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C^n-PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF CLASS r IN THE α-NORM UNDER THE LIGHT OF MEASURE THEORY

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Abstract. In this paper we present many interesting results such as completeness and composition theorems in the α norm. Moreover, under some conditions, we establish the existence and uniqueness of $C^n-(\mu,\nu)$ pseudo-almost automorphic solutions of class r in the α-norm for some partial functional differential equations in Banach space when the delay is distributed. An example is given to illustrate our results.

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1. Introduction

The aim of this work, is to study the existence and uniqueness of C^n- weighted pseudo almost automorphic functions in the α -norm for the following partial functional differential equation

$$u'(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1}$$

where $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators on Banach space X,
$C_\alpha = ([-r,0], D(A^\alpha))$, $0 < \alpha < 1$, denotes the pace of continuous functions from $[-r,0]$ to $D(A^\alpha)$, A^α is fractional α-power of A. This operator will be describe later and

$$\|\varphi\|_{C_\alpha} = \|A^\alpha\varphi\|_C([-r,0], X).$$

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For $t \geq 0$, $u \in C([-r,a], D(A^{\alpha}))$, $r > 0$ and u_t denotes history function of C_{α} defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \theta \leq 0,$$

L is a bounded linear operator from C_{α} into X and $f : \mathbb{R} \rightarrow X$ is a continuous function.

Some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1) have been made.

In [10], the authors investigated to the existence of C^n-almost periodic solutions and C^n-almost automorphic solutions for partial neutral functional differential equations solutions ($n \geq 1$). They established that the existence of a bounded integral solution on $\mathbb{R}_+$ implies the existence of C^n-almost periodic solutions and C^n-almost automorphic solutions. They also proved the uniqueness of strict solutions of C^n-almost automorphic when the exponential dichotomy holds for the homogeneous linear equation.

Recently in [5], the authors presented a new approach to study weighted pseudo-almost periodic functions using measure theory. They introduce a new concept of weighted periodic functions which is more general than the classical one. They established many interesting results on the functional space of such functions, such as completeness and composition theorems. The theory of their work generalizes the classical results on weighted pseudo-almost automorphic functions. More recently, in [12], Miailou Napo et al. established the existence and uniqueness of the solution of class r for some neutral partial functional differential equations in Banach spaces when the delay is distributed.

The aim of this work is to prove the existence and uniqueness of C^n-(μ,ν) pseudo-almost atomorphic solutions of the equation of class r for the equation (1) in the α-norm when the delay is distributed. Our present approach is based on the variation constants formula and the spectral decomposition of the phase space developed by Adimy and co-authors [2] and a new approach developed in [4].

This work is organized as follows, in Section 2 we recall some preliminary results about analytic semigroups and fractional power associated to its generator will be used throughout this work. In Section 3, we recall some preliminary results on spectral decomposition. In Section 4, we recall some preliminary results on C^n-(μ,ν)-pseudo-almost automorphic functions and partial functional differential equations which will be used in this work. In Section 5, we give some properties of C^n-(μ,ν)-pseudo-almost automorphic functions of class r. In Section 6, we discuss the main result of this paper. Using the strict contraction principle, we show the existence and uniqueness of C^n-(μ,ν)-pseudo-almost automorphic solutions of class r in the α-norm of the equation (1).The last section is devoted to an application for some model arising in the population dynamics.

2. Analytic Semigroup

Let $(X, \|.\|)$ be a Banach space, let α be a constant such that $0 < \alpha < 1$ and let $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A-\sigma I)$ with σ large enough such that $0 \in \rho(A-\sigma I)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α. For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. [14] *Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have:*
i) $T(t) : X \to D(A^\alpha)$ for every $t > 0$,
ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$,
iii) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \text{ for } t > 0,$$

iv) if $0 < \alpha \leq \beta < 1$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$,
v) there exists $N_\alpha > 0$ such that

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \text{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$. Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

3. Spectral decomposition

To equation (1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \geq 0, \\ \\ u_0 = \varphi \in C_\alpha, \end{cases} \tag{2}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.
For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_α by

$$\mathcal{U}(t) = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0, \\ \\ v_0 = \varphi \in C_\alpha. \end{cases}$$

Proposition 3.1. [1] *Let $\mathcal{A}_\mathcal{U}$ defined on C_α by*

$$\begin{cases} D(\mathcal{A}_\mathcal{U}) = \{\varphi \in C_\alpha, \varphi' \in C_\alpha, \varphi(0) \in (D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = -A\varphi(0) + L(\varphi)\} \\ \\ \mathcal{A}_\mathcal{U}\varphi = \varphi' \in D(\mathcal{A}_\mathcal{U}). \end{cases}$$

Then $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_t \geq 0$ on C_α .
Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 \text{ if } \theta \in [-r, 0[\\ \\ c \text{ if } \theta = 0. \end{cases}$$

Consider the extension $\mathcal{A}_\mathcal{U}$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_\mathcal{U}}) = \left\{\varphi \in C^1([-r, 0], X_\alpha) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\right\} \\ \\ \widetilde{\mathcal{A}}_\mathcal{U}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

We make the following assertion:
($\mathbf{H}_0$) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X and satisfies $0 \in \rho(A)$.

Lemma 3.2. [3] *Assume that ($\boldsymbol{H}_0$) holds. Then, $\mathcal{A}_\mathcal{U}$ satisfies the Hile-Yosida condition on $C_\alpha \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_\mathcal{U}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to equation (2).

Theorem 3.3. [1] *Assume that ($\boldsymbol{H}_0$) holds. Then for all $\varphi \in C_\alpha$, the solution u of equation (2) is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda\to+\infty}\int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}_\mathcal{U}})^{-1}$ for $\lambda > \widetilde{\omega}$.

Definition 3.4. We say a semigroup, $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_\mathcal{U}) \cap i\mathbb{R} = \varnothing.$$

For the sequel, we make the following assumption:
$(\mathbf{H}_1)$ $(T(t))$ is compact on $\overline{D(A)}$ for $t > 0$.
We get the following result on the spectral decomposition of the phase space C_α.

Proposition 3.5. *Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. If the semigroup $\mathcal{U}(t)_{t\geq 0}$ is hyperbolic,then the space C_α is decomposed as a direct sum*

$$C_\alpha = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi|_{C_\alpha} \leq \overline{M}e^{-\omega t}|\varphi|_{C_\alpha} \ \textit{for}\, t \geq 0 \ \textit{and} \ \varphi \in S,$$
$$|\mathcal{U}(t)\varphi|_{C_\alpha} \leq \overline{M}e^{-\omega t}|\varphi|_{C_\alpha} \ \textit{for} \ t \leq 0 \ \textit{and} \ \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

4. (μ,ν)-Pseudo almost automorphic functions and (μ,ν)-ergodic functions

In this section, we recall some properties about pseudo almost automorphic functions. Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to X equipped with the uniform topology norm.

Definition 4.1. A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to+\infty} \phi(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} g(t - s_n) = \phi(t),$$

for each $t \in \mathbb{R}$.
We denote by $AA(\mathbb{R}, X)$, the space of all such functions.

Proposition 4.2. [13] *$AA(\mathbb{R}, X)$ equipped with the sup norm is a Banach space.*

Definition 4.3. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called almost automorphic in $t \in \mathbb{R}$ uniformly for each x in X_1 if for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t,x) = \lim_{n\to+\infty} \phi(t+s_n,x) \ in \ \ X_2,$$

is well defined for each $t \in \mathbb{R}$ and each $x \in X_1$ and

$$\lim_{n\to+\infty} g(t-s_n,x) = \phi(t,x) \ in \ \ X_2$$

for each $t \in \mathbb{R}$ and for every $x \in X_1$. Denote by $AA(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

Let $C^n(\mathbb{R}, X_\alpha)$ be a space of all continuous function $h : \mathbb{R} \to X$ which have a continuous n-th derivative on $\mathbb{R}$ and let $C_b^n(\mathbb{R}, X_\alpha)$ be the subspace of $C^n(\mathbb{R}, X_\alpha)$ which consists of all function $h : \mathbb{R} \to X_\alpha$ satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)|_\alpha < \infty,$$

where $h^{(i)}$ denotes the i-th derivative of h. Then $C_b^n(\mathbb{R}, X)$ is a Banach space provided with the norm

$$|h|_{n,\alpha} = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)|_\alpha.$$

Now, we state a new concept of the C^n-almost automorphy, which generalizes the one of the C^n- almost periodicity.

Definition 4.4. A continuous function $h : \mathbb{R} \to X_\alpha$ is said to be C^n-almost automorphic for $n \geq 1$, if for $i = 1, 2, \ldots, n$ the i-th derivative $h^{(i)}$ of h is almost automorphic. We denote by $AA^{(n)}(\mathbb{R}, X_\alpha)$ the space of C^n-almost automorphic functions.

Definition 4.5. A continuous function $h : \mathbb{R} \to X$ is said to be C^n-compact almost automorphic for $n \geq 1$ if for $i = 1, 2, \ldots, n$ the i-th derivative $h^{(i)}$ of h is Compact almost automorphic. We denote by $AA_c^{(n)}(\mathbb{R}, X_\alpha)$ the space of C^n-almost automorphic functions.

By [8], since $AA(\mathbb{R}, X_\alpha)$ and $AA_c(\mathbb{R}, X_\alpha)$ are Banach space, we get also the following result.

Proposition 4.6. *([10] The spaces $AA^{(n)}(\mathbb{R}, X_\alpha)$ and $AA_c^{(n)}(\mathbb{R}, X_\alpha)$ equipped the norm $|.|_{n,\alpha}$ are Banach spaces.*

In the sequel, we use some preliminary results concerning the (μ,ν)-pseudo almost automorphic functions. The symbol $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu)$ stands for the space of functions. We have

$$\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu) = \Big\{ u \in BC(\mathbb{R}; X_\alpha) : \lim_{\tau\to+\infty} \frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{+\tau} |u(t)|_\alpha d\mu(t) = 0 \Big\}.$$

To study distributed delay differential equations for which the history belongs to $C([-r,0];X_\alpha)$, we need to introduce the space

$$\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,r)=\Big\{u\in BC(\mathbb{R};X_\alpha):\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(t)|_\alpha\Big)d\mu(t)=0\Big\}.$$

In addition to above-mentioned spaces, we consider the following spaces

$$\mathscr{E}(\mathbb{R}\times X_\alpha,\mu,\nu)=\Big\{u\in BC(\mathbb{R}\times X_\alpha;X_\alpha):\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}|u(t,x)|_\alpha d\mu(t)=0\Big\},$$

$$\mathscr{E}(\mathbb{R}\times X_\alpha,\mu,\nu,r)=\Big\{u\in BC(\mathbb{R}\times X_\alpha;X_\alpha):\lim_{\tau\to+\infty}\frac{1}{\nu[-\tau,\tau]}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(t,x)|_\alpha\Big)d\mu(t)=0\Big\},$$

where in both cases the limit(as $\tau\to+\infty$)is uniform in compact subset of X_α. In view of previous definitions, it is clear that the space $\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,r)$ is continuously embedded in $\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$. On the other hand, one can observe that a ρ-weighted pseudo almost automorphic functions is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t)=\rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu[-\tau,\tau]=2\tau$ for all $\tau\geq 0$.

Example 4.7. [5] Let ρ be a nonnegative B-measurable function. Denote by μ the positive measure defined by

$$\mu(A)=\int_A\rho(t)dt \ \text{ for } \ A\in\mathscr{B}, \tag{3}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 4.8. A function $h\in C_b^n(\mathbb{R},X_\alpha)$ is said to be C^n-(μ,ν)-ergodic if $h^{(i)}\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$ for $i=1,2,\dots,n$. We denote by $\mathscr{E}^{(n)}(\mathbb{R};X_\alpha,\mu,\nu,r)$ the space of C^n-(μ,ν)-ergodic functions.

Definition 4.9. A function $h\in C_b^n(\mathbb{R},X_\alpha)$ is said to be C^n-(μ,ν)-ergodic of class r if $h^{(i)}\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu,r)$ for $i=1,2,\dots,n$.
We denote by $\mathscr{E}^{(n)}(\mathbb{R};X_\alpha,\mu,\nu,r)$ the space of C^n-(μ,ν)-ergodic functions.

From μ, $\nu\in\mathcal{M}$, we formulate the following hypotheses.

($\mathbf{H}_2$) Let μ,$\nu\in\mathcal{M}$ be such that

$$\limsup_{\tau\to+\infty}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\delta<\infty.$$

We have the following result.

Lemma 4.10. *Assume* $(\boldsymbol{H}_2)$ *holds and let* $f \in C_b^n(\mathbb{R}; X_\alpha)$.
Then $f \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$ *if and only if for any* $\varepsilon > 0$ *and for* $i = 1, 2, \dots, n$ *the following equation holds*

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu[-\tau, \tau]} = 0,$$

where

$$M_{\tau,\varepsilon}(f^{(i)}) = \Big\{ t \in [-\tau, \tau] : |f^{(i)}(t)|_\alpha \geq \varepsilon \Big\}.$$

Proof. Suppose $f \in \mathscr{E}^{(i)}(\mathbb{R}; X_\alpha, \mu, \nu)$. Then, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} |f^{(i)}(t)|_\alpha d\mu(t) &= \frac{1}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|_\alpha d\mu(t) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|_\alpha d\mu(t) \\
&\geq \frac{1}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|_\alpha d\mu(t) \\
&\geq \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f^{(i)})} d\mu(t) \\
&\geq \frac{\varepsilon \mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau, \tau])}.
\end{aligned}$$

Consequently

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau, \tau])} = 0.$$

Suppose $f \in C_b^n(\mathbb{R}; X_\alpha)$ such that for any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu([\mathcal{M}_{\tau,\varepsilon}(f^{(i)}])}{\nu[-\tau, \tau])} = 0.$$

We can assume $|f(t)|_\alpha \leq N$ for all $t \in \mathbb{R}$. Using $(\mathbf{H}_2)$, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} |f^{(i)}(t)|_\alpha d\mu(t) &= \frac{1}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|_\alpha d\mu(t) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f^{(i)})} |f^{(i)}(t)|_\alpha d\mu(t) \\
&\leq \frac{N}{\nu([-\tau, \tau])} \int_{\mathcal{M}_{\tau,\varepsilon}(f^{(i)})} d\mu(t) \\
&\quad + \frac{\varepsilon}{\nu([-\tau, \tau])} \int_{[-\tau,\tau] \setminus \mathcal{M}_{\tau,\varepsilon}(f^{(i)})} d\mu(t) \\
&\leq \frac{N \mu(\mathcal{M}_{\tau,\varepsilon})(f^{(i)})}{\nu([-\tau, \tau])} + \frac{\varepsilon \mu([-\tau, \tau]}{\nu([-\tau, \tau])}.
\end{aligned}$$

Which implies that

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} |f^{(i)}(t)|_\alpha d\mu(t) \leq \delta \varepsilon \text{ for any } \varepsilon > 0,$$

Therefore $f^{(i)} \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$ for $i = 1, 2, \dots, n$ and
hence $f \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$. □

Lemma 4.11. *Assume that* $(\boldsymbol{H}_2)$ *holds. The space* $\mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ *endowed with the uniform topology norm is a Banach space.*

Definition 4.12. A bounded continuous function $\phi \in C_b^n(\mathbb{R}, X_\alpha)$ is called C^n-(μ,ν)-pseudo-almost automorphic(respectively C^n-(μ,ν)-pseudo compact almost automorphic) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$ (respectively $\phi_1 \in AA_c^{(n)}(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$). We denote by $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$ (respectively $PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$) the space of all such functions.

Definition 4.13. A bounded continuous function $\phi \in C_b^n(\mathbb{R}, X_\alpha)$ is called C^n-(μ,ν)-pseudo-almost automorphic of class r (respectively C^n- (μ,ν)-pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively $\phi_1 \in AA_c^{(n)}(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$). We denote by $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$) the space of all such functions.

Definition 4.14. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R} \times X_\alpha, X_\alpha)$ is called uniformly C^n-(μ,ν)-pseudo almost automorphic(respectively uniformly C^n- (μ,ν)-pseudo compact almost automorphic) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu)$ (respectively $\phi_1 \in AA_c^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$). We denote by $PAA^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$ (respectively $PAA_c^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$) the space of all such functions.

Definition 4.15. Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R} \times X_\alpha, X_\alpha)$ is called uniformly C^n-(μ,ν)-pseudo almost automorphic of class r (respectively uniformly C^n-(μ,ν)-pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively $\phi_1 \in AA_c^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$). We denote by $PAA^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c^{(n)}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$) the space of all such functions.

5. Properties of C^n-(μ,ν)-Pseudo almost automorphic functions of class r

Lemma 5.1. *Let* $\mu, \nu \in \mathcal{M}$. *The space* $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ *endowed with the norm* $|.|_{n,\alpha}$.

Proof. Let (x_m) be a sequence in $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ such that the limit $x = \lim_{m \longrightarrow \infty} x_m$ belongs to $BC^n(\mathbb{R}X_\alpha)$. For each m, let $x_m = y_m + z_m$ where $y_m \in AA^{(n)}(\mathbb{R}X_\alpha)$ and $z_m \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. Since $y_m \in AA^{(n)}(\mathbb{R}X_\alpha)$, we have by Definition 4.4, $y_m^{(i)} \in AA(\mathbb{R}X_\alpha)$. Consequently, $y \in AA^{(n)}(\mathbb{R}X_\alpha)$ by Definition 4.4. Since $z_m \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, Definition 4.8 implies that $z_m^{(i)} \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ and $(z^{(i)})_m$ converge to some $z \in BC(\mathbb{R}; X_\alpha)$. We have also

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z^{(i)}(t)|_\alpha\Big)d\mu(t)\\
&\qquad\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(t)-z^{(i)}(t)|_\alpha\Big)d\mu(t)\\
&\qquad\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(t)|_\alpha\Big)d\mu(t)\\
&\qquad\le \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_m^{(i)}(t)-z^{(i)}(t)|_\alpha\Big)d\mu(t)\\
&\qquad\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(t)|_\alpha\Big)d\mu(t)\\
&\qquad\le \|z_n^{(i)}-z^{(i)}\|_{\infty,\alpha}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\\
&\qquad\quad+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(t)|_\alpha\Big)d\mu(t).
\end{aligned}$$

Where

$$\|z_m^{(i)}-z^{(i)}\|_{\infty,\alpha}=\sup_{t\in\mathbb{R}}|z_n^{(i)}(t)-z^{(i)}(t)|_\alpha$$

Then we get $z^{(i)} \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ for $i = 1, 2, \ldots, n$, so $z \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$. It follows that $x \in PAA^{(n)}(\mathbb{R}, X_\alpha, \mu, \nu, r)$. □

Next result is a characterization of C^n-(μ,ν)-ergodic functions of class r.

Theorem 5.2. *Let $\mu, \nu \in \mathcal{M}$such that $(\boldsymbol{H}_2)$ holds and let $\mu, \nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \varnothing$). Then for every $f \in C_b^n(\mathbb{R}; X_\alpha)$, the following assertions are equivalent*

i) $f \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$,

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|_\alpha\Big)d\mu(t)=0$ *for* $i=1,2,\ldots,n$,

iii) for any $\varepsilon>0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big(\{t\in[-\tau,\tau]\setminus I : |f^{(i)}(\theta)|_\alpha>\varepsilon\}\Big)}{\nu([-\tau,\tau]\setminus I)}=0.$
for $i = 1, 2, \ldots, n$.

Proof. *i)* $\Leftrightarrow$ *ii)*. Denote By $A = \mu(I)$, $B = \displaystyle\int_I\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|_\alpha\Big)d\mu(t)$.

We have $A,\ B \in \mathbb{R}$.

Since the interval I is bounded and the function $f^{(i)}$ is bounded continuous. For $\tau > 0$, such that $I \subset [-\tau, \tau]$ and $\nu([-\tau, \tau] \setminus I) > 0$, we have

$$\frac{1}{\nu([-\tau,\tau]) \setminus I} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta|_\alpha \Big)$$

$$\begin{aligned} &= \frac{1}{\nu[-\tau,\tau]-A} \Big[\int_{[-\tau,\tau]} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) - B \Big] \\ &= \frac{\nu([-\tau,\tau])}{\nu[-\tau,\tau]-A} \Big[\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) - \frac{B}{\nu([-\tau,\tau])} \Big]. \end{aligned}$$

From above equalities and the fact $\nu(\mathbb{R}) = +\infty$, we deduce $ii)$ is equivalent to

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) = 0, \ i = 1, 2, \ldots, n,$$

that $i)$. $iii) \Rightarrow ii)$ Denote by A^ε_τ and B^ε_τ the following sets

$A^\varepsilon_\tau = \Big\{ t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha > \varepsilon \Big\}$
and
$B^\varepsilon_\tau = \Big\{ t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \le \varepsilon \Big\}$.
Assume that $ii)$ holds, that is

$$\lim_{\tau\to+\infty} \frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]) \setminus I)} = 0. \tag{4}$$

From the equality

$$\begin{aligned} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) \quad &= \quad \int_{A^\varepsilon_\tau} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) \\ &+ \int_{B^\varepsilon_\tau} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t), \end{aligned}$$

we deduce that for τ sufficient large

$$\begin{aligned} \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) \quad &\le \quad \|f\|_{\infty,\alpha} \times \frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau] \setminus I)} \\ &+ \varepsilon \frac{\mu(B^\varepsilon_\tau)}{\nu([-\tau,\tau] \setminus I)}, \end{aligned}$$

for τ sufficiently large, where $\|f\|_{\infty,\alpha} = \sup_{t\in\mathbb{R}} |f^{(i)}(t)|_\alpha$.

Since $\mu(\mathbb{R}) = \nu(\mathbb{R}) = \infty$ and by using $(\mathbf{H}_2)$, we have

$$\frac{1}{\nu([-\tau,\tau] \setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha \Big) d\mu(t) \le \delta\varepsilon,$$

then for any $\varepsilon > 0$ and $i = 1, 2, \ldots, n$. Consequently $ii)$ holds.
$ii) \Rightarrow iii)$

$$\int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha\Big) d\mu(t) \geq \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha\Big) d\mu(t)$$

$$\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha\Big) d\mu(t) \geq \varepsilon \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}$$

$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} |f^{(i)}(\theta)|_\alpha\Big) d\mu(t) \geq \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},$$

we obtain equation (4), that is $iii)$. $\square$

In what follows, we prove some preliminary results concerning the composition of C^n-(μ,ν)-pseudo almost automorphic functions of class r.

Theorem 5.3. *Let $\mu,\ \nu \in \mathcal{M}$and $\phi \in PAA^{(n)}(\mathbb{R}\times X_\alpha, \mu, \nu, r)$ and $h \in PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. Assume that there exists a function $L_\phi : \mathbb{R} \to [0,\infty[$ such that*

$$|\phi(t,x_1) - \phi(t,x_2)|_\alpha \leq L_\phi |x_1 - x_2|_\alpha, \ \textit{for}\ t\in\mathbb{R}\ \textit{and for}\ x_1, x_2 \in X_\alpha. \quad (5)$$

If

$$\limsup_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} L_\phi(\theta)\Big) d\mu(t) < \infty \ \textit{and for each}\ \ \xi \in \mathscr{E}(\mathbb{R},\mathbb{R},\mu,\nu)$$

we have

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} L_\phi(\theta)\Big) \xi(t) d\mu(t) = 0, \quad (6)$$

then the function $t \to \phi(t, h(t))$ belongs to $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}\times X_\alpha, X_\alpha)$; $\phi_2 \in \mathscr{E}^{(n)}(\mathbb{R}\times X_\alpha; \mu, \nu, r)$ and $h_1 \in AA^{(n)}(\mathbb{R}; X_\alpha)$, $h_2 \in \mathscr{E}^{(n)}(\mathbb{R}, X_\alpha, \mu, \nu, r)$. Then $\phi_1^{(i)} \in AA(\mathbb{R}\times X_\alpha, X_\alpha)$; $\phi_2^{(i)} \in \mathscr{E}(\mathbb{R}\times X_\alpha; \mu, \nu, r)$ and $h_1^{(i)} \in AA(\mathbb{R}; X_\alpha)$, $h_2^{(i)} \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ for $i = 1, 2, \ldots, n$. Consider the following decomposition

$$\phi^{(i)}(t, h^{(i)}(t)) = \phi_1^{(i)}(t, h_1^{(i)}(t)) + \Big[\phi^{(i)}(t, h^{(i)}(t)) - \phi^{(i)}(t, h_1^{(i)}(t))\Big] + \phi_2^{(i)}(t, h_1^{(i)}(t)).$$

From [4], we know the $\phi_1^{(i)}(., h_1^{(i)}(.)) \in AA(\mathbb{R}, X_\alpha)$ for $i = 1, 2, \ldots, n$. Its remains to prove that both $\phi^{(i)}(., h^{(.)}(t)) - \phi^{(i)}(., h_1^{(i)}(.))$ and $\phi_2^{(i)}(., h_1^{(.)}(t))$ belong to $\mathscr{E}(\mathbb{R}, X_\alpha)$. By using equation (5.3), we have

$$\frac{\mu\Big(\Big\{t\in[-\tau,\tau] : \sup_{\theta\in[t-r,t]} |\phi^{(i)}(\theta, h^{(i)}(\theta)) - \phi^{(i)}(\theta, h_1^{(i)}(\theta))|_\alpha > \varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}$$

$$\leq \frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\sup_{\theta\in[t-r,t]}(L_\phi(\theta)|h_2^{(i)}(\theta)|_\alpha)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}$$

$$\leq \frac{\mu\Big(\Big\{t\in[-\tau,\tau]:(\sup_{\theta\in[t-r,t]}L_\phi(\theta))(\sup_{\theta\in[t-r,t]}|h_2^{(i)}(\theta)|_\alpha)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}.$$

Since $h_2^{(i)}$ is (μ,ν)-ergodic of class r, Theorem 5.2 and equation (5.3) yield that for above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:(\sup_{\theta\in[t-r,t]}L_\phi(\theta))(\sup_{\theta\in[t-r,t]}|h_2^{(i)}(\theta)|_\alpha)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0,$$

and then, we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]:\sup_{\theta\in[t-r,t]}|\phi^{(i)}(\theta,h^{(i)}(\theta))-\phi(\theta,h_1^{(i)}(\theta))|_\alpha>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0. \tag{7}$$

By Theorem 5.2 and equation (7) it follows that for $i=1,2,\ldots,n$, $t\mapsto\phi^{(i)}(t,h^{(i)}(t))-\phi^{(i)}(t,h_1^{(i)}(t))$ is (μ,ν)-ergodic of class r.
Now to complete the proof is enough to prove that $t\to\phi_2^{(i)}(t,h^{(i)}(t))$ is (μ,ν)-ergodic class of r.
Since $\phi_2^{(i)}$ is uniformly continuous on the compact set $K_i=\overline{\{h_1^{(i)}(t),t\in\mathbb{R}\}}$ with the respect of second variable x, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that for all $t\in\mathbb{R}$, $\xi_1^{(i)}$ and $\xi_2^{(i)}\in K_i$, one has

$$|\xi_1^{(i)}-\xi_2^{(i)}|\leq\delta\Rightarrow|\phi_2^{(i)}(t,\xi_1^{(i)})-\phi^{(i)}(\xi_2^{(i)})|_\alpha\leq\varepsilon.$$

Therefore there exists $m(\varepsilon)$ and $\{z_k^{(i)}\}_{k=1}^{m(\varepsilon)}\subset K$ such that

$$K_i\subset\bigcup_{k=1}^{m(\varepsilon)}B_\delta(z_k^{(i)},\delta)$$

and then

$$|\phi_2^{(i)}(t,h_1^{(i)}(t))|_\alpha\leq\varepsilon+\sum_{i=1}^{m(\varepsilon)}|\phi_2^{(i)}(t,z_i)|_\alpha.$$

Since

$$\forall k\in\Big\{1,\ldots,m(\varepsilon)\Big\},\ \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,z_k^{(i)})|_\alpha\Big)d\mu(t)=0,$$

we deduce that

$$\forall\varepsilon>0,\ \limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(t,h_1^{(i)}(t))|_\alpha\Big)d\mu(t)\leq\varepsilon\delta,$$

that implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(t,h_1^{(i)}(t))|_\alpha\Big)d\mu(t)=0.$$

Consequently for $i=1,2,\dots,n$, the function $t\to\phi_2^{(i)}(t,h_1^{(i)}(t))$ is (μ,ν)-ergodic class of r. By use of Definition 4.4 and Definition 4.8, it follows that the function $t\mapsto\phi(t,h(t))$ belongs to $PAA^{(n)}(\mathbb{R},X_\alpha,\mu,\nu,r)$. $\square$

For $\mu\in\mathcal{M}$ and $\delta\in\mathbb{R}$, we denote μ_δ the positive measure on $(\mathbb{R},\mathscr{B})$ defined by

$$\mu_\delta(A)=\mu([a+\delta:a\in A]). \tag{8}$$

($\mathbf{H}_3$) For all a, b and $c\in\mathbb{R}$ such that $0\le a<b<c$, there exist δ_0 and $\alpha_0>0$ such that

$$|\delta|\ge\delta_0\Rightarrow\mu(a+\delta,b+\delta)\ge\alpha_0\mu(\delta,c+\delta).$$

For all $\tau\in\mathbb{R}$ there exist $\beta>0$ and a bounded interval Isuch that

$$\mu\Big(\{a+\tau:a\in A\}\big)\le\beta\mu(A)\text{ when }A\in\mathscr{B}\text{ and satisfies }A\cap I=\varnothing.$$

We have the following result due to [5].

Lemma 5.4. [5] *Hypothesis ($\boldsymbol{H}_4$) implies ($\boldsymbol{H}_3$).*

Proposition 5.5 (6,9). *$\mu,\nu\in\mathcal{M}$ satisfy ($\boldsymbol{H}_3$) and $f\in PAA(\mathbb{R},X;\mu,\nu)$ be such that*

$$f=g+h,$$

where $g\in AA(\mathbb{R};X)$ and $h\in\mathscr{E}(\mathbb{R},X;\mu,\nu)$. Then

$$\{g(t),t\in\mathbb{R}\}\subset\overline{\{f(t),t\in\mathbb{R}\}}\textit{(the closure of the range of f).}$$

Corollary 5.6. [6] *Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of a (μ,ν)-pseudo almost automorphic function in the form $f=g+\phi$ where $g\in AA(\mathbb{R};X_\alpha)$ and $\phi\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$, is unique.*

The following corollary is a consequence of Theorem 5.2.

Proposition 5.7. *Let $\mu,\nu\in\mathcal{M}$. Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of a (μ,ν)-pseudo-almost automorphic function in the form $\phi=\phi_1+\phi_2$, where $\phi_1\in AA^{(n)}(\mathbb{R};X_\alpha)$ and $\phi_2\in\mathscr{E}^{(n)}(\mathbb{R};X_\alpha,\mu,\nu)$, is unique.*

Proof. Let $\phi=\phi_1+\phi_2$, where $\phi_1\in AA^{(n)}(\mathbb{R};X_\alpha)$ and $\phi_2\in\mathscr{E}^{(n)}(\mathbb{R};X_\alpha,\mu,\nu)$. Then $\phi_1^{(i)}\in AA(\mathbb{R};X_\alpha)$ and $\phi_2^{(i)}\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$ for $i=1,2,\dots,n$. As consequence Corollary 5.6, the decomposition of a $\alpha-(\mu,\nu)$-pseudo-almost automorphic function $\phi^{(i)}=\phi_1^{(i)}+\phi_2^{(i)}$, where $\phi_1^{(i)}\in AA(\mathbb{R};X_\alpha)$ and $\phi_2^{(i)}\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$ is unique. Since $PAA(\mathbb{R};X_\alpha,\mu,\nu)\subset PAA(\mathbb{R};X,\mu,\nu)$ and $PAA(\mathbb{R};X_\alpha,\mu,\nu,r)\subset PAA(\mathbb{R};X_\alpha,\mu,\nu,r)$. The decomposition of a (μ,ν)-pseudo-almost automorphic function $\phi^{(i)}=\phi_1^{(i)}+\phi_2^{(i)}$ of class r, where $\phi_1^{(n)}\in AA(\mathbb{R};X_\alpha)$ and $\phi_2^{(i)}\in\mathscr{E}(\mathbb{R};X_\alpha,\mu,\nu)$ is unique. Consequently we get the desired result. $\square$

Definition 5.8. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2 , denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha\mu_1(A) \leq \mu_2\beta\mu_1(A)$, when $A \in \mathscr{B}$ satisfies $A \cap I = \varnothing$.

From [5]$\sim$ is binary equivalent relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathbb{R}$ will then be denoted by

$$cl(\mu) = \big\{\overline{\omega} \in \mathcal{M} : \mu \sim \overline{\omega}\big\}.$$

Theorem 5.9. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1\nu_2$, then $PAA(\mathbb{R}; X_\alpha\mu_1, \nu_1, r) = PAA(\mathbb{R}; X_\alpha, \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist constants α_1, α_2, β_1, $\beta_2 > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha_1(A) \leq \mu_2 \leq \beta_1\mu(A)$ and $\alpha_2\nu_1(A) \leq \nu_2 \leq \beta_2\nu(A)$ for each $A \in \mathscr{B}$ satisfies $A \cap I = \varnothing$ i.e

$$\frac{1}{\beta_1\nu(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu(A)}.$$

Let $f \in C_b^n(\mathbb{R}, X_\alpha)$. Since $\mu_1 \sim \mu_2$ and β is the Lebesgue σ-field, we obtain for τ sufficiently Large that

$$\begin{aligned}
&\frac{\alpha_1\mu_1\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} |f^{(i)}(\theta)|_\alpha > \varepsilon\}\Big)}{\beta_2\mu_2([-\tau, \tau] \setminus I)} \\
&\leq \frac{\mu_2\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} |f^{(i)}(\theta)|_\alpha > \varepsilon\}\Big)}{\nu_2([-\tau, \tau] \setminus I)} \\
&\leq \frac{\beta_1\mu_1\Big(\{t \in [-\tau, \tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} |f^{(i)}(\theta)|_\alpha > \varepsilon\}\Big)}{\alpha_2\nu([-\tau, \tau] \setminus I)}.
\end{aligned}$$

By using Theorem 5.2, we deduce that $\mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu_1, \nu_1, r) = \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu_2, \nu_2, r)$. From the definition of a (μ, ν)-pseudo almost automorpic function, we deduce that $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu_1, \nu_1, r) = PAA^{(n)}(\mathbb{R}; X_\alpha, \mu_1, \nu_1, r)$. $\square$

Lemma 5.10. [4] *Let $\mu, \nu \in \mathcal{M}$ satisfies $(\boldsymbol{H}_4)$. Then $PAA(\mathbb{R}; X, \mu, \nu)$ is invariant by translation that is $f \in PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ implies $f_\gamma \in PAA(\mathbb{R}; X, \mu, \nu)$ for all $\gamma \in \mathbb{R}$.*

Corollary 5.11. [16] *Let $\mu, \nu \in \mathcal{M}$ satisfies $(\boldsymbol{H}_4)$. Then $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ is invariant by translation that is $f \in PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ implies $f_\gamma \in PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ for all $\gamma \in \mathbb{R}$.*

Corollary 5.12. *Let $\mu, \nu \in \mathcal{M}$ satisfies $(\boldsymbol{H}_4)$. Then $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ is invariant by translation that is $f \in PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ implies $f_\gamma \in PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ for all $\gamma \in \mathbb{R}$.*

Lemma 5.13. [4] *Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. Then the measures μ and μ_δ are equivalent for all $\delta \in \mathbb{R}$.*

Lemma 5.14. [5] *$(\boldsymbol{H}_4)$ implies*

$$\text{for all } \sigma > 0,\ \limsup_{\tau \to +\infty} \frac{\mu([-\tau - \sigma, \tau + \sigma])}{\nu([-\tau, \tau])} < \infty.$$

We have the following result

Theorem 5.15. *Assume that $(\boldsymbol{H}_4)$ holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, then the function $t \to \phi_t$ belongs to $PAA_c^{(n)}(C([-r, 0], X_\alpha); \mu, \nu, r)$.*

Proof. Assume that $\phi = g + h$, where $g \in AA^n(\mathbb{R}; X_\alpha)$ and $h \in \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu\nu, r)$. We can see that $u_t^{(i)} = g_t^{(i)} + h_t^{(i)}$. we want to show that $g_t \in AA(\mathbb{R}; X_\alpha)$ and $h_t \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ for $i = 1, 2, \ldots, n$, and by [9] the function $g_t^{(i)}$ belongs to $AA_c(C([-r, 0]; X_\alpha)$ which implies that $g_t \in AA_c^{(n)}(C([-r, 0]; X_\alpha)$. Let $i = 1, 2, \ldots, n$ $\delta \in \mathbb{R}$, μ_δ and ν_δ be the positive measures defined by equation (8). Let us denote

$$M_\delta(\tau) = \frac{1}{\nu_\delta([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu_\delta(t).$$

By using Lemma 5.13, it follows that μ_δ and μ are equivalent and ν_δ and ν are also equivalent. By using Theorem 5.9, we have $\mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu_\delta, \nu_\delta, r) = \mathscr{E}^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, therefore $h^{(i)} \in \mathscr{E}(\mathbb{R}; X_\alpha, \mu_\delta, \nu_\delta, r)$ that is

$$\lim_{\tau \to +\infty} M_\delta(\tau) = 0, \quad \text{for all } \delta \in \mathbb{R}.$$

On the other hand for $\tau > 0$, we have

$$\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} \Big(\sup_{\theta \in [-r, 0]} |h^{(i)}(\theta + \xi)|_\alpha \Big) d\mu(t)$$

$$\begin{aligned}
&\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-2r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu(t) \\
&\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-2r, t-r]} |h(\theta)|_\alpha + \sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \Big) d\mu(t) \\
&\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-2r, t-r]} |h(\theta)|_\alpha \Big) d\mu(t) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \Big) d\mu(t) \\
&\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{+\tau-r} \Big(\sup_{\theta \in [t-r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu(t) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu(t) \\
&\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{+\tau+} \Big(\sup_{\theta \in [t-r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu(t + r) \\
&\quad + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} |h^{(i)}(\theta)|_\alpha \Big) d\mu(t)
\end{aligned}$$

$$\leq \quad \frac{\mu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\Big(\frac{1}{\mu([-\tau-r,\tau+r])}\int_{-\tau-r}^{+\tau+}\Big(\sup_{\theta\in[t-r,t]}|h^{(i)}(\theta)|_\alpha\Big)d\mu(t+r)\Big)$$
$$+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}|h^{(i)}(\theta)|_\alpha\Big)d\mu(t).$$

Consequently

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\Big(\sup_{\theta\in[-r,0]}|h^{(i)}(\theta+\xi)|_\alpha\Big)d\mu(t)\\
&\leq \quad \frac{\mu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times M_\delta(\tau+r)\\
&\qquad +\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}|h^{(i)}(\theta)|_\alpha\Big)d\mu(t),
\end{aligned}$$

which shows usind Lemma 5.13 and Lemma 5.14 that $\phi_t^{(i)}$ belongs to $PAA_c(C([-r,0];X_\alpha),\mu,\nu,r)$. Thus we obtain the desired result. □

6. C^n-(μ,ν)-Pseudo almost automorphic class of r

Proposition 6.1. [1] *Assume that* $(\boldsymbol{H}_0)$ *and* $(\boldsymbol{H}_1)$ *hold and the semigroup* $(U(t))_{t\geq 0}$ *is hyperbolic. If f is bounded on* $\mathbb{R}$*, then there exists a unique bounded solution u of equation* (1) *on* $\mathbb{R}$*, given by*

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))ds,$$

where $\widetilde{B}_\lambda=\lambda(\lambda I-\mathcal{A}_{\mathcal{U}})^{-1}$ *for*$\lambda>\widetilde{\omega}$, Π^s *and* Π^u *are projections of* C_α *onto the stable and unstable subspace respectively.*

Proposition 6.2. [7] *If* $f\in BC(\mathbb{R},X_\alpha)$ *there exists a unique bounded solution u of equation* (1) *on* $\mathbb{R}$*, given by* $u_t=\Gamma f(t)$*.*

Proposition 6.3. [16] *If* $h\in AA_c(\mathbb{R},X)$ *then the function* $t\mapsto\Gamma h(t)(0)$ *belongs to* $AA_c(\mathbb{R},X_\alpha)$*.*

Corollary 6.4. *If* $h\in AA_c^{(n)}(\mathbb{R},X)$ *then the function* $t\mapsto\Gamma h(t)(0)$ *belongs to* $AA_c^{(n)}(\mathbb{R},X_\alpha)$*.*

Proof. In fact $h\in AA_c^{(n)}(\mathbb{R},X)$, we have $h^{(i)}\in AA_c(\mathbb{R},X)$ for $i=1,2,\ldots,n$. Thus the function $t\mapsto\Gamma f^{(i)}(t)(0)$ belongs to $AA_c(\mathbb{R}X_\alpha)$ for $i=1,2,\ldots,n$. □

Theorem 6.5. *Let* $\mu,\nu\in\mathcal{M}$, $(\boldsymbol{H}_3)$ *holds and* $g\in\mathscr{E}^{(n)}(\mathbb{R},X_\alpha,\mu,\nu,r)$*, then the function* $t\mapsto\Gamma h(t)(0)$ *belongs to* $\mathscr{E}^{(n)}(\mathbb{R},X_\alpha,\mu,\nu,r)$*.*

Proof. In fact, for $g \in \mathscr{E}^{(n)}(\mathbb{R}, X_\alpha, \mu, \nu, r)$, we have $g^{(i)} \in \mathscr{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$ for $i = 1, 2, \ldots, n$. For $\tau > 0$ we get

$$\begin{aligned}
&\int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} |\Gamma g^{(i)}(\theta)|_\alpha \Big) d\mu(t) \\
&\qquad \leq \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{\theta} |\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g^{(i)}(s)|_\alpha ds \\
&\qquad\quad + \lim_{\lambda\to+\infty} \int_{\theta}^{+\infty} |\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g^{(i)}(s)|_\alpha ds \Big](0) \Big) d\mu(t) \\
&\qquad \leq \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{\theta} \|\mathcal{A}_{\mathcal{U}}^{\alpha}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g^{(i)}(s)\| ds \\
&\qquad\quad + \lim_{\lambda\to+\infty} \int_{\theta}^{+\infty} \|\mathcal{U}_{\mathcal{U}}^{\alpha}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g^{(i)}(s)\| ds \Big](0) \Big) d\mu(t) \\
&\qquad \leq \overline{M}\widetilde{M} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha} |\Pi^s||g^{(i)}(s)| ds \Big) d\mu(t) \\
&\qquad\quad + \overline{M}\widetilde{M} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{\theta}^{+\infty} \frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha} |\Pi^u||g^{(i)}(s)| ds \Big) d\mu(t) \\
&\qquad \leq K \Big[\int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha} |g^{(i)}(s)| ds \Big) d\mu(t) \\
&\qquad\quad + \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{\theta}^{+\infty} \frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha} |g^{(i)}(s)| ds \Big) d\mu(t) \Big],
\end{aligned}$$

where $K = \max(\overline{M}\widetilde{M}|\Pi^s|, \overline{M}\widetilde{M}|\Pi^u|)$.

On the one hand using Fubini's Theorem, we have

$$\begin{aligned}
\int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(t-s)}}{(\theta-s)^\alpha} |g^{(i)}(s)| ds \Big) d\mu(t) &\leq \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{0}^{+\infty} \frac{e^{-\omega s}}{s^\alpha} |g^{(i)}(\theta-s)| ds \Big) d\mu(t) \\
&\leq \int_{0}^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \Big(\sup_{\theta\in[t-r,t]} \int_{-\tau}^{\tau} |g^{(i)}(\theta-s)| ds \Big) d\mu(t).
\end{aligned}$$

By the Lebesgue dominated convergence Theorem and by using Corollary 5.12, it follows that

$$\lim_{\tau\to+\infty} \int_{0}^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \frac{1}{\nu([-\tau,\tau])} \Big(\sup_{\theta\in[t-r,t]} \int_{-\tau}^{\tau} |g^{(i)}(\theta-s)| ds \Big) d\mu(t) = 0.$$

On the other hand by Fubini's theorem, we also have

$$\begin{aligned}
&\int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(s-\theta)^\alpha} |g^{(i)}(s)| ds \Big) d\mu(t) \\
&\qquad \leq \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \int_{0}^{+\infty} \frac{e^{-\omega s}}{s^\alpha} |g^{(i)}(s+\theta)| ds \Big) d\mu(t)
\end{aligned}$$

$$\leq \quad \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha}\Big(\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}|g^{(i)}(s+\theta)|ds\Big)d\mu(t).$$

Resoning like above, it follows that

$$\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-\omega s}}{s^\alpha}\frac{1}{\nu([-\tau,\tau])}\Big(\sup_{\theta\in[t-r,t]}\int_{-\tau}^{\tau}|g^{(i)}(s+\theta)|ds\Big)d\mu(t)=0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\Gamma g^{(i)}(\theta)|_\alpha)d\mu(t)=0,$$

which implies that for $i=1,2,\dots,n$ the function $\Gamma f^{(i)}(t)(0)$ belongs to $\mathscr{E}(\mathbb{R},X_\alpha,\mu,\nu,r)$. Thus, we obtain the desired result. □

We assume the following hypothesis in order to ensure the existence of the C^n-(μ,ν)-pseudo almost automorphic solution of the class of r.
$(\mathbf{H}_5)$ $f:\mathbb{R}\to X$ is compact $cl-(\mu,\nu)$-pseudo almost automorphic of class r.

Proposition 6.6. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_3)$ *and* $(\boldsymbol{H}_5)$ *hold. Then* (1) *has a unique compact* C^n-$cl(\mu,\nu)$*-pseudo almost automorphic solution of class* r.

Proof. Since f is a C^n-(μ,ν)-pseudo almost automorphic function, f has a decomposition $f=f_1+f_2$ where $f_1\in AA_c^{(n)}(\mathbb{R},X)$ and $f_2\in\mathscr{E}^{(n)}(\mathbb{R};X\mu,\nu)$. Using Proposition 6.2, Corollary 6.4 and Theorem 6.5, we get the desired result. □

Next, we want to prove that the following problem has C^n-(μ,ν)-pseudo almost automorphic solution of class r.

$$u'(t)=-Au(t)+L(u_t)+f(t,u_t)\text{ for }t\in\mathbb{R},\tag{9}$$

where $f:\mathbb{R}\times C_\alpha\to X$ is continuous.
$(\mathbf{H}_6)$ The unstable space $U\equiv\{0\}$.
$(\mathbf{H}_7)$ Let $\mu,\nu\in\mathcal{M}$ and $f:\mathbb{R}\times C_\alpha\to X$ $cl(\mu,\nu)$-pseudo almost automorphic class of r such that there exists a positive constant L_f such that

$$|f(t,\varphi_1)-f(t,\varphi_2)|\leq L_f\|\varphi_1-\varphi_2\|_{C_\alpha}\text{ for all }t\in\mathbb{R}$$

$\varphi_1,\varphi_2\in C_\alpha$ and L_f satisfies (5.3).

Theorem 6.7. *Assume* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$, $(\boldsymbol{H}_6)$ *and* $(\boldsymbol{H}_7)$ *holds. If*

$$\frac{\overline{M}\widetilde{M}|\Pi^s|L_f\Gamma(1-\alpha)}{\omega^{1-\alpha}}<1,$$

then equation (9) *has a unique* C^n- $cl(\mu,\nu)$*-pseudo compact almost automorphic solution of class* r.

Proof. Let x be a function in $PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, from Theorem 5.15, the function $t \to x_t$ belongs to $PAA_c^{(n)}(C_\alpha; \mu, \nu, r)$. Hence Theorem 5.3 implies that the function $g(.) := f(., x)$ is in $PAA^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u = 0$. Consider the mapping

$$\mathcal{H} : PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r) \to PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\tau \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s, x_s))ds\Big](0).$$

From the Proposition 6.1, Proposition 6.2 and taking into account Theorem 6.5, it suffices now to show that the the operator $\mathcal{H}$ has fixed point in $PAA^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

Let $x_1, x_2 \in PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, then $x_1^{(i)}, x_2^{(i)} \in PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ for $i = 1, 2, \dots, n$, we have

$$\begin{aligned}
|(\mathcal{H}x_1^{(i)})(t) - (\mathcal{H}x_2^{(i)})(t)|_\alpha &\leq \Big|\lim_{\tau \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}X_0(f(s, x_{1s}^{(i)}) - f(s, x_{2s}^{(i)})))ds\Big|_\alpha \\
&\leq \lim_{\tau \to +\infty} \int_{-\infty}^{t} |\mathcal{A}_{\mathcal{U}}^\alpha \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}X_0(f(s, x_{1s}^{(i)}) - f(s, x_{2s}^{(i)})))|ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s L_f|x_1^{(i)} - x_2^{(i)}|_\alpha \int_{-\infty}^{t} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} ds \\
&\leq \overline{M}\widetilde{M}|\Pi^s|L_f\Big(\int_0^{+\infty} e^{-\omega s}s^\alpha ds\Big)|x_1^{(i)} - x_2^{(i)}|_\alpha \\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|L_f\Gamma(1-\alpha)}{\omega^{1-\alpha}}|x_1^{(i)} - x_2^{(i)}|_\alpha,
\end{aligned}$$

which implies that

$$\sum_{i=0}^{n} |(\mathcal{H}x_1^{(i)})(t) - (\mathcal{H}x_2^{(i)})(t)|_\alpha \leq \frac{\overline{M}\widetilde{M}|\Pi^s|L_f\Gamma(1-\alpha)}{\omega^{1-\alpha}}|x_1 - x_2|_{\alpha,n}.$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed-point theorem, $\mathcal{H}$ has a unique fixed point u in $PAA_c^{(n)}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. We conclude that equation (9), has one and only one C^n-$cl(\mu, \nu)$-pseudo compact almost automorphic solution of class r. □

7. Application

For illustration, we propose to study the existence of solutions for the following model

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial t}z(t,x) = -\dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x)d\theta + x\exp\big(\sin(at)\sin(bt)\big) + \cos(t) & \\ \qquad\qquad + \ h(t, \dfrac{\partial}{\partial x}z(t+\theta,x)\Big) & \text{for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0 & \text{for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \end{array}\right. \tag{10}$$

where $a, b \in \mathbb{R}$, the function $G : [-r, 0] \to \mathbb{R}$ is continuous function and $h : \mathbb{R} \times \mathbb{R} \to$ is lipschitz continuous with the respect of the second argument. To rewrite (10) in abstract form, we introduce the space $X = L^2([0, \pi]; \mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^{\pi} |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $A : X \to X$ be defined by

$$\left\{\begin{array}{l} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \\ Ay = y''. \end{array}\right.$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 : \ n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n\geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns), \ s \in [0, \pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2(y, e_n)e_n, \ y \in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X : \ \sum_{n=1}^{+\infty} n(y, e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y, e_n)e_n, \ y \in D(A).$$

Lemma 7.1. [15] *If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and*

$$\|y\| = \|y'\| = \|A^{\frac{1}{2}}y\|.$$

It is well known that $-A$ is the generator of a compact analytic semigroup semigroup $(T(t))_{t\geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x, e_n)e_n,\ x \in X.$$

Then $(\mathbf{H}_0)$ and $(\mathbf{H}_1)$ are satisfies. here we choose $\alpha = \dfrac{1}{2}$.
We define $f : \mathbb{R} \times C_{\frac{1}{2}} \to X$ and $L : C_{\frac{1}{2}} \to X$ as follows:

$$f(t,\varphi)(x) = x\exp\big(\sin(at)\sin(bt)\big) + \cos(t) + h(t, \frac{\partial}{\partial x} z(t+\theta, x)\Big), \quad \text{for } t \in \mathbb{R},\ x \in [0,\pi]$$

and $L(\varphi)(x) = \displaystyle\int_{-r}^{0} G(\theta)\varphi(\theta, x)d\theta,$ for $-r \leq \theta \leq 0$.
Let us pose $v(t) = z(t,x)$. Then equation (10) takes the following abstract form

$$v'(t) = -Av(t) + L(v_t) + f(t, v_t) \text{ for } t \in \mathbb{R}. \tag{11}$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0 \\ e^t & \text{for } t \leq 0 \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\mu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \rho_2(t)dt \text{ for } A \in \mathscr{B}.$$

From [5] $\mu, \nu \in \mathcal{M}$ satisfies Hypothesis $(\mathbf{H}_4)$.

$$\lim_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-r}^{0} e^t dt + \int_0^{\tau} dt}{2\displaystyle\int_0^{\tau} t dt} = \limsup_{\tau\to+\infty} \frac{1 + e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H}_2)$ is satisfied.
Since $A^{\frac{1}{2}}\Big(x\exp(\sin(at)\sin(bt))\Big) = \Big(x\exp(\sin(at)\sin(bt))\Big)' = \exp(\sin(at)\sin(bt))$.
By Mophou et al.[11], $t \mapsto \exp(\sin(at)\sin(bt))$ belongs $AA_c^{(n)}(\mathbb{R}; X)$, if a and b are incommensurate real numbers (i.e a and b are relatively prime), it follows that $t \to x\exp(\sin(at)\sin(bt))$ belongs $AA_c^{(n)}(\mathbb{R}; X_{\frac{1}{2}})$.

On other hand, for all $t \in \mathbb{R}$ and $i = 0, 1, \ldots, n$, we have $|\cos^{(i+1)}(t)| \leq 1$, which implies

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta \int[t-r,t]}|\cos^{(i)}(\theta)|_{\frac{1}{2}}dt &= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}|A^{\frac{1}{2}}\cos^{(i)}(\theta)|dt \\
&= \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}|\cos^{(i+1)}(\theta)|dt \\
&\leq \frac{1}{\nu([-\tau,\tau])}\Big(\int_{-\tau}^{0}\sup_{\theta\in[t-r,t]}|\cos^{(i+1)}(\theta)|e^{t}dt \\
&\quad + \int_{0}^{\tau}\sup_{\theta\in[t-r,t]}|\cos^{(i+1)}(\theta)|dt\Big) \\
&\leq \frac{1}{\nu([-\tau,\tau])}\Big(\int_{-\tau}^{0}e^{t}dt + \int_{0}^{\tau}dt\Big) \\
&= \frac{1+e^{-\tau}+\tau}{\tau^2} \to 0 \text{ as } \tau \to \infty.
\end{aligned}$$

It follows that $t \to \cos^{(i)} t$ belongs to $\mathscr{E}^{(n)}(\mathbb{R}, X_{\frac{1}{2}}, \mu, \nu, r)$,
f belongs $PAA^{(n)}(\mathbb{R},, X_{\frac{1}{2}}, \mu, \nu, r)$. Moreover, L is a bounded linear operator from $C_{\frac{1}{2}}$ to X.

Let k be the lipschiz constant of h, then for every $\varphi_1, \varphi_2 \in C_{\frac{1}{2}}$ and $t \geq 0$, we have

$$\begin{aligned}
\|f(t,\phi_1)(x) - f(t,\phi_2)(x)\| &= \Big(\int_0^{\pi}|h(\theta,\frac{\partial}{\partial x}\varphi_1(\theta,x)) - h(\theta,\frac{\partial}{\partial x}\varphi_2(\theta,x))|^2 ds\Big)^{\frac{1}{2}} \\
&\leq L_h\Big(\int_0^{\pi}|\frac{\partial}{\partial x}\varphi_1(\theta,x)) - \frac{\partial}{\partial x}\varphi_2(\theta,x))|^2 ds\Big)^{\frac{1}{2}} \\
&\leq L_h\sup_{\theta\in[-r,0]}\Big(\int_0^{\pi}|\frac{\partial}{\partial x}\varphi_1(\theta,x)) - \frac{\partial}{\partial x}\varphi_2(\theta,x))|^2 ds\Big)^{\frac{1}{2}} \\
&\leq L_h\|\varphi_1 - \varphi_2\|_{C_{\frac{1}{2}}}.
\end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost automorphic of class r.

Lemma 7.2. [10] *If* $\int_{-r}^{0}|G(\theta)|d\theta < 1$, *then the seimigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic and the unstable* $U \equiv \{0\}$.

For example, let us pose $G(\theta) = \dfrac{\theta^2-1}{(\theta^2+1)^2}$, we can see that

$$\int_{-r}^{0}|G(\theta)|d\theta = \int_{-r}^{0}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta = [\frac{\theta}{\theta^2+1}]_{-r}^{0} = \frac{r}{r^2+1} < 1 \text{ if } r < 1.$$

$$\int_{-r}^{0}|G(\theta)|d\theta=\int_{-r}^{0}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta=\int_{-r}^{-1}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta+\int_{-1}^{0}\frac{\theta^2-1}{(\theta^2+1)^2}d\theta=1-\frac{r}{r^2+1}<1 \text{ if } r\geq 1.$$

By Theorem 6.7 we deduce the following result.

Theorem 7.3. *Under the above assumptions, if the Lipschitz constant $Lip(f)$ of f satisfies the inequality*

$$k<\frac{w^{1-\alpha}}{\overline{M}\widetilde{M}|\Pi^s|\Gamma(1-\alpha)},$$

then equation (11) *has a unique compact C^n - $cl(\mu,\nu)$-pseudo almost automorphic solution v of class r.*

Proof. Let us pose $k=Lip(f)$. Then we have

$$\begin{aligned}
|(\mathcal{H}x_1^{(i)})(t)-(\mathcal{H}x_2^{(i)})(t)|_\alpha &\leq \Big|\lim_{\tau\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}X_0(f(s,x_{1s}^{(i)})-f(s,x_{2s}^{(i)})))ds\Big|_\alpha\\
&\leq \lim_{\tau\to+\infty}\int_{-\infty}^{t}|\mathcal{A}_{\mathcal{U}}^{\alpha}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}X_0(f(s,x_{1s}^{(i)})-f(s,x_{2s}^{(i)})))|ds\\
&\leq \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}|f(t,x_{1s}^{(i)})-f(t,x_{2s}^{(i)})|ds\\
&\leq \overline{M}\widetilde{M}|\Pi^s k|x_1^{(i)}-x_2^{(i)}|_\alpha\int_{-\infty}^{t}\frac{e^{-\omega(t-s)}}{(t-s)^\alpha}ds\\
&\leq \overline{M}\widetilde{M}|\Pi^s|k\Big(\int_0^{+\infty}e^{-\omega s}s^\alpha ds\Big)|x_1^{(i)}-x_2^{(i)}|_\alpha\\
&\leq \frac{\overline{M}\widetilde{M}|\Pi^s|k\Gamma(1-\alpha)}{\omega^{1-\alpha}}|x_1^{(i)}-x_2^{(i)}|_\alpha,
\end{aligned}$$

which implies that

$$\sum_{i=0}^{n}|(\mathcal{H}x_1^{(i)})(t)-(\mathcal{H}x_2^{(i)})(t)|_\alpha\leq\frac{\overline{M}\widetilde{M}|\Pi^s|k\Gamma(1-\alpha)}{\omega^{1-\alpha}}|x_1-x_2|_{\alpha,n}.$$

Consequently, if $k<\dfrac{w^{1-\alpha}}{\overline{M}\widetilde{M}|\Pi^s|\Gamma(1-\alpha)}$, then $\mathcal{H}$ is a strict contraction. □

8. Discussion

In this work, under some appropriate conditions, we establish the existence and uniqueness of C^n-(μ,ν)-pseudo almost automorphic solutions of class r in the α-norm for some functional partial differential equations in a Banach spaces. It is well known that the study under the α-norm is more general than the classical one (see for example [10]).

However, to obtain our results we use the hypothesis that the operator $T(t)$ is compact. The next challenge is to establish the existence and uniqueness of the solution of class r in the α norm without using the compactness of the analytic semigroup $(T(t))_{t\geq 0}$.

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