two inequalities are sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_s(\varphi)$. Then from (2.18), we obtain

$$a_2 = \frac{B_1 p_1}{8}, \quad a_3 = \frac{1}{24}((B_2 - B_1)p_1^2 + 2B_1 p_2). \tag{3.2}$$

The rotationally invariant property of the classes $\mathcal{C}_s(\varphi)$ and $\mathcal{P}$ allows to take $p_1 \in [0,2]$. Using the formula $2p_2 = p_1^2 + (4 - p_1^2)\zeta$ (see [15]) in (3.2), we get

$$-|a_3|^2 = -\frac{1}{576}(B_2^2 p_1^4 + B_1^2(4-p_1^2)^2|\zeta|^2 + 2B_1B_2p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta}),$$

$$2\operatorname{Re}(a_2^2\bar{a}_3) = \frac{B_1^2p_1^2}{768}(-B_1p_1^2 + B_2p_1^2 + B_1(p_1^2 + (4-p_1^2)\operatorname{Re}\bar{\zeta})).$$

These above values together with (1.4) leads to

$$\begin{aligned} T_{3,1}(f) &= \left(\frac{B_1^2B_2}{768} - \frac{B_2^2}{576}\right)p_1^4 - \frac{1}{576}B_1^2(4-p_1^2)^2|\zeta|^2 \\ &\quad + \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)p_1^2(4-p_1^2)\operatorname{Re}\bar{\zeta} - \frac{B_1^2p_1^2}{32} + 1 \\ &=: F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}). \end{aligned}$$

As $\operatorname{Re}\bar{\zeta} \geq -|\zeta|$, hence $F(p_1, |\zeta|, \operatorname{Re}\bar{\zeta}) \geq F(p_1, |\zeta|, -|\zeta|) := G(x,y)$, where

$$\begin{aligned} G(x,y) =& \left(\frac{B_1^2B_2}{768} - \frac{B_2^2}{576}\right)x^2 - \frac{1}{576}B_1^2(4-x)^2y^2 - \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)x(4-x)y \\ & - \frac{B_1^2}{32}x + 1 \end{aligned}$$

for $x = p_1^2 \in [0,4]$ and $y = |\zeta| \in [0,1]$. Whenever $3B_1^2 \geq 8B_1B_2$, we have

$$\frac{\partial G(x,y)}{\partial y} = -\frac{1}{288}B_1^2(4-x)^2y - \left(\frac{3B_1^3 - 8B_1B_2}{2304}\right)x(4-x) \leq 0.$$

Therefore, $G(x,y)$ is a decreasing function of y and $G(x,y) \geq G(x,1) =: I(x)$, where

$$\begin{aligned} I(x) = 1 &- \frac{B_1(3B_1^2 + 10B_1 - 8B_2)}{576}x - \frac{B_1^2}{36} \\ &+ \frac{x^2(3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2)}{2304}. \end{aligned}$$

An elementary calculation reveals that $I'(x) = 0$ at

$$x_0 = \frac{2B_1(3B_1^2 + 10B_1 - 8B_2)}{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}$$

and

$$I''(x) = \frac{3B_1^3 + 3B_1^2B_2 - 4B_1^2 - 8B_1B_2 - 4B_2^2}{1152}.$$

Since $3B_1^2 \geq 8B_2$ and $B_1 > 0$, therefore numerator of x_0 is always positive. Also, note that, denominator x_0 and numerator of $I''(x)$ is same, therefore sign of x_0 and $I''(x)$ changes simultaneously. Here, two cases arise:

Case I: $0 < x_0 < 4$. In this case $I''(x) > 0$, so the minimum of $I(x)$ attains at x_0, which gives

$$\begin{aligned} T_{3,1}(f) &\geq I(x_0) \\ &= 1 - \frac{B_1^3(B_1^3 + 12B_1^2 + 4B_1 - 32B_2)}{64(3B_1^3 + B_1^2(3B_2 - 4) - 8B_1B_2 - 4B_2^2)}. \end{aligned}$$

Case II: $x_0 < 0$ or $x_0 > 4$. In this case $I(x)$ has no critical point. Thus

$$\begin{aligned} T_{3,1}(f) &\geq \min\{I(0), I(4)\} \\ &= \min\left\{1 - \frac{B_1^2}{36}, 1 - \frac{B_1^2}{8} + \frac{B_1^2B_2}{48} - \frac{B_2^2}{36}\right\}. \end{aligned}$$

For the case $x_0 = 4$, we have $T_{3,1}(f) \geq I(4)$.

The sharpness of these bounds follows from the functions $\tilde{g}_2(z)$ and $\tilde{g}_3(z)$ defined by

$$\frac{(2z\tilde{g}_2'(z))'}{(\tilde{g}_2(z) - \tilde{g}_2(-z))'} = \varphi(z) \quad \text{and} \quad \frac{(2z\tilde{g}_3'(z))'}{(\tilde{g}_3(z) - \tilde{g}_3(-z))'} = \varphi(z^2),$$

respectively. Since

$$T_{3,1}(\tilde{g}_2) = 1 - \frac{B_1^2}{8} + \frac{B_1^2B_2}{48} - \frac{B_2^2}{36} \quad \text{and} \quad T_{3,1}(\tilde{g}_3) = 1 - \frac{B_1^2}{36},$$

which completes the proof. □

4. Some special cases

If $\varphi(z) = (1 + Az)/(1 + Bz)$, the classes $\mathcal{S}_s^*(\varphi)$ and $\mathcal{C}_s(\varphi)$ reduce to the classes $\mathcal{S}_s^*[A, B]$ and $\mathcal{C}_s[A, B]$, respectively. Theorems 3.5 and 3.6 immediately give the following sharp bound for the class $\mathcal{S}_s^*[A, B]$ and $\mathcal{C}_s[A, B]$.

Corollary 4.1. (1) *If $f \in \mathcal{S}_s^*[A, B]$ and $A - B \leq |B^2 - AB|$, then $T_{3,1}(f) \leq 1$.*
(2) *If $f \in \mathcal{C}_s[A, B]$ and $A - B \leq |B^2 - AB|$, then $T_{3,1}(f) \leq 1$.*

Theorems 3.7 and 3.8 yield the following lower bound of $T_{3,1}(f)$ for these classes.

Corollary 4.2. *If $f \in \mathcal{S}_s^*[A, B]$ such that $A^2 - B^2 > 0$, then the following estimates hold*:

$$T_{3,1}(f) \geq \begin{cases} \min\left\{1 - \frac{1}{4}(A - B)^2, 1 - \frac{1}{4}(A - B)^2(AB + 2)\right\}, & \sigma_1 \notin [0, 4], \\ 1 - \frac{1}{4}(A - B)^2(AB + 2), & \sigma_1 = 4, \\ 1 + \frac{(A^2 - 2A(B - 2) + B^2 + 4B - 4)(A - B)^2}{16(1 - A)(1 - B)}, & \sigma_1 \in (0, 4), \end{cases}$$

where

$$\sigma_1 = -\frac{2(A + B)}{(1 - A)(1 - B)}.$$

First two inequalities are sharp.

Corollary 4.3. *If $f \in \mathcal{C}_s[A,B]$ and $3A^2+2AB-5B^2 \geq 0$, then the following estimates hold*:

$$T_{3,1}(f) \geq \begin{cases} \min\left\{1-\dfrac{(A-B)^2}{36}, 1-\dfrac{(A-B)^2(B^2+3AB+18)}{144}\right\}, & \sigma_2 \notin [0,4], \\ 1-\dfrac{(A-B)^2(B^2+3AB+18)}{144}, & \sigma_2 = 4, \\ 1-\dfrac{(A^2-2A(B-6)+B^2+20B+4)(A-B)^2}{64(1-B)(3A+B-4)}, & \sigma_2 \in (0,4), \end{cases}$$

where

$$\sigma_2 = \frac{2(3A+5(B+2))}{(1-B)(3A+B-4)}.$$

First two inequalities are sharp.

For $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$ and $(1+z)/(1-z)$ in $\mathcal{S}_s^*(\varphi)$, we obtain the class $\mathcal{S}_s^*(\alpha)$ and Sakaguchi's class, $\mathcal{S}_s^*$, respectively, where $\alpha \in [0,1]$. For more detail of these classes, we refer to [16,22]. Theorems 3.5 and 3.7 yield the following sharp lower and upper bounds of $T_{3,1}(f)$ for these classes, proved by Kumar and Kumar [11].

Remark 4.4. (1) If $f \in \mathcal{S}_s^*(\alpha)$, then $(3-2\alpha)\alpha^2 \leq T_{3,1}(f) \leq 1$ [11, Theorem 2.2].

(2) If $f \in \mathcal{S}_s^*$, then $0 \leq T_{3,1}(f) \leq 1$ [11, Corollary 2.3].

For other subclasses of $\mathcal{S}_s^*$, the following sharp bounds follow from Theorem 3.7.

Corollary 4.5. *If $f \in \mathcal{S}_{s,SG}^*$, then $T_{3,1}(f) \geq 2009/2304$.*

Remark 4.6. (1) If $f \in \mathcal{S}_{s,L}^*$, then $T_{3,1}(f) \geq 221/256$ [11, Theorem 3.1].

(2) If $f \in \mathcal{S}_{s,RL}^*$, then $T_{3,1}(f) \geq (863-444\sqrt{2})/256$ [11, Theorem 3.3].

Theorems 3.6 and 3.8 give the following corollaries for different subclasses of $\mathcal{C}_c$.

Corollary 4.7. (1) *If $f \in \mathcal{C}_s[A,B]$ and $A-B \leq |B^2-AB|$, then $T_{3,1}(f) \leq 1$.*

(2) *If $f \in \mathcal{C}_s(\alpha)$, then $T_{3,1}(f) \leq 1$.*

(3) *If $f \in \mathcal{C}_s$, then $T_{3,1}(f) \leq 1$.*

All these bounds are sharp.

Corollary 4.8. (1) *If $f \in \mathcal{C}_{s,SG}$, then $T_{3,1}(f) \geq 31/32$.*

(2) *If $f \in \mathcal{C}_{s,L}$, then $T_{3,1}(f) \geq 4459/4608$.*

(3) *If $f \in \mathcal{C}_{s,RL}$, then $T_{3,1}(f) \geq (-3731+5835\sqrt{2})/4608$.*

All these bounds are sharp.

Acknowledgments. The work of the Surya Giri is supported by University Grant Commission, New-Delhi, India under UGC-Ref. No. 1112/(CSIR-UGC NET JUNE 2019).

References

[1] O. P. Ahuja, K. Khatter, and V. Ravichandran, *Toeplitz determinants associated with Ma-Minda classes of starlike and convex functions*, Iran. J. Sci. Technol. Trans. A Sci. **45** (2021), no. 6, 2021–2027. https://doi.org/10.1007/s40995-021-01173-6

[2] K. Bano, M. Raza, and D. K. Thomas, *On the coefficients of $\mathcal{B}_1(\alpha)$ Bazilevič functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), Paper No. 7, 12 pp. https://doi.org/10.1007/s13398-020-00947-8

[3] N. E. Cho, B. Kowalczyk, and A. Lecko, *Sharp bounds of some coefficient functionals over the class of functions convex in the direction of the imaginary axis*, Bull. Aust. Math. Soc. **100** (2019), no. 1, 86–96. https://doi.org/10.1017/s0004972718001429

[4] K. Cudna, O. S. Kwon, A. Lecko, Y. J. Sim, and B. Śmiarowska, *The second and third-order Hermitian Toeplitz determinants for starlike and convex functions of order α*, Bol. Soc. Mat. Mex. (3) **26** (2020), no. 2, 361–375. https://doi.org/10.1007/s40590-019-00271-1

[5] R. N. Das and P. Singh, *On subclasses of schlicht mapping*, Indian J. Pure Appl. Math. **8** (1977), no. 8, 864–872.

[6] K. Gangania and S. S. Kumar, *A generalized Bohr-Rogosinski phenomenon*, Iran. J. Sci. **47** (2023), no. 1, 187–198. https://doi.org/10.1007/s40995-022-01398-z

[7] P. Goel and S. S. Kumar, *Certain class of starlike functions associated with modified sigmoid function*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 1, 957–991. https://doi.org/10.1007/s40840-019-00784-y

[8] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math. **28** (1973), 297–326. https://doi.org/10.4064/ap-28-3-297-326

[9] K. Khatter, V. Ravichandran, and S. S. Kumar, *Estimates for initial coefficients of certain starlike functions with respect to symmetric points*, in Applied analysis in biological and physical sciences, 385–395, Springer Proc. Math. Stat., 186, Springer, New Delhi, 2016. https://doi.org/10.1007/978-81-322-3640-5_24

[10] V. Kumar and N. E. Cho, *Hermitian-Toeplitz determinants for functions with bounded turning*, Turkish J. Math. **45** (2021), no. 6, 2678–2687. https://doi.org/10.3906/mat-2101-104

[11] S. Kumar and V. Kumar, *Sharp estimates on the third order Hermitian-Toeplitz determinant for Sakaguchi classes*, Commun. Korean Math. Soc. **37** (2022), no. 4, 1041–1053. https://doi.org/10.4134/CKMS.c210332

[12] V. Kumar, R. Srivastava, and N. E. Cho, *Sharp estimation of Hermitian-Toeplitz determinants for Janowski type starlike and convex functions*, Miskolc Math. Notes **21** (2020), no. 2, 939–952. https://doi.org/10.18514/mmn.2020.3361

[13] A. Lecko, Y. J. Sim, and B. Śmiarowska, *The fourth-order Hermitian Toeplitz determinant for convex functions*, Anal. Math. Phys. **10** (2020), no. 3, Paper No. 39, 11 pp. https://doi.org/10.1007/s13324-020-00382-3

[14] A. Lecko and B. Śmiarowska, *Sharp bounds of the Hermitian Toeplitz determinants for some classes of close-to-convex functions*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 5, 3391–3412. https://doi.org/10.1007/s40840-021-01122-x

[15] R. J. Libera and E. J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257. https://doi.org/10.2307/2043698

[16] P. T. Mocanu, *Certain classes of starlike functions with respect to symmetric points*, Mathematica (Cluj) **32(55)** (1990), no. 2, 153–157.

[17] Z. Nehari and E. Netanyahu, *On the coefficients of meromorphic schlicht functions*, Proc. Amer. Math. Soc. **8** (1957), 15–23. https://doi.org/10.2307/2032803

[18] M. Obradović and N. Tuneski, *Hermitian Toeplitz determinants for the class $\mathcal{S}$ of univalent functions*, Armen. J. Math. **13** (2021), Paper No. 4, 10 pp. https://doi.org/10.52737/18291163-2021.13.4-1-10
[19] V. Ravichandran, *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **20** (2004), no. 1, 31–37.
[20] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), no.1, 72–75. https://doi.org/10.2969/jmsj/01110072
[21] T. N. Shanmugam, C. Ramachandran, and V. Ravichandran, *Fekete-Szegő problem for subclasses of starlike functions with respect to symmetric points*, Bull. Korean Math. Soc. **43** (2006), no. 3, 589–598. https://doi.org/10.4134/BKMS.2006.43.3.589
[22] J. Thangamani, *On starlike functions with respect to symmetric points*, Indian J. Pure Appl. Math. **11** (1980), no. 3, 392–405.
[23] K. Ye and L.-H. Lim, *Every matrix is a product of Toeplitz matrices*, Found. Comput. Math. **16** (2016), no. 3, 577–598. https://doi.org/10.1007/s10208-015-9254-z

Surya Giri
Department of Applied Mathematics
Delhi Technological University
New Delhi 110042, India
Email address: suryagiri456@gmail.com

S. Sivaprasad Kumar
Department of Applied Mathematics
Delhi Technological University
New Delhi 110042, India
Email address: spkumar@dce.ac.in