PAIR MEAN CORDIAL LABELING OF SOME UNION OF GRAPHS

R. PONRAJ* AND S. PRABHU

ABSTRACT. Let a graph $G = (V, E)$ be a $(p, q)$ graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p+1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \ldots, \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in $M$ to the different elements of $V$ when $p$ is even and different labels in $M$ to $p - 1$ elements of $V$ and repeating a label for the remaining one vertex when $p$ is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge $uv$ of $G$, there exists a labeling $\frac{\lambda(u) + \lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u) + \lambda(v) + 1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{S}_{\lambda_1} - \bar{S}_{\lambda_1'}| \leq 1$ where $\bar{S}_{\lambda_1}$ and $\bar{S}_{\lambda_1'}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph $G$ with a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling behavior of some union of graphs.

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Key words and phrases : Path, cycle, wheel graph, shell graph and pair mean cordial labeling.

1. Introduction

In this paper, a finite, simple, connected and undirected graph is known as a graph $G$. We use the terminologies, fundamental concepts and notations in graph theory as in [7] and referring to the study on graph labeling in [6]. In [3], the concept of cordial labeling was first established and also studied some cordial related graphs in [1,2,4,5,8-13,19-24]. We have introduced the notion of pair mean cordial labeling in [14] and examined the pair mean cordial labeling...
behavior of several graphs in [14-18]. In this paper, we investigate the pair mean cordial labeling behavior of some union of graphs such as $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $P_m \cup W_n$, $C_m \cup C_n$, $C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$, $S_m \cup W_n$.

2. Preliminaries

Definition 2.1. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

Definition 2.2. The union of two graphs $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 2.3. The Shell $S_n$ is the graph obtained by taking $n - 3$ concurrent chord in cycle $C_n$. The vertex at which all the chords are concurrent is called the apex vertex.

Definition 2.4. A Wheel $W_n$ is a graph with $n + 1$ vertices, formed by connecting a single vertex to all the vertices of the cycle $C_n$. It is denoted by $W_n = C_n + K_1$.

3. Pair Mean Cordial Labeling

Definition 3.1. Let a graph $G = (V, E)$ be a $(p, q)$ graph. Define

$$\rho = \left\{ \frac{p}{2} \right\} \text{ if } p \text{ is even}$$

and $M = \{\pm 1, \pm 2, \ldots, \pm \rho\}$ called the set of labels. Consider a mapping $\lambda: V \to M$ by assigning different labels in $M$ to the different elements of $V$ when $p$ is even and different labels in $M$ to $p - 1$ elements of $V$ and repeating a label for the remaining one vertex when $p$ is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge $uv$ of $G$, there exists a labeling $\frac{\lambda(u) + \lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u) + \lambda(v) + 1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|S_{\lambda_1} - S_{\lambda_2}| \leq 1$ where $S_{\lambda_1}$ and $S_{\lambda_2}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph $G$ with a pair mean cordial labeling is called a pair mean cordial graph.

Theorem 3.2. The graph $P_m \cup P_n$ is pair mean cordial for all $m, n \geq 1$.

Proof. Let $P_m$ be the path $u_1 u_2 \ldots u_m$ and $P_n$ be the path $v_1 v_2 \ldots v_n$. Then $P_m \cup P_n$ has $m + n$ vertices and $m + n - 2$ edges. We have the following two cases arise:

Case (i): $m$ is odd

Let us assign the labels $1, 2, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_m$ and $-1, -2, \ldots, -\frac{m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. Then there are two subcases that arise:

Subcase (i): $n$ is odd

Let us now assign the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{-n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively.
Subcase (ii): $n$ is even

Furthermore we give the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Thus we assign the label $\frac{-m-n+1}{2}$ to the vertex $v_n$.

Case (ii): $m$ is even

In this case, we give the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $2, 3, \ldots, \frac{m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. There are two subcases that arise:

Subcase (i): $n$ is odd

Now, we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Finally, we assign the labels $1, \frac{-m-n+1}{2}$ to the vertices $v_{n-1}, v_n$ respectively.

Subcase (ii): $n$ is even

Next we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. More over assign the label 1 to the vertex $v_n$.

The following table shows that this vertex labeling $\lambda$ is a pair mean cordial of $P_m \cup P_n$ for all $m, n \geq 1$.

$$
\begin{array}{|c|c|c|}
\hline
\text{Nature of } m \text{ and } n & S_{\lambda_1} & S_{\lambda_1'} \\
\hline
m \text{ is odd and } n \text{ is odd} & \frac{m+n-2}{2} & \frac{m+n-2}{2} \\
\hline
m \text{ is odd and } n \text{ is even} & \frac{m+n-3}{2} & \frac{m+n-1}{2} \\
\hline
m \text{ is even and } n \text{ is odd} & \frac{m+n-3}{2} & \frac{m+n-1}{2} \\
\hline
m \text{ is even and } n \text{ is even} & \frac{m+n-2}{2} & \frac{m+n-2}{2} \\
\hline
\end{array}
$$

Table 1

**Theorem 3.3.** The graph $P_m \cup C_n$ is pair mean cordial for all $m \geq 1$ and $n \geq 3$.

**Proof.** Let $P_m$ be the path $u_1 u_2 \ldots u_m$ and $C_n$ be the cycle $v_1 v_2 \ldots v_n v_1$. Then the graph $P_m \cup C_n$ has $m+n$ vertices and $m+n-1$ edges. We have the following two cases arise:

Case (i): $m$ is odd

There are two subcases that arise:

Subcase (i): $n$ is odd

In this case, assign the labels to the vertices $u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq n$ as in subcase (i) of case (i) of theorem 3.1.

Subcase (ii): $n$ is even

Furthermore, assign the labels to the vertices $u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq n-1$ as in subcase (ii) of case (i) of theorem 3.1. Finally assign the label $\frac{m+3}{2}$ to the vertex $v_n$.

Case (ii): $m$ is even

There are two subcases that arise:
Subcase (i): $n$ is odd
In this case, assign the labels to the vertices $u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq n - 2$ as in subcase (i) of case (ii) of theorem 3.1. Finally, we assign the labels $1, 1$ to the vertices $v_{n-1}, v_n$ respectively.

Subcase (ii): $n$ is even
Furthermore, assign the labels to the vertices $u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq n$ as in subcase (ii) of case (ii) of theorem 3.1.

The following table shows that this vertex labeling $\lambda$ is a pair mean cordial of $P_m \cup C_n$ for all $m \geq 1$ and $n \geq 3$.

<table>
<thead>
<tr>
<th>Nature of $m$ and $n$</th>
<th>$S_{\lambda_1}$</th>
<th>$S_{\lambda_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ is odd and $n$ is odd</td>
<td>$\frac{m+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
<tr>
<td>$m$ is odd and $n$ is even</td>
<td>$\frac{m+1}{2}$</td>
<td>$\frac{m+n-1}{2}$</td>
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<tr>
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<td>$\frac{m+1}{2}$</td>
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</tr>
<tr>
<td>$m$ is even and $n$ is even</td>
<td>$\frac{m+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
</tbody>
</table>

Table 2

Theorem 3.4. The graph $P_m \cup S_n$ is pair mean cordial for all $m \geq 1$ and $n \geq 4$.

Proof. Let $V(P_m \cup S_n) = \{u_i, v_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(P_m \cup S_n) = \{u_i u_{i+1}, v_j v_{j+1}, v_n v_1 : 1 \leq i \leq m-1$ and $1 \leq j \leq n-1\} \cup \{v_1 v_2 : 1 \leq j \leq n-3\}$.

Hence it has $m + n$ vertices and $m + 2n - 4$ edges. We have the following two cases arise:

Case (i): $m$ is odd
Let us assign the labels $1, 2, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_m$ and $-1, -2, \ldots, -\frac{m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. Then there are two subcases that arise:

Subcase (i): $n$ is odd
First we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots, -\frac{m-n+3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Furthermore, assign the label $\frac{n-m}{2}$ to the vertex $v_n$.

Subcase (ii): $n$ is even
Now we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and $-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots, -\frac{m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. More over, assign the label $\frac{m+n-1}{2}$ to the vertex $v_{n-1}$.

Case (ii): $m$ is even
Let us assign the labels $1, 2, \ldots, \frac{m}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $-1, -2, \ldots, -\frac{m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-2}$ respectively. Hence there are two subcases that arise:

Subcase (i): $n$ is odd
In this case, we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $-\frac{m}{2}, -\frac{m-2}{2}, \ldots, -\frac{m-n+3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$.
respectively. Also assign the label $\frac{m+n-1}{2}$ to the vertex $v_n$.

**Subcase (ii):** $n$ is even

Now we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $-\frac{m}{2}, -\frac{m-2}{2}, \ldots, -\frac{m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively.

The following table shows that this vertex labeling $\lambda$ is a pair mean cordial of $P_m \cup S_n$ for all $m \geq 1$ and $n \geq 4$.

<table>
<thead>
<tr>
<th>Nature of $m$ and $n$</th>
<th>$\mathcal{S}_{\lambda_1}$</th>
<th>$\mathcal{S}_{\lambda_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ is odd and $n$ is odd</td>
<td>$\frac{m+2n-2}{2}$</td>
<td>$\frac{m+2n-6}{2}$</td>
</tr>
<tr>
<td>$m$ is odd and $n$ is even</td>
<td>$\frac{m+2n-2}{2}$</td>
<td>$\frac{m+2n-6}{2}$</td>
</tr>
<tr>
<td>$m$ is even and $n$ is odd</td>
<td>$\frac{m+2n-4}{2}$</td>
<td>$\frac{m+2n-4}{2}$</td>
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<tr>
<td>$m$ is even and $n$ is even</td>
<td>$\frac{m+2n-4}{2}$</td>
<td>$\frac{m+2n-4}{2}$</td>
</tr>
</tbody>
</table>

Table 3

**Example 3.5.** A pair mean cordial labeling of $P_8 \cup S_9$ is shown in Figure 1.

![FIGURE 1](image_url)

**Theorem 3.6.** The graph $P_m \cup W_n$ is pair mean cordial for all $m \geq 2$ and $n \geq 3$ except for $m = 3$ and $n$ is even.

**Proof.** Let $V(P_m \cup W_n) = \{u_i, v, v_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(P_m \cup W_n) = \{vuv_j : 1 \leq j \leq n\} \cup \{u_i, u_{i+1}, v_j, v_{j+1}, v_n, v_1 : 1 \leq i \leq m-1$ and $1 \leq j \leq n-1\}$. Hence $P_m \cup W_n$ has $m + n + 1$ vertices and $m + 2n - 1$ edges. We have the following three cases arise:

**Case (i):** $m = 3$

There are two subcases that arise:

**Subcase (i):** $n$ is odd

Let us consider $\lambda(u_1) = 1, \lambda(u_2) = 1$ and $\lambda(u_3) = \frac{-n-3}{2}$. Next we give the labels $2, 3, \ldots, \frac{n+3}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $-1, -2, \ldots, -\frac{n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Then assign the label $\frac{-n-1}{2}$ to the vertex $v$.

**Subcase (ii):** $n$ is even

Suppose $P_3 \cup W_n$ is pair mean cordial. Now if the edge $uv$ get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Then the maximum number
of edges label with 1 is \( n \). That is \( \bar{S}_{\lambda_1} \leq n \). Thus \( \bar{S}_{\lambda_1} \geq n + 2 \). Therefore \( \bar{S}_{\lambda_1} - \bar{S}_{\lambda_1} \geq n + 2 - n = 2 > 1 \), a contradiction.

**Case (ii):** \( m \) is odd

There are two subcases that arise:

**Subcase (i):** \( n \) is odd

In this case, we give the labels 1, 2, \( \frac{m-1}{2} \) respectively to the vertices \( u_1, u_3, \ldots, u_{m-2} \) and \(-1, -2, \ldots, \frac{-m+3}{2} \) to the vertices \( u_2, u_4, \ldots, u_{m-3} \) respectively. Assign the labels \( \frac{-m+3}{2}, \frac{-m+n}{2} \) to the vertices \( u_{m-1}, u_m \) respectively. Next we give the labels \( \frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_n \) and \(-\frac{m+1}{2}, -\frac{m+3}{2}, \ldots, -\frac{m+n}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-1} \) respectively. Hence assign the label \( \frac{-m-n+2}{2} \) to the vertex \( v \).

**Subcase (ii):** \( n \) is even

Let us consider \( \lambda(u_1) = 2, \lambda(u_2) = -1, \lambda(u_3) = 3 \) and \( \lambda(u_4) = -2 \). Then we give the labels \(-3, -4, \ldots, \frac{-m+1}{2} \) respectively to the vertices \( u_5, u_7, \ldots, u_{m-2} \) and \( 4, 5, \ldots, \frac{m+1}{2} \) to the vertices \( u_6, u_8, \ldots, u_{m-1} \) respectively. More over assign the label 1 to the vertex \( u_m \). Next we give the labels \( \frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n+1}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \(-\frac{m-1}{2}, -\frac{m-3}{2}, \ldots, -\frac{m-n+1}{2} \) to the vertices \( v_2, v_4, \ldots, v_n \) respectively. Hence assign the label \( \frac{-m-n-1}{2} \) to the vertex \( v \).

**Case (iii):** \( m \) is even

Now give the labels 1, 2, \ldots, \( \frac{m}{2} \) respectively to the vertices \( u_1, u_3, \ldots, u_{m-1} \) and \(-1, -2, \ldots, -\frac{m}{2} \) to the vertices \( u_2, u_4, \ldots, u_{m-2} \) respectively. There are three subcases that arise:

**Subcase (i):** \( n \) is odd

Next assign the label \( \frac{-m-n-1}{2} \) to the vertex \( u_m \). Then we assign the labels \( \frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n+1}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_n \) and \(-\frac{m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m+n+3}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-1} \) respectively. Hence assign the label \( \frac{-m-n+1}{2} \) to the vertex \( v \).

**Subcase (ii):** \( n \) is even

Now assign the label \( \frac{-m-n}{2} \) to the vertex \( u_m \). Next we assign the labels \( \frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \(-\frac{m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+3}{2} \) to the vertices \( v_2, v_4, \ldots, v_n \) respectively. Hence assign the label \( \frac{m-n+2}{2} \) to the vertex \( v \).

The following table shows that this vertex labeling \( \lambda \) is a pair mean cordial of \( P_m \cup W_n \) for all \( m \geq 2 \) and \( n \geq 3 \) except for \( m = 3 \) and \( n \) is even.

\[ \square \]

**Remark 3.1.** The graph \( P_1 \cup W_n \) is pair mean cordial for all \( n \geq 5 \) and \( n \) is odd.

**Proof.** The graph \( P_1 \cup W_n \) has \( 2n + 2 \) vertices and \( 2n \) edges. We have the following two cases arise:

**Case (i):** \( n = 3 \) and \( n \) is even

Suppose \( P_1 \cup W_n \) is pair mean cordial. Now if the edge \( uv \) get the label 1, the
Let us take $\lambda(v)$ to the vertices respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $-1, -2, \ldots, -\frac{n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Next assign the label $\frac{m-1}{2}$ to the vertex $v_n$. Finally assign the label $\frac{n+1}{2}$ to the vertex $v$. Hence $S_{\lambda_1} = S_{\lambda_1'} = n$.

Theorem 3.7. The graph $C_m \cup S_n$ is pair mean cordial for all $m, n \geq 4$.

Proof. Let us define $V(C_m \cup S_n) = \{u_i, v_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(C_m \cup S_n) = \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, v_nv_1 : 1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1\} \cup \{v_1v_{j+2} : 1 \leq j \leq n - 3\}$. Hence the graph $C_m \cup S_n$ has $m + n$ vertices and $m + 2n - 3$ edges. We have the following two cases arise:

Case (i): $m$ is odd
There are two subcases that arise:

Subcase (i): $n$ is odd
Take $\lambda(u_1) = 2, \lambda(u_2) = -1, \lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Next, we give the labels $-3, -4, \ldots, -\frac{m+1}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_{m-2}$ and $4, 5, \ldots, \frac{m+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{m-1}$ respectively. Then we assign the label 1 to the vertex $u_m$. More over, we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-2}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally we assign the label $\frac{n-1}{2}$ to the vertex $v_n$.

Subcase (ii): $n$ is even
In this case, we assign the labels to the vertices $u_i, 1 \leq i \leq m$ as in subcase (iii) of case (1). Next, we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Furthermore, assign the label $\frac{m+n-1}{2}$ to the vertex $v_{n-1}$.

Case (ii): $m$ is even
There are two subcases that arise:

Subcase (i): $n$ is odd
Let us take $\lambda(u_1) = 2, \lambda(u_2) = -1, \lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Next,

<table>
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<tbody>
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<tr>
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<tr>
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<td>$\frac{m+2n-2}{2}$</td>
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<tr>
<td>$m$ is even and $n$ is even</td>
<td>$\frac{m+2n-2}{2}$</td>
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</tr>
</tbody>
</table>
we give the labels \(-3, -4, \ldots, -\frac{m}{2}\) respectively to the vertices \(u_5, u_7, \ldots, u_{m-1}\) and \(4, 5, \ldots, \frac{m+2}{2}\) to the vertices \(u_6, u_8, \ldots, u_m\) respectively. We assign the labels \(\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}\) respectively to the vertices \(v_1, v_3, \ldots, v_{n-4}\) and \(-\frac{m-2}{2}, -\frac{m-4}{2}, \ldots, -\frac{m-n+1}{2}\) to the vertices \(v_2, v_4, \ldots, v_{n-1}\) respectively. Also assign the label \(\frac{m+n-1}{2}\) to the vertices \(v_{n-2}, v_n\) respectively.

**Subcase (ii):** \(n\) is even

In this case, we assign the labels to the vertices \(u_i, 1 \leq i \leq m\) as in subcase (i) of case (ii). Next, we assign the labels \(\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}\) respectively to the vertices \(v_1, v_3, \ldots, v_{n-3}\) and \(-\frac{m-2}{2}, -\frac{m-4}{2}, \ldots, -\frac{m-n}{2}\) to the vertices \(v_2, v_4, \ldots, v_n\) respectively. Now, we assign the label 1 to the vertex \(v_{n-1}\).

The following table shows that this vertex labeling \(\lambda\) is a pair mean cordial of \(C_m \cup S_n\) for all \(m, n \geq 4\).

\[
\begin{array}{|c|c|c|}
\hline
\text{Nature of } m \text{ and } n & S_{\lambda_1} & S_{\lambda_2} \\
\hline
m \text{ is odd and } n \text{ is odd} & \frac{m+2n-3}{2} & \frac{m+2n-3}{2} \\
\hline
m \text{ is odd and } n \text{ is even} & \frac{m+2n-3}{2} & \frac{m+2n}{2} \\
\hline
m \text{ is even and } n \text{ is odd} & \frac{m+2n-4}{2} & \frac{m+2n-2}{2} \\
\hline
m \text{ is even and } n \text{ is even} & \frac{m+2n-4}{2} & \frac{m+2n-2}{2} \\
\hline
\end{array}
\]

**Table 5**

**Remark 3.2.** The graph \(C_3 \cup S_n\) is pair mean cordial iff \(n\) is even.

**Proof.** If \(n\) is odd, suppose that \(C_3 \cup S_n\) is pair mean cordial. Now if the edge \(uv\) get the label 1, the possibilities are \(\lambda(u) + \lambda(v) = 1\) or \(\lambda(u) + \lambda(v) = 2\). Hence the maximum number of edges label with 1 is \(n - 1\). That is \(S_{\lambda_1} \leq n - 1\). Then \(S_{\lambda_2} \geq q - (n - 1) = n + 1\). Therefore \(S_{\lambda_2} - S_{\lambda_1} \geq n + 1 - (n - 1) = 2 > 1\), a contradiction.

Further if \(n\) is even, let us consider \(\lambda(u) = 1, \lambda(u_2) = 1\) and \(\lambda(u_3) = -\frac{m-n+1}{2}\). Then we give the labels \(2, 3, \ldots, \frac{m+n-1}{2}\) respectively to the vertices \(v_1, v_3, \ldots, v_{n-4}\) and \(-1, -2, \ldots, -\frac{m-n+3}{2}\) to the vertices \(v_2, v_4, \ldots, v_n\) respectively. \(S_{\lambda_1} = S_{\lambda_2} = n\).

\[
\square
\]

**Theorem 3.8.** The graph \(C_m \cup C_n\) is pair mean cordial for all \(m \geq 3\) and \(n \geq 4\).

**Proof.** Let \(C_m\) be the cycle \(u_1u_2 \ldots u_mu_1\) and \(C_n\) be the cycle \(v_1v_2 \ldots v_nv_1\). Then the graph \(C_m \cup C_n\) has \(m + n\) vertices and \(m + n\) edges. We have the following three cases arise:

**Case (i):** \(m = 3\)

Let us consider \(\lambda(u_1) = 2, \lambda(u_2) = -1\) and \(\lambda(u_3) = 3\). There are two subcases that arise:

**Subcase (i):** \(n\) is odd

Let \(\lambda(v_1) = -2, \lambda(v_2) = 4\) and \(\lambda(v_3) = -3\). Then we give the labels \(-4, -5, \ldots,\)
respectively to the vertices \( v_4, v_6, \ldots, v_{n-1} \) and \( 5, 6, \ldots, \frac{m+n}{2} \) to the vertices \( v_5, v_7, \ldots, v_{n-2} \) respectively. Finally assign the label 1 to the vertex \( v_n \).

**Subcase (ii):** \( n \) is even

If \( n = 4 \), define \( \lambda(v_1) = -2 \), \( \lambda(v_2) = -3 \), \( \lambda(v_3) = 1 \) and \( \lambda(v_4) = 1 \). Therefore \( 3 \lambda_1 = 3 \) and \( 3 \lambda_2 = 4 \).

If \( n > 4 \), then we give the labels \(-4, -5, \ldots, \frac{-m-n}{2} \) respectively to the vertices \( v_4, v_6, \ldots, v_{n-2} \) and \( 5, 6, \ldots, \frac{m+n}{2} \) to the vertices \( v_5, v_7, \ldots, v_{n-3} \) respectively. Also assign the labels \( 1, \frac{-m-n+1}{2}, \frac{-m-n+3}{2}, \ldots, \frac{m+n}{2} \) to the vertices \( v_{n-1}, v_n \) respectively.

**Case (ii):** \( m \) is odd

Let us take \( \lambda(u_1) = 2 \), \( \lambda(u_2) = -1 \), \( \lambda(u_3) = 3 \) and \( \lambda(u_4) = -2 \). Then we give the labels \(-3, -4, \ldots, \frac{-m+1}{2} \) respectively to the vertices \( u_5, u_7, \ldots, u_{m-2} \) and \( 4, 5, \ldots, \frac{m+1}{2} \) to the vertices \( u_6, u_8, \ldots, u_{m-1} \) respectively. Next assign the label 1 to the vertex \( u_m \). Then there are two subcases that arise:

**Subcase (i):** \( n \) is odd

In this case, we give the labels \( \frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_n \) and \( \frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-1} \) respectively.

**Subcase (ii):** \( n \) is even

Also we give the labels \( \frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \( \frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-2} \) respectively. Thus assign the label \( \frac{-m-n+1}{2} \) to the vertex \( v_n \).

**Case (iii):** \( m \) is even

Let us take \( \lambda(u_1) = 2 \), \( \lambda(u_2) = -1 \), \( \lambda(u_3) = 3 \) and \( \lambda(u_4) = -2 \). Thus we give the labels \(-3, -4, \ldots, \frac{-m}{2} \) respectively to the vertices \( u_5, u_7, \ldots, u_{m-1} \) and \( 4, 5, \ldots, \frac{m+2}{2} \) to the vertices \( u_6, u_8, \ldots, u_m \) respectively. There are two subcases that arise:

**Subcase (i):** \( n \) is odd

Now, we give the labels \( \frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-2} \) and \( \frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-3} \) respectively. Next we assign the labels \( 1, \frac{-m-n+1}{2} \) to the vertices \( v_{n-1}, v_n \) respectively.

**Subcase (ii):** \( n \) is even

In this case, we give the labels \( \frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \( \frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-2} \) respectively. Finally assign the label 1 to the vertex \( v_n \).

The following table shows that this vertex labeling \( \lambda \) is a pair mean cordial of \( C_m \cup C_n \) for all \( m \geq 3 \) and \( n \geq 4 \).
Table 6

<table>
<thead>
<tr>
<th>Nature of m and n</th>
<th>(\bar{S}_{\lambda_1})</th>
<th>(\bar{S}_{\lambda_1^c})</th>
</tr>
</thead>
<tbody>
<tr>
<td>m is odd and n is odd</td>
<td>(\frac{m+n-1}{2})</td>
<td>(\frac{m+n}{2})</td>
</tr>
<tr>
<td>m is odd and n is even</td>
<td>(\frac{m+n-1}{2})</td>
<td>(\frac{m+n}{2})</td>
</tr>
<tr>
<td>m is even and n is odd</td>
<td>(\frac{m+n}{2})</td>
<td>(\frac{m+n+1}{2})</td>
</tr>
<tr>
<td>m is even and n is even</td>
<td>(\frac{m+n}{2})</td>
<td>(\frac{m+n+1}{2})</td>
</tr>
</tbody>
</table>

Example 3.9. A pair mean cordial labeling of \(C_8 \cup C_7\) is shown in Figure 2.

Remark 3.3. \(C_3 \cup C_3\) is not a pair mean cordial graph.

Proof. Suppose that \(C_3 \cup C_3\) is pair mean cordial. Now if the edge \(uv\) get the label 1, the possibilities are \(\lambda(u) + \lambda(v) = 1\) or \(\lambda(u) + \lambda(v) = 2\). Hence the maximum number of edges label with 1 is 2. That is \(\bar{S}_{\lambda_1} \leq n - 1\). Then \(\bar{S}_{\lambda_1^c} \geq q - (2) = 4\). Therefore \(\bar{S}_{\lambda_1^c} - \bar{S}_{\lambda_1} \geq 4 - 2 = 2 > 1\), a contradiction. □

Theorem 3.10. The graph \(W_m \cup C_n\) is pair mean cordial for all \(m \geq 3\) and \(n \geq 5\).

Proof. Let \(V(W_m \cup C_n) = \{u, u_i, v_j : 1 \leq i \leq m; 1 \leq j \leq n\}\) and \(E(W_m \cup C_n) = \{u_i u_{i+1}, u_m v_1, v_j v_{j+1}, v_n v_1 : 1 \leq i \leq m - 1\}\). Hence \(W_m \cup C_n\) has \(m + n + 1\) vertices and \(2m + n\) edges. We have the following two cases arise:

Case (i): \(m\) is odd

Now, we give the labels \(2, 3, \ldots, \frac{m+3}{2}\) respectively to the vertices \(u_1, u_3, \ldots, u_m\) and \(-1, -2, \ldots, -\frac{m+1}{2}\) to the vertices \(u_2, u_4, \ldots, u_{m-1}\) respectively. Next assign the label \(-\frac{m-1}{2}\) to the vertex \(u\). There are two subcases that arise:

Subcase (i): \(n\) is odd

Then we assign the labels \(-\frac{m-3}{2}, -\frac{m-5}{2}, \ldots, -\frac{m-n}{2}\) respectively to the vertices \(v_1, v_3, \ldots, v_{n-2}\) and \(\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n}{2}\) to the vertices \(v_2, v_4, \ldots, v_{n-3}\) respectively. Finally, assign the labels 1, 1 to the vertices \(v_{n-1}, v_n\) respectively.

Subcase (ii): \(n\) is even

□
Then assign the labels \( \frac{m+5}{2}, \frac{-m-3}{2}, \frac{m+7}{2}, \frac{-m-5}{2} \) respectively to the vertices \( v_1, v_2, v_3, v_4 \). Next, we assign the labels \( \frac{-m-7}{2}, \frac{-m-9}{2}, \ldots, \frac{-m-n-1}{2} \) respectively to the vertices \( v_5, v_7, \ldots, v_{n-1} \) and \( \frac{m+9}{2}, \frac{m+11}{2}, \ldots, \frac{m+n+1}{2} \) to the vertices \( v_6, v_8, \ldots, v_{n-2} \) respectively. Finally, we assign the label 1 to the vertex \( v_n \).

**Case (ii):** \( m \) is even

First, we give the labels \( 2, 3, \ldots, \frac{m+2}{2} \) respectively to the vertices \( u_1, u_3, \ldots, u_{m-1} \) and \(-1, -2, \ldots, \frac{-m}{2} \) to the vertices \( u_2, u_4, \ldots, u_m \) respectively. There are two subcases that arise:

**Subcase (i):** \( n \) is odd

Now assign the label \( \frac{m+4}{2} \) to the vertex \( u \). Then we assign the labels \( \frac{-m-2}{2}, \frac{m+6}{2}, \frac{-m-4}{2} \) respectively to the vertices \( v_1, v_2, v_3 \). Moreover we assign the labels \( \frac{-m-6}{2}, \frac{-m-8}{2}, \ldots, \frac{-m-n-1}{2} \) respectively to the vertices \( v_4, v_6, \ldots, v_{n-1} \) and \( \frac{m+8}{2}, \frac{m+10}{2}, \ldots, \frac{m+n+1}{2} \) to the vertices \( v_5, v_7, \ldots, v_{n-2} \) respectively. Finally, assign the label 1 to the vertex \( v_n \).

**Subcase (ii):** \( n \) is even

In this case, assign the label \( \frac{m+2}{2} \) to the vertex \( u \). Then we assign the labels \( \frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \( \frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-2} \) respectively. Hence assign the label 1 to the vertex \( v_n \).

The following table shows that this vertex labeling \( \lambda \) is a pair mean cordial of \( W_m \cup C_n \) for all \( m \geq 3 \) and \( n \geq 5 \).

<table>
<thead>
<tr>
<th>Nature of ( m ) and ( n )</th>
<th>( \bar{S}_{\lambda_1} )</th>
<th>( \bar{S}_{\lambda_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) is odd and ( n ) is odd</td>
<td>( \frac{2m+n-1}{2} )</td>
<td>( \frac{2m+n+1}{2} )</td>
</tr>
<tr>
<td>( m ) is odd and ( n ) is even</td>
<td>( \frac{2m+n}{2} )</td>
<td>( \frac{2m+n}{2} )</td>
</tr>
<tr>
<td>( m ) is even and ( n ) is odd</td>
<td>( \frac{2m-n-1}{2} )</td>
<td>( \frac{2m-n+1}{2} )</td>
</tr>
<tr>
<td>( m ) is even and ( n ) is even</td>
<td>( \frac{2m+n}{2} )</td>
<td>( \frac{2m+n}{2} )</td>
</tr>
</tbody>
</table>

Table 7

**Remark 3.4.** The graph \( W_m \cup C_3 \) is pair mean cordial iff \( m \) is odd.

*Proof.* If \( m \) is odd, let us assign labels to the vertices as \( u, u_i, v_j, 1 \leq i \leq m \) and \( 1 \leq j \leq 3 \) in case (i) of subcase (i) of theorem 3.7. If \( m \) is even, assume \( W_m \cup C_3 \) is pair mean cordial. Now if the edge \( uv \) get the label 1, the possibilities are \( \lambda(u) + \lambda(v) = 1 \) or \( \lambda(u) + \lambda(v) = 2 \). Hence the maximum number of edges label with 1 is \( m \). That is \( \bar{S}_{\lambda_1} \leq m \). Then \( \bar{S}_{\lambda_2} \geq q - m = m + 3 \). Therefore \( \bar{S}_{\lambda_2} - \bar{S}_{\lambda_1} \geq m + 3 - m = 3 > 1 \), a contradiction.

**Remark 3.5.** The graph \( W_m \cup C_4 \) is pair mean cordial iff \( m \) is even.

*Proof.* If \( m \) is even, assign labels to the vertices as \( u, u_i, v_j, 1 \leq i \leq m \) and \( 1 \leq j \leq 4 \) in case (ii) of subcase (ii) of theorem 3.7. If \( m \) is odd, suppose
\[ W_m \cup C_4 \text{ is pair mean cordial. Now if the edge } uv \text{ get the label } 1, \text{ the possibilities are } \lambda(u) + \lambda(v) = 1 \text{ or } \lambda(u) + \lambda(v) = 2. \text{ Hence the maximum number of edges label with } 1 \text{ is } m + 1. \text{ That is } \bar{S}_{\lambda_1} \leq m + 1. \text{ Then } \bar{S}_{\lambda^*_1} \geq q - (m + 1) = m + 3. \text{ Therefore } \bar{S}_{\lambda^*_1} - \bar{S}_{\lambda_1} \geq m + 3 - (m + 1) = 2 > 1, \text{ a contradiction.} \]

**Theorem 3.11.** The graph \( S_m \cup S_n \) is not a pair mean cordial graph for all \( m, n \geq 4 \).

**Proof.** Let us define \( V(S_m \cup S_n) = \{u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \) and \( E(S_m \cup S_n) = \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, v_nv_1 : 1 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n - 1\} \cup \{u_1u_2, v_1v_{j+1}, v_nv_1 : 1 \leq i \leq m - 3 \text{ and } 1 \leq j \leq n - 3\} \). Hence \( S_m \cup S_n \) has \( m + n \) vertices and \( 2m + 2n - 6 \) edges. Suppose \( S_m \cup S_n \) is pair mean cordial. Now if the edge \( uv \) get the label 1, the possibilities are \( \lambda(u) + \lambda(v) = 1 \) or \( \lambda(u) + \lambda(v) = 2 \). Hence the maximum number of edges label with 1 is \( m + n - 4 \). That is \( \bar{S}_{\lambda_1} \leq m + n - 4 \). Then \( \bar{S}_{\lambda^*_1} \geq q - (m + n - 4) = m + n - 2 \). Therefore \( \bar{S}_{\lambda^*_1} - \bar{S}_{\lambda_1} \geq m + n - 2 - (m + n - 4) = 2 > 1, \text{ a contradiction.} \)

**Theorem 3.12.** The graph \( S_m \cup W_n \) is pair mean cordial for all \( m \geq 4 \) and \( n \geq 3 \) except for \( m + n \) is odd.

**Proof.** Let \( V(S_m \cup W_n) = \{u_i, v, j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \) and \( E(S_m \cup W_n) = \{v_j : 1 \leq j \leq n\} \cup \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, v_nv_1 : 1 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n - 1\} \cup \{v_1v_{j+2} : 1 \leq j \leq n - 3\} \). Hence \( S_m \cup W_n \) has \( m + n + 1 \) vertices and \( 2m + 2n - 3 \) edges. We have the following two cases arise:

**Case (i):** \( m + n \) is odd

Suppose that \( S_m \cup W_n \) is pair mean cordial. Now if the edge \( uv \) get the label 1, the possibilities are \( \lambda(u) + \lambda(v) = 1 \) or \( \lambda(u) + \lambda(v) = 2 \). Hence the maximum number of edges label with 1 is \( m + n - 3 \). That is \( \bar{S}_{\lambda_1} \leq m + n - 3 \). Then \( \bar{S}_{\lambda^*_1} \geq m + n - (m + n - 3) = 3 > 1, \text{ a contradiction.} \)

**Case (ii):** \( m + n \) is even

There are two subcases that arise:

**Subcase (i):** \( m \) and \( n \) is even

In this case, we give the labels \( 2, 3, \ldots, \frac{m+2}{2} \) respectively to the vertices \( u_1, u_3, \ldots, u_{m-1} \) and \( -1, -2, \ldots, -\frac{m+2}{2} \) to the vertices \( v_2, u_4, \ldots, u_{m-2} \) respectively. More over assign the label 1 to the vertex \( u_m \). Next we give the labels \( -\frac{m-2}{2}, -\frac{m-4}{2}, \ldots, -\frac{m-n+2}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \( \frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-2} \) respectively. Furthermore assign the label \( -\frac{m-n}{2} \) to the vertex \( v_1 \). Finally, assign the label \( \frac{m+n}{2} \) to the vertex \( v \).

**Subcase (ii):** \( m \) and \( n \) is odd

First give the labels \( 2, 3, \ldots, \frac{m+1}{2} \) respectively to the vertices \( u_1, u_3, \ldots, u_{m-2} \) and \( -1, -2, \ldots, -\frac{m+3}{2} \) to the vertices \( v_2, u_4, \ldots, u_{m-3} \) respectively. More over assign the labels \( -\frac{m-1}{2}, -1, 1, \ldots, -\frac{m-n+2}{2} \) respectively to the vertices \( v_1, v_3, \ldots, v_{n-1} \) and \( \frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-2}{2} \) to the vertices \( v_2, v_4, \ldots, v_{n-1} \) respectively. Furthermore
assign the label $\frac{m+n}{2}$ to the vertex $v$. In both cases, $\bar{S}_{\lambda_1} = m + n - 2$ and $\bar{S}_{\lambda_1^*} = m + n - 1$.

\[\square\]

Example 3.13. A pair mean cordial labeling of $S_9 \cup W_9$ is shown in Figure 3.

![Figure 3](image-url)

Theorem 3.14. The graph $W_m \cup W_n$ is not a pair mean cordial graph for all $m, n \geq 3$.

Proof. Let $V(W_m \cup W_n) = \{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(W_m \cup W_n) = \{u_iu_{i+1}, v_jv_{j+1} : 1 \leq i \leq m - 1 \text{ and } 1 \leq j \leq n - 1\}$. Hence $W_m \cup W_n$ has $m + n + 2$ vertices and $2m + 2n$ edges. We have the following two cases arise:

Case (i): $m + n$ is odd

Suppose $W_m \cup W_n$ is pair mean cordial. Now if the edge $uv$ get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label with 1 is $m + n - 1$. That is $\bar{S}_{\lambda_1} \leq m + n - 1$. Then $\bar{S}_{\lambda_1^*} \geq q - (m + n - 1) = m + n - 1$. Therefore $\bar{S}_{\lambda_1^*} - \bar{S}_{\lambda_1} \geq m + n - 1 - (m + n - 1) = 2 > 1$, a contradiction.

Case (ii): $m + n$ is even

Assume that $W_m \cup W_n$ is pair mean cordial. Now if the edge $uv$ get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label with 1 is $m + n - 2$. That is $\bar{S}_1 \leq m + n - 2$. Then $\bar{S}_1^* \geq q - (m + n - 2) = m + n + 2$. Therefore $\bar{S}_1^* - \bar{S}_1 \geq m + n + 2 - (m + n - 2) = 4 > 1$, a contradiction. $\square$

4. Discussion

Since its initial proposal in 1987 by Cahit[3], cordial labeling is now a popular field of research in graph labeling. Many authors examined the different kinds of cordial labeling in [1,2,4,5,8-13,19-23]. The concept of mean labeling was introduced in [24], while the pair difference cordial labeling was first proposed in [13]. Our introduction of the pair mean cordial labeling in [14] was motivated by these two concepts. The current paper presents the results of the pair mean cordial labeling behavior of union of few graphs, which include $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $P_m \cup W_n$, $S_m \cup C_n$, $C_m \cup C_n$, $C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$ and $S_m \cup W_n$. 
5. Limitation of Research

Investigating the pair mean cordial labeling behavior of the scorpion graph, spider graph, generalized Peterson graph, generalized Heawood graph, cubic diamond k-chain graph, swastik graph, broken wheel graph and n-cube graph on a large number of vertices is currently challenging to study.

6. Future Research

Future research to examine the pair mean cordial labeling behavior of union of path with other graphs like bull graph, shackle graph, jahangir graph, olive graph, coconut graph, step ladder and shadow graph.

7. Conclusion

In this paper, we have investigated the pair mean cordial labeling behavior of some union of graphs such as $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $C_m \cup W_n$, $C_m \cup C_n$, $C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$ and $S_m \cup W_n$. Future research should focus on examining the pair mean cordial labeling behavior of many graphs including generalized web graph, banana tree, x-tree, coconut tree, windmill graph, lollipop graph, broom graph and polar grid graph.

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