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FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN OPERATOR

CHOUKRI DERBAZI, ABDELKRIM SALIM*, HADDA HAMMOUCHE,
MOUFFAK BENCHOHRA

Abstract. In this paper, we study the existence of solutions for hybrid fractional differential equations with p-Laplacian operator involving fractional Caputo derivative of arbitrary order. This work can be seen as an extension of earlier research conducted on hybrid differential equations. Notably, the extension encompasses both the fractional aspect and the inclusion of the p-Laplacian operator. We build our analysis on a hybrid fixed point theorem originally established by Dhage. In addition, an example is provided to demonstrate the effectiveness of the main results.

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1. Introduction

Fractional differential equations have garnered a lot of significance and interest seeing as they have been shown to be effective tools in the modeling of a wide range of phenomena in engineering and sciences [1, 2, 3, 17, 22, 23, 29, 32, 25, 26, 27, 28, 19, 7, 8, 4, 5, 6].

Numerous authors have investigated fractional-order boundary value problems with the p-Laplacian operator recently. We direct the reader to [16, 15, 20, 30] and the sources within.

The importance of hybrid differential equations arises from the fact that they incorporate various dynamic systems as particular cases. The perturbations of original differential equations in various ways are included in this class of hybrid

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*Corresponding author.

fractional differential equations. For some recent advancements in the hybrid equations, see [21, 14, 31, 24, 13] and the references therein.

Motivated by the preceding publications, in this study, we investigate the existence of solutions to the following problem:

$$\begin{cases} {}^cD^{\alpha_2}_{0^+}\left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-\sum_{\iota=1}^{n} I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right) = f(t,y(t)), \;\; t\in J=[0,1], \\ \left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-\sum_{\iota=1}^{n} I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)^{(i)}\Big|_{t=0} = 0, \quad i=0,2,3\ldots,m-1, \\ \left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-\sum_{\iota=1}^{n} I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)\Big|_{t=1} = 0, \\ \left(\frac{y(t)-\sum_{\iota=1}^{n} I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)^{(\jmath)}\Big|_{t=0} = 0, \quad \text{for } \jmath=2,3\ldots,m-1, \\ {}^cD^{\xi}_{0^+}\left[\frac{y(t)-\sum_{\iota=1}^{n} I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right]_{t=1} = 0, \quad y(0)=0, \end{cases} \tag{1}$$

where ${}^cD^{\varsigma}_{0^+}$ is the Caputo fractional derivative of order $\varsigma \in \{\alpha_1,\alpha_2,\xi\}$ such that $m-1<\alpha_1,\alpha_2\le m,\ 0<\xi\le 1, I^r_{0^+}$ is the Riemann–Liouville fractional integral of order $r>0, r\in\{\wp_1,\wp_2,\ldots,\wp_n\}, \phi_p(q)$ is a p-Laplacian operator, i.e., $\phi_p(q)=|q|^{p-2}q$ for $p>1, \phi_p^{-1}=\phi_\ell$ where $\frac{1}{p}+\frac{1}{\ell}=1$ and $\varphi\in C\big(J\times\mathbb{R},\mathbb{R}\setminus\{0\}\big), f\in C\big(J\times\mathbb{R},\mathbb{R}\big), h_\iota\in C\big(J\times\mathbb{R},\mathbb{R}\big), 0<\wp_\iota, \iota=1,2,\ldots,n$.

The following is the structure of the paper. Section 2 introduce some notations, definitions and lemmas that will be used later. Then, in Section 3, we present our main existence results. Finally, an example is given to show the effectiveness of the main results.

2. Preliminaries

We start by giving some necessary concepts and findings that will be needed for forthcoming advancements in this study.

Definition 2.1 ([17])**.** The Riemann–Liouville fractional integral of order $\alpha_1>0$ for an integrable function h defined by

$$I^{\alpha_1}_{0^+}h(t)=\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\varrho)^{\alpha_1-1}h(\varrho)\,\mathrm{d}\varrho, \quad \alpha_1>0,$$

exists almost everywhere on $[0,b]$.

Definition 2.2 ([17])**.** Let $\alpha_1>0, m=[\alpha_1]+1$. If $h\in AC^m([0,b])$, then the Caputo fractional derivative of order α_1 defined by

$$ {}^{c}D_{0^+}^{\alpha_1}h(t) = \frac{1}{\Gamma(m-\alpha_1)}\int_0^t (t-\varrho)^{m-\alpha_1-1}h^{(m)}(\varrho)d\varrho, $$

exists almost everywhere on $[0,b]$.

Lemma 2.3 ([17]). *Let $\alpha_1, \eta > 0$, $m = [\alpha_1]+1$, then the following relation holds*

$$ {}^{c}D_{0^+}^{\alpha_1}t^{\eta} = \begin{cases} \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha_1+1)}t^{\eta-\alpha_1}, & (\eta > m-1), \\ \\ 0, \quad \eta \in \{0,\dots,m-1\}. \end{cases} $$

Lemma 2.4 ([17]). *Let $\alpha_1 > \alpha_2 > 0$, and $h \in L^1([0,b])$. Then*

- $I_{0^+}^{\alpha_1}I_{0^+}^{\alpha_2}h(t) = I_{0^+}^{\alpha_1+\alpha_2}h(t)$,
- ${}^{c}D_{0^+}^{\alpha_1}I_{0^+}^{\alpha_1}h(t) = h(t)$,
- ${}^{c}D_{0^+}^{\alpha_2}I_{0^+}^{\alpha_1}h(t) = I_{0^+}^{\alpha_1-\alpha_2}h(t)$.

Lemma 2.5 ([17]). *Let $\alpha_1 > 0$ and $h \in AC^n([0,b])$, then the equation*

$$ \left({}^{c}D_{0^+}^{\alpha_1}h\right)(t) = 0 $$

has solution

$$ h(t) = \sum_{\jmath=0}^{n-1}\varepsilon_j t^{\jmath}, \quad \varepsilon_j \in \mathbb{R}, \jmath = 0\dots n-1, $$

where $n-1 < \alpha_1 < n$.

Lemma 2.6 ([17]). *Let $\alpha_1 > 0$ and $h \in AC^n([0,b])$; then*

$$ I_{0^+}^{\alpha_1}\left({}^{c}D_{0^+}^{\alpha_1}h(t)\right) = h(t) + \sum_{\jmath=0}^{n-1}\varepsilon_j t^{\jmath}, $$

for some $\varepsilon_j \in \mathbb{R}, \jmath = 0,1,2,\dots,n-1$, where $n = [\alpha_1]+1$.

Lemma 2.7 ([17]). *For all $\alpha_1 > 0$ and $r > -1$, we have*

$$ \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\varrho)^{\alpha_1-1}\varrho^r \mathrm{d}\varrho = \frac{\Gamma(r+1)}{\Gamma(\alpha_1+r+1)}t^{\alpha_1+r} \quad t \geq 0. $$

The next lemma has an important role in this paper.

Lemma 2.8 ([9, 18]). *Let $\beta_1, \beta_2 \in \mathbb{R}, \theta > 0$*

(i) If $0 < \theta \leq 1$, then

$$ (|\beta_1| + |\beta_2|)^{\theta} \leq |\beta_1|^{\theta} + |\beta_2|^{\theta}. $$

(ii) If $\theta > 1$, then

$$ (|\beta_1| + |\beta_2|)^{\theta} \leq 2^{\theta-1}(|\beta_1|^{\theta} + |\beta_2|^{\theta}). $$

By Lemma 2.8, we observe that if $\theta > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$ then

$$ (|\beta_1| + |\beta_2|)^{\theta} \leq \max\{1, 2^{\theta-1}\}(|\beta_1|^{\theta} + |\beta_2|^{\theta}). $$

3. Main Results

Let $E = C(J,\mathbb{R})$ be the Banach space of all continuous functions from J to $\mathbb{R}$ with the norm

$$\|y\| = \sup_{t\in J} |y(t)|,$$

and and the multiplication in E by

$$(yw)(t) = y(t)w(t).$$

Lemma 3.1 ([10]). *Let Ω be a closed convex, bounded and nonempty subset of a Banach algebra E, and let $S_1, S_3 : E \longrightarrow E$ and $S_2 : \Omega \longrightarrow E$ be three operators such that*

(i) S_1 and S_3 are Lipschitzian with Lipschitz constants ξ_1 and ξ_2 , respectively;
(ii) S_2 is compact and continuous;
(iii) $y = S_1yS_2w + S_3y \Rightarrow y \in \Omega$ for all $w \in \Omega$;
(iv) $\xi_1\gamma + \xi_2 < 1$ where $\gamma = \big\|S_2(\Omega)\big\|$.

Then the equation $S_1yS_2y + S_3y = y$ has a solution in Ω.

For the sake of brevity, we pose:

$$\begin{aligned}
\varpi_1 &= \frac{(\Gamma(\alpha_2))^{1-\ell}}{\Gamma(\alpha_1)},\\
\varpi_2 &= \frac{\Gamma(2-\xi)(\Gamma(\alpha_2))^{1-\ell}}{\Gamma(\alpha_1-\xi)},\\
\mathcal{M}_f &= (\Gamma(\alpha_2+1))^{1-\ell}\max\{1,2^{\ell-2}\}\|p_f\|\varkappa(r)\left[\frac{\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1+\alpha_2(\ell-1)+1)}+\frac{\Gamma(\ell)}{\Gamma(\alpha_1+\ell)}\right.\\
&\quad\left.+\frac{\Gamma(2-\xi)\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1-\xi+\alpha_2(\ell-1)+1)}+\frac{\Gamma(2-\xi)\Gamma(\ell)}{\Gamma(\alpha_1-\xi+\ell)}\right].
\end{aligned}\tag{2}$$

Lemma 3.2. *The solution of*

$$ {}^cD^{\alpha_2}_{0^+}\left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-\sum_{\iota=1}^n I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right) = f(t),\quad t\in J := [0,1], \tag{3}$$

with boundary conditions

$$\begin{cases}
\left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)^{(i)}\Big|_{t=0} = 0, \quad i = 0,2,3\ldots,m-1,\\[2ex]
\left(\phi_p\left[{}^cD^{\alpha_1}_{0^+}\left(\frac{y(t)-h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)\Big|_{t=1} = 0,\\[2ex]
\left(\frac{y(t)-\sum_{\iota=1}^n I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right)^{(\jmath)}\Big|_{t=0} = 0, \quad \text{for } \jmath = 2,3\ldots,m-1,\\[2ex]
{}^cD^{\xi}_{0^+}\left[\frac{y(t)-\sum_{\iota=1}^n I^{\wp_\iota}_{0^+}h_\iota(t,y(t))}{\varphi(t,y(t))}\right]_{t=1} = 0, \quad y(0)=0,
\end{cases}\tag{4}$$

is given by

$$y(t) = \Big\{ \varpi_1 \int_0^t (t-\varrho)^{\alpha_1 - 1} \phi_\ell \left(\int_0^\varrho (\varrho - \tau)^{\alpha_2 - 1} f(\tau) \mathrm{d}\tau - \varrho \int_0^1 (1-\tau)^{\alpha_2 - 1} f(\tau) \mathrm{d}\tau \right) \mathrm{d}\varrho$$
$$- \varpi_2 \int_0^1 (1-\varrho)^{\alpha_1 - \xi - 1} \phi_\ell \left(\int_0^\varrho (\varrho - \tau)^{\alpha_2 - 1} f(\tau) \mathrm{d}\tau - \varrho \int_0^1 (1-\tau)^{\alpha_2 - 1} f(\tau) \mathrm{d}\tau \right) \mathrm{d}\varrho \Big\}$$
$$\times \varphi(t, y(t))$$
$$+ \sum_{\iota=1}^n \frac{1}{\Gamma(\wp_\iota)} \int_0^t (t-\varrho)^{\wp_\iota - 1} h_\iota(\varrho, y(\varrho)) \mathrm{d}\varrho,$$

Proof. Step 1. The problem

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\alpha_1} \left[\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right] = f(t), \\ \left(\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right)^{(\jmath)} \Big|_{t=0} = 0, \quad \text{for } \jmath = 2, 3 \ldots, m-1, \\ {}^c\mathcal{D}_{0^+}^{\xi} \left[\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right]_{t=1} = 0, \quad y(0) = 0, \end{cases} \tag{5}$$

has a unique solution satisfying

$$y(t) = \left[\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\varrho)^{\alpha_1 - 1} f(\varrho) \mathrm{d}\varrho - \frac{\Gamma(2-\xi) t}{\Gamma(\alpha_1 - \xi)} \int_0^1 (1-\varrho)^{\alpha_1 - \xi - 1} f(\varrho) \mathrm{d}\varrho \right]$$
$$\times \varphi(t, y(t)) + \sum_{\iota=1}^n \frac{1}{\Gamma(\wp_\iota)} \int_0^t (t-\varrho)^{\wp_\iota - 1} h_\iota(\varrho, y(\varrho)) \mathrm{d}\varrho. \tag{6}$$

In fact, applying $I_{0^+}^{\alpha_1}(\cdot)$ to both sides of (5) and using Lemma 2.6, we obtain

$$\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} = I_{0^+}^{\alpha_1} f(t) + \sum_{\iota=1}^m d_\iota t^{\iota - 1}. \tag{7}$$

By the use of $\left(\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right)^{(\jmath)} \Big|_{t=0} = 0$, for $\jmath = 2, 3 \ldots, m-1$ we get $d_3 = d_4 = \cdots = d_m = 0$ and hence (7) takes the form

$$\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} = I_{0^+}^{\alpha_1} f(t) + d_1 + d_2 t. \tag{8}$$

Since $y(0) = 0$ then $d_1 = 0$. Applying the operator ${}^c\mathcal{D}_{0^+}^{\xi}(\cdot)$ to (8), we get

$${}^c\mathcal{D}_{0^+}^{\xi} \left[\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right] = I_{0^+}^{\alpha_1 - \xi} f(t) + d_2 \frac{t^{1-\xi}}{\Gamma(2-\xi)}.$$

The boundary condition ${}^c\mathcal{D}_{0^+}^{\xi} \left[\frac{y(t) - \sum_{\iota=1}^n I_{0^+}^{\wp_\iota} h_\iota(t, y(t))}{\varphi(t, y(t))} \right]_{t=1} = 0$, implies

$$d_2 = -\Gamma(2-\xi) I_{0^+}^{\alpha_1 - \xi} f(1).$$

Thus, (8) becomes

$$\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\wp_\iota} h_\iota(t,y(t))}{\varphi(t,y(t))} = I_{0^+}^{\alpha_1} f(t) - \Gamma(2-\xi) t I_{0^+}^{\alpha_1-\xi} f(1),$$

which implies that

$$\begin{aligned} y(t) = &\left[\frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\varrho)^{\alpha_1-1} f(\varrho)\mathrm{d}\varrho - \frac{\Gamma(2-\xi)t}{\Gamma(\alpha_1-\xi)}\int_0^1 (1-\varrho)^{\alpha_1-\xi-1} f(\varrho)\mathrm{d}\varrho\right] \\ &\times \varphi(t,y(t)) + \sum_{\iota=1}^{n} \frac{1}{\Gamma(\wp_\iota)}\int_0^t (t-\varrho)^{\wp_\iota-1} h_\iota(\varrho,y(\varrho))\mathrm{d}\varrho. \end{aligned}$$

Step 2. Let $w = {}^cD_{0^+}^{\alpha_1}\left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\wp_\iota} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)$ and $\aleph = \phi_p(w)$. Clearly, $w = \phi_\ell(\aleph)$. Then, the solution of the following boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\alpha_2}\aleph(t) = f(t), \ \ t \in J, \\ \\ \aleph(0) = \aleph''(0) = \aleph^{(3)}(0) = \cdots = \aleph^{(m-1)}, \quad \aleph(1) = 0, \end{cases} \tag{9}$$

can be written as

$$\aleph(t) = I_{0^+}^{\alpha_2} f(t) - t I_{0^+}^{\alpha_2} f(1), \tag{10}$$

In fact, applying the Riemann-Liouville fractional integral operator of order α_2 to both sides of (9) and using Lemma 2.6, we have

$$\aleph(t) = I_{0^+}^{\alpha_2} f(t) + \sum_{\iota=1}^{m} \varepsilon_\iota t^{\iota-1}. \tag{11}$$

Using $\aleph^{(i)}(0) = 0$, for $i = 0, 2, 3 \ldots, m-1$ we get $\varepsilon_1 = \varepsilon_3 = \cdots = \varepsilon_m = 0$ and hence (11) becomes

$$\aleph(t) = I_{0^+}^{\alpha_2} f(t) + \varepsilon_2 t, \tag{12}$$

and $\aleph(1) = 0$, gives

$$\varepsilon_2 = -I_{0^+}^{\alpha_2} f(1).$$

Therefore, we have

$$\aleph(t) = I_{0^+}^{\alpha_2} f(t) - t I_{0^+}^{\alpha_2} f(1). \tag{13}$$

Consequently, the solution of (1) verifies

$$\begin{cases} {}^cD_{0^+}^{\alpha_1}\left[\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\wp_\iota} h_\iota(t,y(t))}{\varphi(t,y(t))}\right] = \phi_p^{-1}\left(I_{0^+}^{\alpha_2} f(t) - t I_{0^+}^{\alpha_2} f(1)\right), \ \ t \in J, \\ \left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\wp_\iota} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)^{(\jmath)}\Bigg|_{t=0} = 0, \quad \text{for } \jmath = 2, 3 \ldots, m-1, \\ \\ {}^cD_{0^+}^{\xi}\left[\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\wp_\iota} h_\iota(t,y(t))}{\varphi(t,y(t))}\right]_{t=1} = 0, \quad y(0) = 0, \end{cases} \tag{14}$$

As in Step 1, the solution of (14) can be written as:

$$\begin{aligned}y(t) = &\bigg\{\varpi_1 \int_0^t (t-\varrho)^{\alpha_1-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau - \varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau\right)\mathrm{d}\varrho\\ &-\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau - \varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau\right)\mathrm{d}\varrho\bigg\}\\ &\times\varphi(t,y(t))\\ &+\sum_{\iota=1}^n \frac{1}{\Gamma(\wp_\iota)}\int_0^t (t-\varrho)^{\wp_\iota-1}h_\iota(\varrho,y(\varrho))\mathrm{d}\varrho.\end{aligned}$$

□

Now we list some hypotheses as follows:

(Cd_1) The functions $\varphi : J\times\mathbb{R}\longrightarrow\mathbb{R}\setminus\{0\}$, f, and $h_\iota : J\times\mathbb{R}\longrightarrow\mathbb{R}$, $\iota = 1,2,3\dots,n$, are continuous.

(Cd_2) There exist positive functions $\xi_\varphi(t)$ and $\lambda_\iota(t)$, $\iota = 1,2,3\dots,n$, with bound $\|\xi_\varphi\|$ and $\|\lambda_\iota\|$, $\iota = 1,2,3\dots,n$, respectively, where

$$\big|h_\iota(t,y)-h_\iota(t,w)\big| \le \lambda_\iota(t)|y-w|, \quad \iota = 1,2,3\dots,n, \tag{15}$$

and

$$\big|\varphi(t,y)-\varphi(t,w)\big| \le \xi_\varphi(t)|y-w|, \text{ for each } (t,y,w)\in J\times\mathbb{R}\times\mathbb{R}. \tag{16}$$

(Cd_3) There exist a function $p_f\in L^\infty(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\varkappa:\mathbb{R}_+\longrightarrow\mathbb{R}_+-\{0\}$ such that

$$\big|f\big(t,y\big)\big| \le \phi_p(p_f(t)\varkappa(|y|)), \text{ for each } t\in J \text{ and all } y\in\mathbb{R}. \tag{17}$$

Theorem 3.3. *Assume that conditions (Cd_1)-(Cd_3) hold. If*

$$\|\xi_\varphi\|\mathcal{M}_f + \sum_{\iota=1}^n \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)} < 1, \tag{18}$$

then the problem (1) has at least one solution on J.

Proof. Let us consider the subset Ω of E given by

$$\Omega = \big\{y\in E : \|y\|_E \le r\big\},$$

where

$$r \ge \frac{\Phi_0\mathcal{M}_f + \sum_{\iota=1}^n \frac{\Psi_\iota}{\Gamma(\wp_\iota+1)}}{1-\|\xi_\varphi\|\mathcal{M}_f - \sum_{\iota=1}^n\frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)}}, \tag{19}$$

and

$$\Phi_0 = \sup_{t\in J}|\varphi(t,0)|, \Psi_\iota = \sup_{t\in J}|h_\iota(t,0)|, \quad \iota = 1,2,3\dots,n. \tag{20}$$

Clearly Ω is closed, convex and bounded subset of the Banach space E. By Lemma 3.2, problem (1) is equivalent to the equation

$$y(t) = \varphi(t,y(t))\bigg\{\varpi_1\int_0^t (t-\varrho)^{\alpha_1-1}\phi_\ell\bigg(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau$$

$$
\begin{aligned}
&-\varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau\Big)\,\mathrm{d}\varrho\\
&-\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau\right.\\
&\left.-\varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau)\mathrm{d}\tau\right)\mathrm{d}\varrho\Big\}\\
&+\sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t(t-\varrho)^{\wp_\iota-1}h_\iota(\varrho,y(\varrho))\mathrm{d}\varrho. \qquad (21)
\end{aligned}
$$

Consider the operators $S_1, S_3 : E \longrightarrow E$ and $S_2 : \Omega \longrightarrow E$ given by

$$S_1y(t) = \varphi(t, y(t)),\ t \in J.$$

$$
\begin{aligned}
S_2y(t) = \varpi_1&\int_0^t(t-\varrho)^{\alpha_1-1}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}f(\tau,y(\tau))\mathrm{d}\tau\right.\\
&\left.-\varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}\varrho\\
&-\varpi_2 t\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}f(\tau,y(\tau))\mathrm{d}\tau\right.\\
&\left.-\varrho\int_0^1(1-\tau)^{\alpha_2-1}f(\tau,y(\tau))\mathrm{d}\tau\right)\mathrm{d}\varrho,\ t\in J,
\end{aligned}
$$

and

$$S_3y(t) = \sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t(t-\varrho)^{\wp_\iota-1}h_\iota(\varrho,y(\varrho))\mathrm{d}\varrho,\ t\in J.$$

Then, (21) can be written as

$$y(t) = S_1y(t)S_2y(t) + S_3y(t), \quad t \in J.$$

We will demonstrate that S_1, S_2 and S_3 verify all the requirements of Lemma 3.1.

Step 1: We prove that S_1 and S_3 are Lipschitzian on E. Let $y, w \in E$, then by (Cd_2), for $t \in J$, we get

$$|S_1y(t) - S_1w(t)| = \big|\varphi(t, y(t)) - \varphi(t, w(t))\big| \le \xi_\varphi(t)|y(t) - w(t)|,$$

for all $t \in J$. Thus

$$\|S_1y - S_1w\| \le \|\xi_\varphi\|\|y - w\|,$$

for all $y, w \in E$. Therefore, S_1 is a Lipschitzian on E with Lipschitz constant $\|\xi_\varphi\|$.

Now for $S_3 : E \longrightarrow E, y, w \in E$, we have

$$|S_3y(t) - S_3w(t)| = \left|\sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t(t-\varrho)^{\wp_\iota-1}h_\iota(\varrho,y(\varrho))\mathrm{d}\varrho\right.$$

$$
\begin{aligned}
&-\sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t (t-\varrho)^{\wp_\iota-1}h_\iota(\varrho,w(\varrho))\mathrm{d}\varrho\Bigg| \\
&\le \sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t (t-\varrho)^{\wp_\iota-1}|h_\iota(\varrho,y(\varrho))-h_\iota(\varrho,w(\varrho))|\mathrm{d}\varrho \\
&\le \sum_{\iota=1}^{n}\frac{1}{\Gamma(\wp_\iota)}\int_0^t (t-\varrho)^{\wp_\iota-1}\lambda_\iota(\varrho)|y(\varrho)-w(\varrho)|\mathrm{d}\varrho \\
&\le \|y-w\|\sum_{\iota=1}^{n}\frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)},
\end{aligned}
$$

for all $t \in J$. Then we have

$$
\|S_3 y - S_3 w\| \le \sum_{\iota=1}^{n}\frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)}\|y-w\|,
$$

Hence, $S_3 : E \longrightarrow E$ is a Lipschitzian on E with Lipschitz constant $\sum_{\iota=1}^{n}\frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)}$.
Step 2: We demonstrate that S_2 is a completely continuous operator from Ω into E. The continuity of S_2 follows from the continuity of f and $\phi_\ell(\cdot)$.

Now, we will show that the $S_2(\Omega)$ is a uniformly bounded in Ω. For any $y \in \Omega$, we have

$$
\begin{aligned}
|S_2y(t)| \le{}& \varpi_1 \int_0^t (t-\varrho)^{\alpha_1-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right. \\
&\left.+\varrho\int_0^1 (1-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}\varrho \\
&+\varpi_2 \int_0^1 (1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right. \\
&\left.+\varrho\int_0^1 (1-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}\varrho \\
\le{}& \varpi_1 \int_0^t (t-\varrho)^{\alpha_1-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}\phi_p(p_f(\tau)\varkappa(|y(\tau)|))\mathrm{d}\tau\right. \\
&\left.+\varrho\int_0^1 (1-\tau)^{\alpha_2-1}\phi_p(p_f(\tau)\varkappa(|y(\tau)|))\mathrm{d}\tau\right)\mathrm{d}\varrho \\
&+\varpi_2 \int_0^1 (1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}\phi_p(p_f(\tau)\varkappa(|y(\tau)|))\mathrm{d}\tau\right. \\
&\left.+\varrho\int_0^1 (1-\tau)^{\alpha_2-1}\phi_p(p_f(\tau)\varkappa(|y(\tau)|))\mathrm{d}\tau\right)\mathrm{d}\varrho \\
\le{}& \varpi_1\|p_f\|\varkappa(\|y\|) \\
&\times\int_0^t (t-\varrho)^{\alpha_1-1}\phi_\ell\left(\int_0^\varrho (\varrho-\tau)^{\alpha_2-1}\mathrm{d}\tau + \varrho\int_0^1 (1-\tau)^{\alpha_2-1}\mathrm{d}\tau\right)\mathrm{d}\varrho
\end{aligned}
$$

$$
\begin{aligned}
&+\varpi_2\|p_f\|\varkappa(\|y\|)\\
&\times\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}\mathrm{d}\tau+\varrho\int_0^1(1-\tau)^{\alpha_2-1}\mathrm{d}\tau\right)\mathrm{d}\varrho\\
\le{}&\|p_f\|\varkappa(\|y\|)\left[\varpi_1\int_0^t(t-\varrho)^{\alpha_1-1}\phi_\ell\left(\frac{\varrho^{\alpha_2}}{\alpha_2}+\frac{\varrho}{\alpha_2}\right)\mathrm{d}\varrho\right.\\
&\left.+\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\frac{\varrho^{\alpha_2}}{\alpha_2}+\frac{\varrho}{\alpha_2}\right)\mathrm{d}\varrho\right]\\
\le{}&\frac{\|p_f\|\varkappa(\|y\|)}{{\alpha_2}^{\ell-1}}\left[\varpi_1\int_0^t(t-\varrho)^{\alpha_1-1}\left(\varrho^{\alpha_2}+\varrho\right)^{\ell-1}\mathrm{d}\varrho\right.\\
&\left.+\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\left(\varrho^{\alpha_2}+\varrho\right)^{\ell-1}\mathrm{d}\varrho\right]\\
\le{}&\frac{\max\{1,2^{\ell-2}\}\|p_f\|\varkappa(\|y\|)}{{\alpha_2}^{\ell-1}}\left[\varpi_1\int_0^t(t-\varrho)^{\alpha_1-1}\left(\varrho^{\alpha_2(\ell-1)}+\varrho^{\ell-1}\right)\mathrm{d}\varrho\right.\\
&\left.+\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\left(\varrho^{\alpha_2(\ell-1)}+\varrho^{\ell-1}\right)\mathrm{d}\varrho\right]\\
\le{}&(\Gamma(\alpha_2+1))^{1-\ell}\max\{1,2^{\ell-2}\}\|p_f\|\varkappa(r)\\
&\times\left[\frac{\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1+\alpha_2(\ell-1)+1)}+\frac{\Gamma(\ell)}{\Gamma(\alpha_1+\ell)}\right.\\
&\left.+\frac{\Gamma(2-\xi)\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1-\xi+\alpha_2(\ell-1)+1)}+\frac{\Gamma(2-\xi)\Gamma(\ell)}{\Gamma(\alpha_1-\xi+\ell)}\right].
\end{aligned}
$$

Thus $\|S_2y\|\le\mathcal{M}_f$ with $\mathcal{M}_f$ given in (2) for all $y\in\Omega$. Consequently, S_2 is uniformly bounded on Ω. Moreover, we have

$$
\begin{aligned}
|(S_2y)'(t)|\le{}&(\alpha_1-1)\varpi_1\int_0^t(t-\varrho)^{\alpha_1-2}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right.\\
&\left.+\varrho\int_0^1(1-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}\varrho\\
&+\varpi_2\int_0^1(1-\varrho)^{\alpha_1-\xi-1}\phi_\ell\left(\int_0^\varrho(\varrho-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right.\\
&\left.+\varrho\int_0^1(1-\tau)^{\alpha_2-1}|f(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}\varrho
\end{aligned}
$$

Some computations give

$$
\begin{aligned}
|(S_2y)'(t)|\le{}&(\Gamma(\alpha_2+1))^{1-\ell}\max\{1,2^{\ell-2}\}\|p_f\|\varkappa(r)\\
&\times\left[\frac{\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1-1+\alpha_2(\ell-1))}+\frac{\Gamma(\ell)}{\Gamma(\alpha_1-1+\ell)}\right.\\
&\left.+\frac{\Gamma(2-\xi)\Gamma(\alpha_2(\ell-1)+1)}{\Gamma(\alpha_1-\xi+\alpha_2(\ell-1)+1)}+\frac{\Gamma(2-\xi)\Gamma(\ell)}{\Gamma(\alpha_1-\xi+\ell)}\right]:=\bar{\gamma}
\end{aligned}
$$

Now, for $t_1, t_2 \in J$ with $t_1 < t_2$, we get

$$|S_2y(t_2) - S_2y(t_1)| \leq \int_{t_1}^{t_2} |(S_2y)'(\varrho)| \, \mathrm{d}\varrho \leq \bar{\gamma}(t_2 - t_1)$$

Therefore, S_2 is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator S_2 is completely continuous.

Step 3: The condition (iii) of Lemma 3.1 is verified.
Let $y \in E$ and $w \in \Omega$ where $y = S_1 y S_2 w + S_3 y$. Then we have

$$\begin{aligned}
|y(t)| &\leq |S_1y(t)| \, |S_2w(t)| + |S_3y(t)| \\
&\leq |\varphi(t, y(t))| \Bigg[\varpi_1 \int_0^t (t-\varrho)^{\alpha_1 - 1} \phi_\ell \Bigg(\int_0^\varrho (\varrho - \tau)^{\alpha_2 - 1} |f(\tau, w(\tau))| \mathrm{d}\tau \\
&\quad + \varrho \int_0^1 (1-\tau)^{\alpha_2 - 1} |f(\tau, w(\tau))| \mathrm{d}\tau \Bigg) \mathrm{d}\varrho \\
&\quad + \varpi_2 \int_0^1 (1-\varrho)^{\alpha_1 - \xi - 1} \phi_\ell \Bigg(\int_0^\varrho (\varrho - \tau)^{\alpha_2 - 1} |f(\tau, w(\tau))| \mathrm{d}\tau \\
&\quad + \varrho \int_0^1 (1-\tau)^{\alpha_2 - 1} |f(\tau, w(\tau))| \mathrm{d}\tau \Bigg) \mathrm{d}\varrho \Bigg] \\
&\quad + \sum_{\iota=1}^{n} \frac{1}{\Gamma(\wp_\iota)} \int_0^t (t-\varrho)^{\wp_\iota - 1} \big| h_\iota\big(t, y(t)\big) \big| \mathrm{d}\varrho \\
&\leq \Big(|\varphi(t, y(t)) - \varphi(t, 0))| + \big|\varphi\big(t, 0)\big)\big| \Big) \mathcal{M}_f \\
&\quad + \sum_{\iota=1}^{n} \frac{1}{\Gamma(\wp_\iota)} \int_0^t (t-\varrho)^{\wp_\iota - 1} \Big(\big| h_\iota\big(t, y(t)\big) - h_\iota\big(t, 0)\big) \big| + \big| h(t, 0) \big| \Big) \mathrm{d}\varrho, \\
&\leq \big(\|\xi_\varphi\| |y(t)| + \Phi_0 \big) \mathcal{M}_f + \sum_{\iota=1}^{n} \frac{|y(t)| \|\lambda_\iota\| + \Psi_\iota}{\Gamma(\wp_\iota + 1)}.
\end{aligned}$$

Thus,

$$|y(t)| \leq \frac{\Phi_0 \mathcal{M}_f + \sum_{\iota=1}^{n} \frac{\Psi_\iota}{\Gamma(\wp_\iota + 1)}}{1 - \|\xi_\varphi\| \mathcal{M}_f - \sum_{\iota=1}^{n} \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota + 1)}}.$$

Thus

$$\|y\| \leq \frac{\Phi_0 \mathcal{M}_f + \sum_{\iota=1}^{n} \frac{\Psi_\iota}{\Gamma(\wp_\iota + 1)}}{1 - \|\xi_\varphi\| \mathcal{M}_f - \sum_{\iota=1}^{n} \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota + 1)}} \leq r.$$

Step 4: Lastly, we prove that $\xi_1 \gamma + \xi_2 < 1$, that is, (iv) of Lemma 3.1 holds.
Since

$$\gamma = \|S_2(\Omega)\| = \sup_{y \in \Omega} \Big\{ \sup_{t \in J} |S_2 y(t)| \Big\} \leq \mathcal{M}_f,$$

and so

$$\|\xi_\varphi\|\gamma + \sum_{\iota=1}^{n} \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)} \le \|\xi_\varphi\|\mathcal{M}_f + \sum_{\iota=1}^{n} \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)} < 1,$$

with

$$\xi_1 = \|\xi_\varphi\|, \xi_2 = \sum_{\iota=1}^{n} \frac{\|\lambda_\iota\|}{\Gamma(\wp_\iota+1)}.$$

Consequently, all requirements of Lemma 3.1 are verified and hence the equation $y = S_1 y S_2 y + S_3 y$ has a solution in Ω. Thus, problem (1) has a solution on J. $\square$

4. An Example

In this part, we provide an illustration to demonstrate the applicability of our study results. Consider the problem for $t \in J$:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}\left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\frac{2k+1}{2}} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right) = \phi_5\left(\frac{1}{(t^2+2)^2}(1+\sin y(t))\right), \\ \left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\frac{2k+1}{2}} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)^{(i)}\Big|_{t=0} = 0, \quad i = 0, 2, \\ \left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\frac{2k+1}{2}} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)\right]\right)\Big|_{t=1} = 0, \\ \left(\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\frac{2k+1}{2}} h_\iota(t,y(t))}{\varphi(t,y(t))}\right)^{(2)}\Big|_{t=0} = 0, \\ {}^cD_{0^+}^{\frac{1}{2}}\left[\frac{y(t)-\sum_{\iota=1}^{n} I_{0^+}^{\frac{2k+1}{2}} h_\iota(t,y(t))}{\varphi(t,y(t))}\right]_{t=1} = 0, \quad y(0) = 0. \end{cases} \tag{22}$$

In this case we take

$$m = 3,\ p = 5,\ \ell = \frac{5}{4},\quad \alpha_1 = \frac{11}{4},\ \alpha_2 = \frac{5}{2},\ \xi = \frac{1}{2}, \wp_\iota = \frac{2k+1}{2}, \iota = 1, 2, \ldots, 10,$$

$$h_\iota\big(t, y(t)\big) = \frac{1}{2(t^2+\iota)^2}\left(y(t) + \sqrt{y^2(t)+1} + e^{-t}\right), \iota = 1, 2, \ldots, 10,$$

$$\varphi\big(t, y(t)\big) = \frac{e^{-2\pi t}\cos(\pi t)}{(e^t+9)^2}\frac{y(t)}{1+y(t)} + \frac{t}{10}$$

$$f\big(t, y(t)\big) = \phi_5\left(\frac{1}{(t^2+2)^2}(1+\sin y(t))\right).$$

We can show that

$$|h_\iota(t,y) - h_\iota(t,w)| \leq \frac{1}{(t^2+\iota)^2}|y-w|, \iota = 1,2,\ldots,10,$$

$$|\varphi(t,y) - \varphi(t,w)| \leq \frac{1}{(e^t+9)^2}|y-w|$$

hence, we have

$$\lambda_\iota(t) = \frac{1}{(t^2+\iota)^2},\ \xi_\varphi(t) = \frac{1}{(e^t+9)^2}.$$

Then,

$$\|\lambda_\iota\| = \frac{1}{\iota^2},\ \|\xi_\varphi\| = \frac{1}{100}$$

$$\Psi_\iota = \sup_{t\in J}|h_\iota(t,0)| = \frac{1}{\iota^2},\ \Phi_0 = \sup_{t\in J}|\varphi(t,0)| = \frac{1}{10}.$$

On the other hand, For each $y \in \mathbb{R}$, $t \in J$ we have

$$\begin{aligned} |f(t,y)| &= \left|\phi_p\left(\frac{1}{(t^2+2)^2}(1+\sin y)\right)\right| \\ &\leq \phi_p\left(\frac{1}{(t^2+2)^2}(1+|y|)\right). \end{aligned}$$

Thus, (Cd_3) is verified with $\frac{1}{(t^2+2)^2}, t \in J$, and $\varkappa(y) = y+1,\ y \in \mathbb{R}_+$. By the Matlab program, we have that (19), (18) are followed with $r \in (0, 6.0086) \cup (119.6157, 135.6242)$. As all conditions of Theorem 3.3 are met, (22) has at least one solution on J.

5. Conclusions

In this research, our aim is to establish the existence of solutions for hybrid fractional differential equations with p-Laplacian operator involving fractional Caputo derivative. Our method for demonstrating the existence of solutions relies on the application of a fixed point theorem due to Dhage. To showcase the practical utility of our key findings and to illustrate that the conditions of our theorems can be met, we provide an illustrative example. Our results in the provided context are novel and add significantly to the literature on this emerging topic of research. Due to the small amount of publications on fractional hybrid differential equations, we believe there are several possible study paths such as coupled systems, problems with infinite delays, and many more.

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Choukri Derbazi is an Associate Professor at the faculty of exact sciences, Frères Mentouri University Constantine. Derbazi received the Ph.D. degree in Mathematics from the university of Ghardaia, Algeria, 2020. His research fields include fractional differential equations and inclusions, control theory and applications, etc.

Laboratoire Equations Différentielles, Department of Mathematics, Faculty of Exact Sciences, Frères Mentouri University Constantine 1, P.O. Box 325, Ain El Bey Way, Constantine, Algeria.
`e-mail: choukriedp@yahoo.com`

Abdelkrim Salim is an Associate Professor at the faculty of Technology, Hassiba Benbouali university of Chlef since 2022. Salim received the master's degree in functional analysis and differential equations from Djillali Liabès University, Algeria, 2016, and Ph.D. degree in mathematical analysis and applications from Djillali Liabes University of Sidi Bel Abbes, Algeria, 2021. His research fields include fractional differential equations and inclusions, control theory and applications, etc.

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.
Faculty of Technology, Hassiba Benbouali University, P.O. Box 151 Chlef 02000, Algeria.
`e-mail: salim.abdelkrim@yahoo.com, a.salim@univ-chlef.dz`

Hadda Hammouche is a Professor at the faculty of Sciences and Technology, University of Ghardaia since 2009. Mrs Hadda received the Ph.D. degree in differential equations from Djillali Liabes University of Sidi Bel Abbes, Algeria, 2009. Her research fields include fractional differential equations and inclusions, control theory and applications, etc.

Laboratory of Mathematics And Applied Sciences, University of Ghardaia, 47000. Algeria.
`e-mail: h.hammouche@yahoo.fr`

Mouffak Benchohra is a Full Professor at the department of mathematics, Djillali Liabes University of Sidi Bel Abbes since October 1994. Benchohra received the master's degree in Nonlinear Analysis from Tlemcen University, Algeria, 1994 and Ph.D. degree in Mathematics from Djillali Liabes University, Sidi Bel Abbes, Algeria. His research fields include fractional differential equations, evolution equations and inclusions, control theory and applications, etc.

Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria.
e-mail: benchohra@yahoo.com