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PAIR MEAN CORDIAL LABELING OF SOME UNION OF GRAPHS

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Abstract. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p - 1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G with a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we investigate the pair mean cordial labeling behavior of some union of graphs.

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1. Introduction

In this paper, a finite, simple, connected and undirected graph is known as a graph G. We use the terminologies, fundamental concepts and notations in graph theory as in [7] and referring to the study on graph labeling in [6]. In [3], the concept of cordial labeling was first established and also studied some cordial related graphs in [1,2,4,5,8-13,19-24]. We have introduced the notion of pair mean cordial labeling in [14] and examined the pair mean cordial labeling

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behavior of several graphs in [14-18]. In this paper, we investigate the pair mean cordial labeling behavior of some union of graphs such as $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $P_m \cup W_n$, $C_m \cup C_n$,$C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$, $S_m \cup W_n$.

2. Preliminaries

Definition 2.1. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

Definition 2.2. The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 2.3. The Shell S_n is the graph obtained by taking $n-3$ concurrent chord in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex.

Definition 2.4. A Wheel W_n is a graph with $n+1$ vertices, formed by connecting a single vertex to all the vertices of the cycle C_n. It is denoted by $W_n = C_n + K_1$.

3. Pair Mean Cordial Labeling

Definition 3.1. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G with a pair mean cordial labeling is called a pair mean cordial graph.

Theorem 3.2. *The graph $P_m \cup P_n$ is pair mean cordial for all $m, n \geq 1$.*

Proof. Let P_m be the path $u_1 u_2 \dots u_m$ and P_n be the path $v_1 v_2 \dots v_n$. Then $P_m \cup P_n$ has $m+n$ vertices and $m+n-2$ edges. We have the following two cases arise:

Case (i): m is odd

Let us assign the labels $1, 2, \dots, \frac{m+1}{2}$ respectively to the vertices $u_1, u_3, \dots, u_m$ and $-1, -2, \dots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \dots, u_{m-1}$ respectively. Then there are two subcases that arise:

Subcase (i): n is odd

Let us now assign the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \dots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \dots, v_n$ and $\frac{m+3}{2}, \frac{m+5}{2}, \dots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \dots, v_{n-1}$ respectively.

Subcase (ii): n is even
Furthermore we give the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Thus we assign the label $\frac{-m-n+1}{2}$ to the vertex v_n.
Case (ii): m is even
In this case, we give the labels $-1, -2, \ldots, \frac{-m}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $2, 3, \ldots, \frac{m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. There are two subcases that arise:
Subcase (i): n is odd
Now, we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Finally, we assign the labels $1, \frac{-m-n+1}{2}$ to the vertices v_{n-1}, v_n respectively.
Subcase (ii): n is even
Next we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. More over assign the label 1 to the vertex v_n.
The following table shows that this vertex labeling λ is a pair mean cordial of $P_m \cup P_n$ for all $m, n \geq 1$. □

Nature of m and n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+n-2}{2}$	$\frac{m+n-2}{2}$
m is odd and n is even	$\frac{m+n-3}{2}$	$\frac{m+n-1}{2}$
m is even and n is odd	$\frac{m+n-3}{2}$	$\frac{m+n-1}{2}$
m is even and n is even	$\frac{m+n-2}{2}$	$\frac{m+n-2}{2}$

Table 1

Theorem 3.3. *The graph $P_m \cup C_n$ is pair mean cordial for all $m \geq 1$ and $n \geq 3$.*

Proof. Let P_m be the path $u_1u_2 \ldots u_m$ and C_n be the cycle $v_1v_2 \ldots v_nv_1$. Then the graph $P_m \cup C_n$ has $m+n$ vertices and $m+n-1$ edges. We have the following two cases arise:
Case (i): m is odd
There are two subcases that arise:
Subcase (i): n is odd
In this case, assign the labels to the vertices u_i, v_j, $1 \leq i \leq m$ and $1 \leq j \leq n$ as in subcase (i) of case (i) of theorem 3.1.
Subcase (ii): n is even
Furthermore, assign the labels to the vertices u_i, v_j, $1 \leq i \leq m$ and $1 \leq j \leq n-1$ as in subcase (ii) of case (i) of theorem 3.1. Finally assign the label $\frac{m+3}{2}$ to the vertex v_n.
Case (ii): m is even
There are two subcases that arise:

Subcase (i): n is odd
In this case, assign the labels to the vertices u_i, v_j, $1 \leq i \leq m$ and $1 \leq j \leq n-2$ as in subcase (i) of case (ii) of theorem 3.1. Finally, we assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively.
Subcase (ii): n is even
Furthermore, assign the labels to the vertices u_i, v_j, $1 \leq i \leq m$ and $1 \leq j \leq n$ as in subcase (ii) of case (ii) of theorem 3.1.
The following table shows that this vertex labeling λ is a pair mean cordial of $P_m \cup C_n$ for all $m \geq 1$ and $n \geq 3$.

□

Nature of m and n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+n-2}{2}$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
m is even and n is odd	$\frac{m+n-1}{2}$	$\frac{m+n-1}{2}$
m is even and n is even	$\frac{m+n-2}{2}$	$\frac{m+n}{2}$

Table 2

Theorem 3.4. *The graph $P_m \cup S_n$ is pair mean cordial for all $m \geq 1$ and $n \geq 4$.*

Proof. Let $V(P_m \cup S_n) = \{u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq \text{j} \leq \text{n}\}$ and $E(P_m \cup S_n) = \{u_i u_{i+1}, v_j v_{j+1}, v_n v_1 : 1 \leq i \leq m-1 \text{ and } 1 \leq \text{j} \leq \text{n}-1\} \cup \{\text{v}_1 \text{v}_{\text{j}+2} : 1 \leq \text{j} \leq \text{n}-3\}$. Hence it has $m+n$ vertices and $m+2n-4$ edges. We have the following two cases arise:
Case (i): m is odd
Let us assign the labels $1, 2, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_m$ and $-1, -2, \ldots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. Then there are two subcases that arise:
Subcase (i): n is odd
First we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Furthermore, assign the label $\frac{-m-n}{2}$ to the vertex v_n.
Subcase (ii): n is even
Now we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. More over, assign the label $\frac{m+n-1}{2}$ to the vertex v_{n-1}.
Case (ii): m is even
Let us assign the labels $1, 2, \ldots, \frac{m}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $-1, -2, \ldots, \frac{-m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-2}$ respectively. Hence there are two subcases that arise:
Subcase (i): n is odd
In this case, we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{-m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$

respectively. Also assign the label $\frac{m+n-1}{2}$ to the vertex v_n.

Subcase (ii): n is even

Now we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{-m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. The following table shows that this vertex labeling λ is a pair mean cordial of $P_m \cup S_n$ for all $m \geq 1$ and $n \geq 4$.

□

Nature of m and n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+2n-2}{2}$	$\frac{m+2n-6}{2}$
m is odd and n is even	$\frac{m+2n-2}{2}$	$\frac{m+2n-6}{2}$
m is even and n is odd	$\frac{m+2n-4}{2}$	$\frac{m+2n-4}{2}$
m is even and n is even	$\frac{m+2n-4}{2}$	$\frac{m+2n-4}{2}$

Table 3

Example 3.5. A pair mean cordial labeling of $P_8 \cup S_9$ is shown in Figure 1.

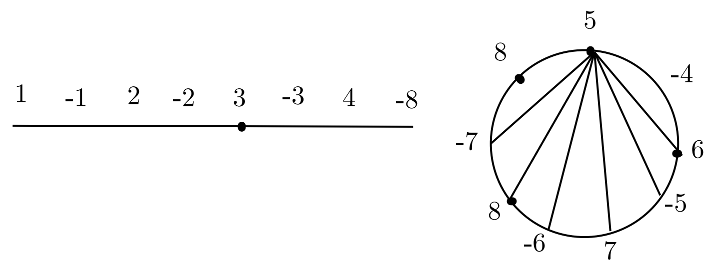

FIGURE 1

Theorem 3.6. *The graph $P_m \cup W_n$ is pair mean cordial for all $m \geq 2$ and $n \geq 3$ except for $m = 3$ and n is even.*

Proof. Let $V(P_m \cup W_n) = \{u_i, v, v_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and $E(P_m \cup W_n) = \{vv_j : 1 \leq j \leq n\} \cup \{u_iu_{i+1}, v_jv_{j+1}, v_nv_1 : 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n-1\}$. Hence $P_m \cup W_n$ has $m+n+1$ vertices and $m+2n-1$ edges. We have the following three cases arise:

Case (i): $m = 3$

There are two subcases that arise:

Subcase (i): n is odd

Let us consider $\lambda(u_1) = 1$, $\lambda(u_2) = 1$ and $\lambda(u_3) = \frac{-n-3}{2}$. Next we give the labels $2, 3, \ldots, \frac{n+3}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $-1, -2, \ldots, \frac{-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Then assign the label $\frac{-n-1}{2}$ to the vertex v.

Subcase (ii): n is even

Suppose $P_3 \cup W_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Then the maximum number of edges label with 1 is n. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n$. Thus $\bar{\mathbb{S}}_{\lambda_1^c} \geq n+2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+2-n = 2 > 1$, a contradiction.

Case (ii): m is odd

There are two subcases that arise:

Subcase (i): n is odd

In this case, we give the labels $1, 2, \ldots, \frac{m-1}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-2}$ and $-1, -2, \ldots, \frac{-m+3}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-3}$ respectively. Assign the labels $\frac{-m+3}{2}, \frac{-m-n}{2}$ to the vertices u_{m-1}, u_m respectively. Next we give the labels $\frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $\frac{-m+1}{2}, \frac{-m-1}{2}, \ldots, \frac{-m-n+4}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Hence assign the label $\frac{-m-n+2}{2}$ to the vertex v.

Subcase (ii): n is even

Let us consider $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Then we give the labels $-3, -4, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_{m-2}$ and $4, 5, \ldots, \frac{m+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{m-1}$ respectively. More over assign the label 1 to the vertex u_m. Next we give the labels $\frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Hence assign the label $\frac{-m-n-1}{2}$ to the vertex v.

Case (iii): m is even

Now give the labels $1, 2, \ldots, \frac{m}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $-1, -2, \ldots, \frac{-m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-2}$ respectively. There are three subcases that arise:

Subcase (i): n is odd

Next assign the label $\frac{-m-n-1}{2}$ to the vertex u_m. Then we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $\frac{-m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Hence assign the label $\frac{-m-n+1}{2}$ to the vertex v.

Subcase (ii): n is even

Now assign the label $\frac{-m-n}{2}$ to the vertex u_m. Next we assign the labels $\frac{m+2}{2}, \frac{m+4}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{-m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Hence assign the label $\frac{m+2}{2}$ to the vertex v.

The following table shows that this vertex labeling λ is a pair mean cordial of $P_m \cup W_n$ for all $m \geq 2$ and $n \geq 3$ except for $m = 3$ and n is even.

□

Remark 3.1. The graph $P_1 \cup W_n$ is pair mean cordial for all $n \geq 5$ and n is odd.

Proof. The graph $P_1 \cup W_n$ has $2n+2$ vertices and $2n$ edges. We have the following two cases arise:

Case (i): $n = 3$ and n is even

Suppose $P_1 \cup W_n$ is pair mean cordial. Now if the edge uv get the label 1, the

Nature of m and n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+2n-1}{2}$	$\frac{m+2n-1}{2}$
m is odd and n is even	$\frac{m+2n-1}{2}$	$\frac{m+2n-1}{2}$
m is even and n is odd	$\frac{m+2n-2}{2}$	$\frac{m+2n}{2}$
m is even and n is even	$\frac{m+2n-2}{2}$	$\frac{m+2n}{2}$

Table 4

possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Then the maximum number of edges label with 1 is $n-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-1$. Thus $\bar{\mathbb{S}}_{\lambda_1^c} \geq n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+1-(n-1) = 2 > 1$, a contradiction.
Case (ii): n is odd
First assign the label 1 to the vertex u. Now, we give the labels $2, 3, \ldots, \frac{n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $-1, -2, \ldots, \frac{-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Next assign the label $\frac{-n-1}{2}$ to the vertex v_n. Finally assign the label $\frac{n+1}{2}$ to the vertex v. Hence $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = n$.

□

Theorem 3.7. *The graph $C_m \cup S_n$ is pair mean cordial for all $m, n \geq 4$.*

Proof. Let us define $V(C_m \cup S_n) = \{u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq \mathrm{j} \leq \mathrm{n}\}$ and $E(C_m \cup S_n) = \{u_i u_{i+1}, u_m u_1, v_j v_{j+1}, v_n v_1 : 1 \leq i \leq m-1 \text{ and } 1 \leq \mathrm{j} \leq \mathrm{n}-1\} \cup \{\mathrm{v_1 v_{j+2}} : 1 \leq \mathrm{j} \leq \mathrm{n}-3\}$. Hence the graph $C_m \cup S_n$ has $m+n$ vertices and $m+2n-3$ edges. We have the following two cases arise:
Case (i): m is odd
There are two subcases that arise:
Subcase (i): n is odd
Take $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Next, we give the labels $-3, -4, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_{m-2}$ and $4, 5, \ldots, \frac{m+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{m-1}$ respectively. Then we assign the label 1 to the vertex u_m. More over, we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+2}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Finally we assign the label $\frac{-m-n}{2}$ to the vertex v_n.
Subcase (ii): n is even
In this case, we assign the labels to the vertices u_i, $1 \leq i \leq m$ as in subcase (iii) of case (1). Next, we assign the labels $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \ldots, v_n$ respectively. Furthermore, assign the label $\frac{m+n-1}{2}$ to the vertex v_{n-1}.
Case (ii): m is even
There are two subcases that arise:
Subcase (i): n is odd
Let us take $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Next,

we give the labels $-3, -4, \dots, \frac{-m}{2}$ respectively to the vertices $u_5, u_7, \dots, u_{m-1}$ and $4, 5, \dots, \frac{m+2}{2}$ to the vertices $u_6, u_8, \dots, u_m$ respectively. We assign the labels $\frac{m+4}{2}, \frac{m+6}{2}, \dots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \dots, v_{n-4}$ and $\frac{-m-2}{2}, \frac{-m-4}{2}, \dots, \frac{-m-n+1}{2}$ to the vertices $v_2, v_4, \dots, v_{n-1}$ respectively. Also assign the label $\frac{m+n-1}{2}, 1$ to the vertices v_{n-2}, v_n respectively.

Subcase (ii): n is even

In this case, we assign the labels to the vertices u_i, $1 \le i \le m$ as in subcase (i) of case (ii). Next, we assign the labels $\frac{m+4}{2}, \frac{m+6}{2}, \dots, \frac{m+n}{2}$ respectively to the vertices $v_1, v_3, \dots, v_{n-3}$ and $\frac{-m-2}{2}, \frac{-m-4}{2}, \dots, \frac{-m-n}{2}$ to the vertices $v_2, v_4, \dots, v_n$ respectively. Now, we assign the label 1 to the vertex v_{n-1}.

The following table shows that this vertex labeling λ is a pair mean cordial of $C_m \cup S_n$ for all $m, n \ge 4$.

□

Nature of m and n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+2n-3}{2}$	$\frac{m+2n-3}{2}$
m is odd and n is even	$\frac{m+2n-3}{2}$	$\frac{m+2n-3}{2}$
m is even and n is odd	$\frac{m+2n-4}{2}$	$\frac{m+2n-2}{2}$
m is even and n is even	$\frac{m+2n-4}{2}$	$\frac{m+2n-2}{2}$

Table 5

Remark 3.2. The graph $C_3 \cup S_n$ is pair mean cordial iff n is even.

Proof. If n is odd, suppose that $C_3 \cup S_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label with 1 is $n-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \le n-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \ge q - (n-1) = n+1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \ge n+1-(n-1) = 2 > 1$, a contradiction.

Further if n is even, let us consider $\lambda(u_1) = 1$, $\lambda(u_2) = 1$ and $\lambda(u_3) = \frac{-m-n+1}{2}$. Then we give the labels $2, 3, \dots, \frac{m+n-1}{2}$ respectively to the vertices $v_1, v_3, \dots, v_{n-1}$ and $-1, -2, \dots, \frac{-m-n+3}{2}$ to the vertices $v_2, v_4, \dots, v_n$ respectively. $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = n$.

□

Theorem 3.8. *The graph $C_m \cup C_n$ is pair mean cordial for all $m \ge 3$ and $n \ge 4$.*

Proof. Let C_m be the cycle $u_1u_2 \dots u_mu_1$ and C_n be the cycle $v_1v_2 \dots v_nv_1$. Then the graph $C_m \cup C_n$ has $m+n$ vertices and $m+n$ edges. We have the following three cases arise:

Case (i): $m = 3$

Let us consider $\lambda(u_1) = 2$, $\lambda(u_2) = -1$ and $\lambda(u_3) = 3$. There are two subcases that arise:

Subcase (i): n is odd

Let $\lambda(v_1) = -2$, $\lambda(v_2) = 4$ and $\lambda(v_3) = -3$. Then we give the labels $-4, -5, \dots,$

$\frac{-m-n}{2}$ respectively to the vertices $v_4, v_6, \ldots, v_{n-1}$ and $5, 6, \ldots, \frac{m+n}{2}$ to the vertices $v_5, v_7, \ldots, v_{n-2}$ respectively. Finally assign the label 1 to the vertex v_n.

Subcase (ii): n is even

If $n = 4$, define $\lambda(v_1) = -2$, $\lambda(v_2) = -3$, $\lambda(v_3) = 1$ and $\lambda(v_4) = 1$. Therefore $\bar{\mathbb{S}}_{\lambda_1} = 3$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 4$.

If $n > 4$, Then we give the labels $-4, -5, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_4, v_6, \ldots, v_{n-2}$ and $5, 6, \ldots, \frac{m+n-1}{2}$ to the vertices $v_5, v_7, \ldots, v_{n-3}$ respectively. Also assign the labels $1, \frac{-m-n+1}{2}$ to the vertices v_{n-1}, v_n respectively.

Case (ii): m is odd

Let us take $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Then we give the labels $-3, -4, \ldots, \frac{-m+1}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_{m-2}$ and $4, 5, \ldots, \frac{m+1}{2}$ to the vertices $u_6, u_8, \ldots, u_{m-1}$ respectively. Next assign the label 1 to the vertex u_m. Then there are two subcases that arise:

Subcase (i): n is odd

In this case, we give the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively.

Subcase (ii): n is even

Also we give the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Thus assign the label $\frac{-m-n+1}{2}$ to the vertex v_n.

Case (iii): m is even

Let us take $\lambda(u_1) = 2$, $\lambda(u_2) = -1$, $\lambda(u_3) = 3$ and $\lambda(u_4) = -2$. Thus we give the labels $-3, -4, \ldots, \frac{-m}{2}$ respectively to the vertices $u_5, u_7, \ldots, u_{m-1}$ and $4, 5, \ldots, \frac{m+2}{2}$ to the vertices $u_6, u_8, \ldots, u_m$ respectively. There are two subcases that arise:

Subcase (i): n is odd

Now, we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n+1}{2}$ respectively to the a vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n-1}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Next we assign the labels $1, \frac{-m-n+1}{2}$ to the vertices v_{n-1}, v_n respectively.

Subcase (ii): n is even

In this case, we give the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Finally assign the label 1 to the vertex v_n.

The following table shows that this vertex labeling λ is a pair mean cordial of $C_m \cup C_n$ for all $m \geq 3$ and $n \geq 4$.

Nature of m and n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
m is odd and n is odd	$\frac{m+n}{2}$	$\frac{m+n}{2}$
m is odd and n is even	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$
m is even and n is odd	$\frac{m+n-1}{2}$	$\frac{m+n+1}{2}$
m is even and n is even	$\frac{m+n}{2}$	$\frac{m+n}{2}$

Table 6

□

Example 3.9. A pair mean cordial labeling of $C_8 \cup C_7$ is shown in Figure 2.

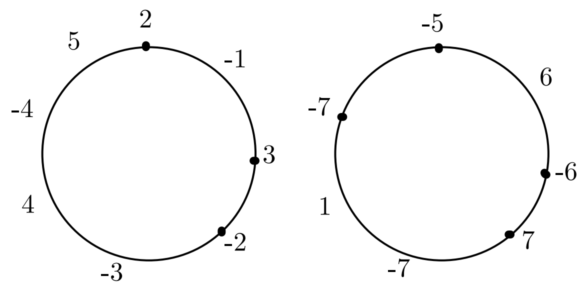

FIGURE 2

Remark 3.3. $C_3 \cup C_3$ is not a pair mean cordial graph.

Proof. Suppose that $C_3 \cup C_3$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges label with 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(2)=4$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4-2=2>1$, a contradiction.

□

Theorem 3.10. *The graph $W_m \cup C_n$ is pair mean cordial for all $m \geq 3$ and $n \geq 5$.*

Proof. Let $V(W_m \cup C_n) = \{u, u_i, v_j : 1 \leq i \leq m and 1 \leq j \leq n\}$ and $E(W_m \cup C_n) = \{uu_i : 1 \leq i \leq m and 1 \leq j \leq n\} \cup \{u_i u_{i+1}, u_m u_1, v_j v_{j+1}, v_n v_1 : 1 \leq i \leq m-1 and 1 \leq j \leq n-1\}$. Hence $W_m \cup C_n$ has $m+n+1$ vertices and $2m+n$ edges. We have the following two cases arise:

Case (i): m is odd

Now, we give the labels $2, 3, \ldots, \frac{m+3}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_m$ and $-1, -2, \ldots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. Next assign the label $\frac{-m-1}{2}$ to the vertex u. There are two subcases that arise:

Subcase (i): n is odd

Then we assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-3}$ respectively. Finally, assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively.

Subcase (ii): n is even

Then assign the labels $\frac{m+5}{2}, \frac{-m-3}{2}, \frac{m+7}{2}, \frac{-m-5}{2}$ respectively to the vertices v_1, v_2, v_3, v_4. Next, we assign the labels $\frac{-m-7}{2}, \frac{-m-9}{2}, \ldots, \frac{-m-n-1}{2}$ respectively to the vertices $v_5, v_7, \ldots, v_{n-1}$ and $\frac{m+9}{2}, \frac{m+11}{2}, \ldots, \frac{m+n+1}{2}$ to the vertices $v_6, v_8, \ldots, v_{n-2}$ respectively. Finally, we assign the label 1 to the vertex v_n.

Case (ii): m is even

First, we give the labels $2, 3, \ldots, \frac{m+2}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $-1, -2, \ldots, \frac{-m}{2}$ to the vertices $u_2, u_4, \ldots, u_m$ respectively. There are two subcases that arise:

Subcase (i): n is odd

Now assign the label $\frac{m+4}{2}$ to the vertex u. Then we assign the labels $\frac{-m-2}{2}, \frac{m+6}{2}, \frac{-m-4}{2}$ respectively to the vertices v_1, v_2, v_3. More over we assign the labels $\frac{-m-6}{2}, \frac{-m-8}{2}, \ldots, \frac{-m-n-1}{2}$ respectively to the vertices $v_4, v_6, \ldots, v_{n-1}$ and $\frac{m+8}{2}, \frac{m+10}{2}, \ldots, \frac{m+n+1}{2}$ to the vertices $v_5, v_7, \ldots, v_{n-2}$ respectively. Finally, assign the label 1 to the vertex v_n.

Subcase (ii): n is even

In this case, assign the label $\frac{m+2}{2}$ to the vertex u. Then we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Hence assign the label 1 to the vertex v_n.

The following table shows that this vertex labeling λ is a pair mean cordial of $W_m \cup C_n$ for all $m \geq 3$ and $n \geq 5$.

□

Nature of m and n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd and n is odd	$\frac{2m+n-1}{2}$	$\frac{2m+n+1}{2}$
m is odd and n is even	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
m is even and n is odd	$\frac{2m+n-1}{2}$	$\frac{2m+n+1}{2}$
m is even and n is even	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$

Table 7

Remark 3.4. The graph $W_m \cup C_3$ is pair mean cordial iff m is odd.

Proof. If m is odd, let us assign labels to the vertices as $u, u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq 3$ in case (i) of subcase (i) of theorem 3.7. If m is even, assume $W_m \cup C_3$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges label with 1 is m. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - m = m + 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m + 3 - m = 3 > 1$, a contradiction.

□

Remark 3.5. The graph $W_m \cup C_4$ is pair mean cordial iff m is even.

Proof. If m is even, assign labels to the vertices as $u, u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq 4$ in case (ii) of subcase (ii) of theorem 3.7. If m is odd, suppose

$W_m \cup C_4$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges label with 1 is $m+1$. That is $\bar{\mathbb{S}}_{\lambda_1} \le m+1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \ge q-(m+1)=m+3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1} \ge m+3-(m+1)=2>1$, a contradiction. □

Theorem 3.11. *The graph $S_m \cup S_n$ is not a pair mean cordial graph for all $m, n \ge 4$.*

Proof. Let us define $V(S_m \cup S_n)=\{u_i, v_j : 1 \le i \le m \text{ and } 1 \le \mathrm{j} \le \mathrm{n}\}$ and $E(S_m \cup S_n)=\{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, v_nv_1 : 1 \le i \le m-1 \text{ and } 1 \le \mathrm{j} \le \mathrm{n}-1\} \cup \{\mathrm{u_1u_{i+2}, v_1v_{j+2} : 1 \le i \le m-3 and\ 1 \le j \le n-3}\}$. Hence $S_m \cup S_n$ has $m+n$ vertices and $2m+2n-6$ edges. Suppose $S_m \cup S_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges label with 1 is $m+n-4$. That is $\bar{\mathbb{S}}_{\lambda_1} \le m+n-4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \ge q-(m+n-4)=m+n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1} \ge m+n-2-(m+n-4)=2>1$, a contradiction.

□

Theorem 3.12. *The graph $S_m \cup W_n$ is pair mean cordial for all $m \ge 4$ and $n \ge 3$ except for $m+n$ is odd.*

Proof. Let $V(S_m \cup W_n)=\{u_i, v, v_j : 1 \le i \le m \text{ and } 1 \le \mathrm{j} \le \mathrm{n}\}$ and $E(S_m \cup W_n)=\{vv_j : 1 \le j \le n\} \cup \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, v_nv_1 : 1 \le i \le m-1 and 1 \le j \le n-1\} \cup \{v_1v_{j+2} : 1 \le j \le n-3\}$. Hence $S_m \cup W_n$ has $m+n+1$ vertices and $2m+2n-3$ edges. We have the following two cases arise:

Case (i): $m+n$ is odd

Suppose that $S_m \cup W_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges label with 1 is $m+n-3$. That is $\bar{\mathbb{S}}_{\lambda_1} \le m+n-3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \ge m+n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1} \ge m+n-(m+n-3)=3>1$, a contradiction.

Case (ii): $m+n$ is even

There are two subcases that arise:

Subcase (i): m and n is even

In this case, we give the labels $2, 3, \ldots, \frac{m+2}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-1}$ and $-1, -2, \ldots, \frac{-m+2}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-2}$ respectively. More over assign the label 1 to the vertex u_m. Next we give the labels $\frac{-m}{2}, \frac{-m-2}{2}, \ldots, \frac{-m-n+2}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-1}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, \frac{m+n}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Furthermore assign the label $\frac{-m-n}{2}$ to the vertex v_n. Finally, assign the label $\frac{m+n}{2}$ to the vertex v.

Subcase (ii): m and n is odd

First give the labels $2, 3, \ldots, \frac{m+1}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{m-2}$ and $-1, -2, \ldots, \frac{-m+3}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-3}$ respectively. More over assign the labels $\frac{-m-n}{2}, 1$ to the vertices $u_{m-1}u_m$ respectively. Next we give the labels $\frac{-m+1}{2}, \frac{-m-1}{2}, \ldots, \frac{-m-n+2}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and $\frac{m+3}{2}, \frac{m+5}{2}, \ldots, \frac{m+n-2}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Furthermore assign the label $\frac{m+n}{2}$ to the vertex v. In both cases, $\bar{\mathbb{S}}_{\lambda_1} = m+n-2$ and $\bar{\mathbb{S}}_{\lambda_1^c} = m+n-1$.

□

Example 3.13. A pair mean cordial labeling of $S_9 \cup W_9$ is shown in Figure 3.

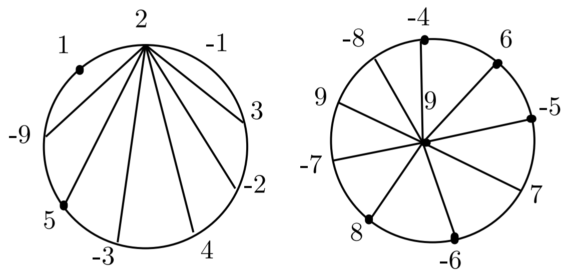

FIGURE 3

Theorem 3.14. *The graph $W_m \cup W_n$ is not a pair mean cordial graph for all $m, n \geq 3$.*

Proof. Let $V(W_m \cup W_n) = \{u, v, u_i, v_j : 1 \leq i \leq m \text{ and } 1 \leq \mathrm{j} \leq \mathrm{n}\}$ and $E(W_m \cup W_n) = \{uu_i, u_m u_1, vv_j, v_n v_1 : 1 \leq i \leq m \text{ and } 1 \leq \mathrm{j} \leq \mathrm{n}\} \cup \{\mathrm{u_i u_{i+1}}, \mathrm{v_j v_{j+1}} : 1 \leq \mathrm{i} \leq \mathrm{m}-1 \text{ and } 1 \leq \mathrm{j} \leq \mathrm{n}-1\}$. Hence $W_m \cup W_n$ has $m+n+2$ vertices and $2m+2n$ edges. We have the following two cases arise:

Case (i): $m+n$ is odd

Suppose $W_m \cup W_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label with 1 is $m+n-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m+n-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(m+n-1) = m+n-1$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m+n-1-(m+n-1) = 2 > 1$, a contradiction.

Case (ii): $m+n$ is even

Assume that $W_m \cup W_n$ is pair mean cordial. Now if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v) = 1$ or $\lambda(u)+\lambda(v) = 2$. Hence the maximum number of edges label with 1 is $m+n-2$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m+n-2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(m+n-2) = m+n+2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m+n+2-(m+n-2) = 4 > 1$, a contradiction. □

4. Discussion

Since its initial proposal in 1987 by Cahit[3], cordial labeling is now a popular field of research in graph labeling. Many authors examined the different kinds of cordial labeling in [1,2,4,5,8-13,19-23]. The concept of mean labeling was introduced in [24], while the pair difference cordial labeling was first proposed in [13]. Our introduction of the pair mean cordial labeling in [14] was motivated by these two concepts. The current paper presents the results of the pair mean cordial labeling behavior of union of few graphs, which include $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $P_m \cup W_n$, $C_m \cup C_n$, $C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$ and $S_m \cup W_n$.

5. Limitation of Research

Investigating the pair mean cordial labeling behavior of the scorpion graph, spider graph, generalized Peterson graph, generalized Heawood graph, cubic diamond k-chain graph, swastik graph, broken wheel graph and n-cube graph on a large number of vertices is currently challenging to study.

6. Future Research

Future research to examine the pair mean cordial labeling behavior of union of path with other graphs like bull graph, shackle graph, jahangir graph, olive graph, coconut graph, step ladder and shadow graph.

7. Conclusion

In this paper, we have investigated the pair mean cordial labeling behavior of some union of graphs such as $P_m \cup P_n$, $P_m \cup C_n$, $P_m \cup S_n$, $P_m \cup W_n$, $C_m \cup C_n$,$C_m \cup S_n$, $W_m \cup W_n$, $W_m \cup C_n$, $S_m \cup S_n$ and $S_m \cup W_n$. Future research should focus on examining the pair mean cordial labeling behavior of many graphs including generalized web graph, banana tree, x-tree, coconut tree, windmill graph, lollipop graph, broom graph and polar grid graph.

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Data availability : Not applicable

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