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PHENOMENA AND PROPERTIES OF ROOTS OF BERNOULLI-FIBONACCI POLYNOMIALS

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Abstract. In this paper, we investigate the distribution of the zeros of the Bernoulli-Fibonacci polynomials by using computer.

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1. Introduction

In this paper, we investigate the distribution of zeros of the Bernoulli-Fibonacci polynomials by using computer. Throughout this paper, we always make use of the following notations: $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

The authors [1, 2, 4] introduced generating functions for Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$B as follow

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}.$$

Now, we give some definitions (for these definitions see [11, 12]) that we will use throughout the article. The F-factorial is defined as

$$F_n! = F_n \cdot F_{n-1} \cdot F_{n-2} \cdots F_1, \quad F_0! = 1.$$

where F_n is n-th Fibonacci numbers. The Fibonomial coefficients are defined as $(0 \le k \le n)$ as

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!}$$

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with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$.

The binomial theorem for the F-analogues (or-Golden binomial theorem) are given by

$$(x+y)_F^n = \sum_{k=0}^{n} (-1)^{\binom{n}{2}} \binom{n}{k}_F x^{n-k} y^k$$

The F-exponential functions $e_F(x)$ and $E_F(x)$ are defined as:

$$e_F(x) = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}, \quad E_F(x) = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{x^n}{F_n!}.$$

The following identity holds

$$e_F^x E_F^x = e_F^{(x+y)_F}$$

The author [6] defined generating functions for Bernoulli-Fibonacci numbers $B_{n,F}$ and Bernoulli-Fibonacci polynomials $B_{n,F}(x)$B as follow

$$\sum_{n=0}^{\infty} B_{n,F} \frac{t^n}{F_n!} = \frac{t}{e_F(t)-1},$$

$$\sum_{n=0}^{\infty} B_{n,F}(x) \frac{t^n}{F_n!} = \frac{t}{e_F(t)-1} e_F(xt).$$

Theorem 1.1. *For $n \geq 1$, we have*

$$(1) \quad B_{n,F}(x) = \sum_{l=0}^{n} \binom{n}{l}_F B_{l,F} x^{n-l}.$$

$$(2) \quad \sum_{l=0}^{n-1} \binom{n}{l}_F B_{l,F}(x) = F_n x^{n-1}.$$

$$(3) \quad \sum_{l=0}^{n} \binom{n}{l}_F B_{l,F}(x) - B_{n,F}(x) = F_n x^{n-1}.$$

$$(4) \quad B_{n,F}(1) = B_{n,F} \quad (n = 2, 3, \ldots).$$

For the first few Bernoulli-Fibonacci numbers we have,

$$B_{0,F} = 1, \quad B_{1,F} = -1, \quad B_{2,F} = \frac{1}{2}, \quad B_{3,F} = -\frac{1}{3},$$
$$B_{4,F} = \frac{3}{10}, \quad B_{5,F} = -\frac{5}{8}, \quad B_{6,F} = \frac{101}{39}, \quad B_{7,F} = -\frac{323}{21},$$
$$B_{8,F} = \frac{21041}{170}, \quad B_{9,F} = -\frac{101485}{66}, \quad B_{10,F} = \frac{10,73481135}{2136},$$
$$B_{11,F} = -\frac{196504771}{144}, \quad B_{12,F} = \frac{1289116727296}{15145}, \quad B_{13,F} = -\frac{3016576720478}{377},$$
$$B_{14,F} = \frac{3041059589599217}{2562}, \quad B_{15,F} = -\frac{388541104908511145}{1316}.$$

2. Zeros of the Bernoulli-Fibonacci polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Bernoulli-Fibonacci polynomials $B_{n,F}(x)$. The Bernoulli-Fibonacci polynomials $B_{n,F}(x)$. can be determined explicitly. A few of them are

$$B_{0,F}(x) = 1,$$
$$B_{1,F}(x) = -1 + x,$$
$$B_{2,F}(x) = \frac{1}{2} - x + x^2,$$
$$B_{3,F}(x) = -\frac{1}{3} + x - 2x^2 + x^3,$$
$$B_{4,F}(x) = \frac{3}{10} - x + 3x^2 - 3x^3 + x^4,$$
$$B_{5,F}(x) = -\frac{5}{8} + \frac{3x}{2} - 5x^2 + \frac{15x^3}{2} - 5x^4 + x^5,$$
$$B_{6,F}(x) = \frac{101}{39} - 5x + 12x^2 - 20x^3 + 20x^4 - 8x^5 + x^6.$$
$$B_{7,F}(x) = -\frac{323}{21} + \frac{101x}{3} - 65x^2 + 78x^3 - \frac{260x^4}{3} + 52x^5 - 13x^6 + x^7,$$
$$B_{8,F}(x) = \frac{21041}{170} - 323x + 707x^2 - \frac{1365x^3}{2} + 546x^4 - 364x^5 + \frac{273x^6}{2} - 21x^7 + x^8,$$
$$B_{9,F}(x) = -\frac{101485}{66} + \frac{21041x}{5} - 10982x^2 + 12019x^3 - 7735x^4 + \frac{18564x^5}{5} - 1547x^6 + 357x^7 - 34x^8 + x^9,$$
$$B_{10,F}(x) = \frac{73481135}{2136} - \frac{507425x}{6} + 231451x^2 - 302005x^3 + \frac{661045x^4}{3} - 85085x^5 + \frac{51051x^6}{2} - 6545x^7 + 935x^8 - 55x^9 + x^{10},$$
$$B_{11,F}(x) = -\frac{196504771}{144} + \frac{73481135x}{24} - \frac{45160825x^2}{6} + \frac{20599139x^3}{2} - \frac{26878445x^4}{3} + \frac{11766601x^5}{3} - \frac{7572565x^6}{8} + \frac{349503x^7}{2} - \frac{83215x^8}{3} + \frac{4895x^9}{2} - 89x^{10} + x^{11}.$$

We investigate the zeros of the Bernoulli-Fibonacci polynomials $B_{n,F}(x) = 0$. by using a computer. We plot the zeros of the Bernoulli-Fibonacci polynomials $B_{n,F}(x) = 0$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 10$.

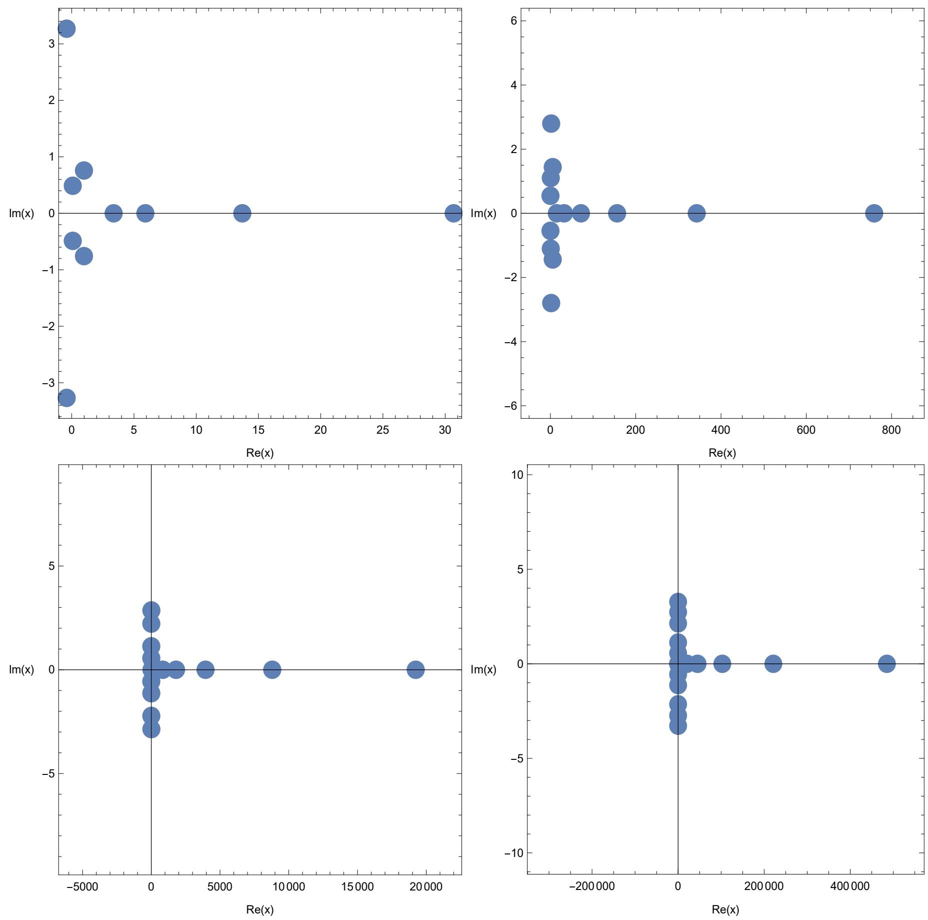

FIGURE 1. Zeros of $B_{n,F}(x) = 0$

In Figure 1(top-right), we choose $n = 20$. In Figure 1(bottom-left), we choose $n = 30$. In Figure 1(bottom-right), we choose $n = 40$.

Stacks of zeros of the Bernoulli polynomials $B_n(x) = 0$ for $1 \leq n \leq 40$ from a 3-D structure are presented(Figure 2).

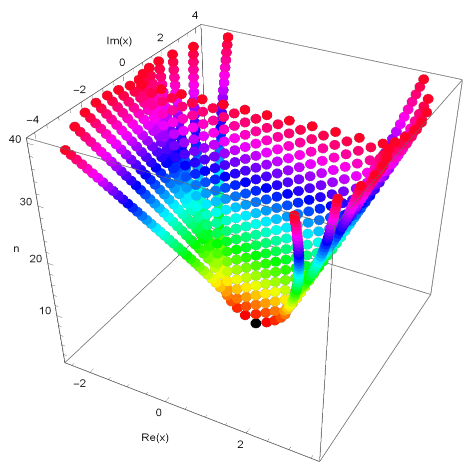

FIGURE 2. Stacks of zeros of $B_n(x) = 0$ for $1 \leq n \leq 40$

Stacks of zeros of the Bernoulli-Fibonacci polynomials $B_{n,F}(x) = 0$ for $1 \leq n \leq 40$ from a 3-D structure are presented(Figure 3).

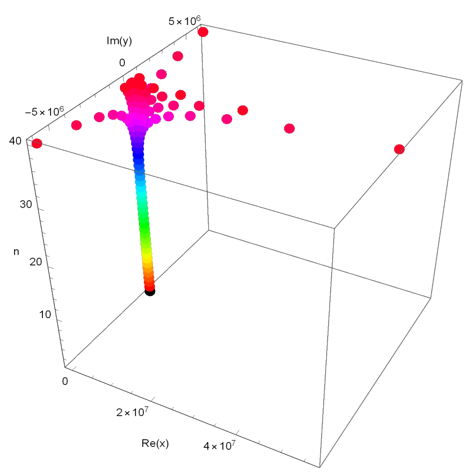

FIGURE 3. Stacks of zeros of $B_{n,F}(x) = 0$ for $1 \leq n \leq 40$

The plot of real zeros of Bernoulli polynomials $B_n(x) = 0$ for $1 \leq n \leq 40$ structure are presented(Figure4).

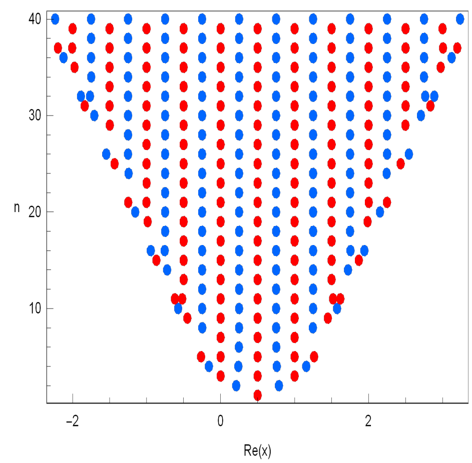

FIGURE 4. Real zeros of $B_n(x) = 0 = 0$ for $1 \leq n \leq 40$

The plot of real zeros of Bernoulli-Fibonacci polynomials $B_{n,F}(x) = 0$ for $1 \leq n \leq 40$ structure are presented(Figure 5).

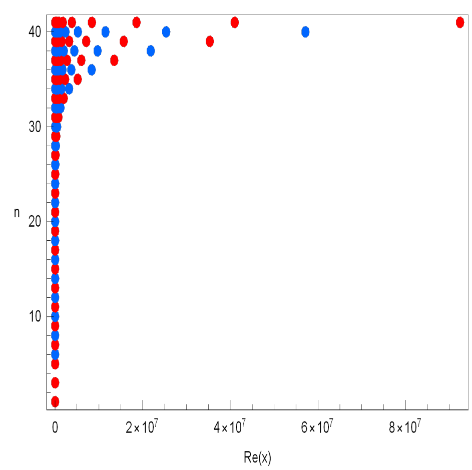

FIGURE 5. Real zeros of $B_{n,F}(x) = 0 = 0$ for $1 \leq n \leq 40$

Next, we calculated an approximate solution satisfying Bernoulli-Fibonacci polynomials $B_{n,F}(x) = 0$ for $x \in \mathbb{C}$. The results are given in Table 1.

Table 1. Approximate solutions of $B_{n,F}(x) = 0$

degree n	x
1	1.0000
2	$0.50000 - 0.50000i$, $0.50000 + 0.50000i$
3	$0.26234 - 0.39638i$, $0.26234 + 0.39638i$, 1.4753
4	$0.14713 - 0.35231i$, $0.14713 + 0.35231i$, $1.3529 - 0.4773i$, $1.3529 + 0.4773i$
5	$0.04058 - 0.41592i$, $0.04058 + 0.41592i$, $0.95161 - 0.53031i$, $0.95161 + 0.53031i$, 3.0156
6	$-0.06994 - 0.54598i$, $-0.06994 + 0.54598i$, $0.71994 - 0.55355i$, $0.71994 + 0.55355i$, 2.4233, 4.2767
7	$-0.11416 - 0.76340i$, $-0.11416 + 0.76340i$, $0.47243 - 0.60161i$, $0.47243 + 0.60161i$ 2.2038, 2.7203, 7.3594
8	$-0.15336 - 1.25842i$, $-0.15336 + 1.25842i$, $0.24703 - 0.50658i$, $0.24703 + 0.50658i$, $1.8748 - 0.5775i$, $1.8748 + 0.5775i$ 5.4031, 11.660
9	$-0.2542 - 2.0107i$, $-0.2542 + 2.0107i$, $0.15483 - 0.45665i$, $0.15483 + 0.45665i$, $1.3733 - 0.7612i$, $1.3733 + 0.7612i$ 4.1436, 8.2847, 19.024

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

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