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ON FULLY MODIFIED q-POLY-EULER NUMBERS AND POLYNOMIALS

C.S. RYOO

Abstract. In this paper, we define a new fully modified q-poly-Euler numbers and polynomials of the first type by using q-polylogarithm function. We derive some identities of the modified polynomials with Gaussian binomial coefficients. We also explore several relations that are connected with the q-analogue of Stirling numbers of the second kind.

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1. Introduction

Throughout this paper, we use the following notations : $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers, respectively.

For $n, q \in \mathbb{R}$, the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q},\ (q \neq 1).$$

We note that $\lim_{q\to 1}[x]_q = x$. The q-factorial of n of order k is defined as

$$[n]_q^{(k)} = [n]_q[n-1]_q \cdots [n-k+1]_q,\ \ (k = 1, 2, 3, \cdots),$$

where $[n]_q$ is q-number(see[**?**]-[8]). Specially, when $k = n$, it is reduced the q-factorial

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q. \tag{1.1}$$

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The q-analogues of the binomial coefficients that are called the Gaussian binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \geq k) \tag{1.2}$$

with q-factorial $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. For $k \in \mathbb{Z}$, the q-analogue of polylogarithm function $Li_{k,q}$ is known by

$$Li_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k} \text{ (see [2],[5],[6]).} \tag{1.3}$$

In [7], the q-exponential functions are defined as:

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}, \quad E_q(t) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_q!}. \tag{1.4}$$

Definition 1.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, q-Euler polynomials is defined by

$$\frac{[2]_q}{e_q(t)+1} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!},$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called q-Euler numbers.

A few of them are

$$E_{0,q}(x) = \frac{1+q}{2},$$

$$E_{1,q}(x) = -\frac{1}{4} - \frac{q}{4} + \frac{x}{2} + \frac{qx}{2},$$

$$\begin{aligned} E_{2,q}(x) = &-\frac{1}{8(1-q)} + \frac{q}{8(1-q)} + \frac{q^2}{8(1-q)} - \frac{q^3}{8(1-q)} - \frac{x}{4(1-q)} \\ &- \frac{qx}{4(1-q)} + \frac{q^2x}{4(1-q)} + \frac{q^3x}{4(1-q)} + \frac{x^2}{2(1-q)} - \frac{q^2x^2}{2(1-q)}. \end{aligned}$$

The (q, r, w)-Stirling numbers of the second kind $S_{q,r,w}(n+r+w, m+r+w)$ are defined by the following generating function

$$\sum_{n=m}^{\infty} S_{q,r,w}(n+r+w, m+r+w) \frac{t^n}{[n]_q!} = \frac{(e_q(t)-1)^m}{[m]_q!} e_q(rt) E_q(wt), \tag{1.5}$$

where $n, m \in \mathbb{Z}_+$ with $n \geq m \geq 0$(see [1],[3],[4]). Setting $r = w = 0$, we get

$$\sum_{n=m}^{\infty} S_q(n, m) \frac{t^n}{[n]_q!} = \frac{(e_q(t)-1)^m}{[m]_q!}. \tag{1.6}$$

In the following section, we define the fully modified q-poly-Euler numbers and polynomials of first type with Gaussian binomial coefficients. We also derive several identities with each other and investigate some properties that are concerned with (q, r, w)-Stirling numbers of the second kind, $S_{q,r,w}(n+r+w, k+r+w)$. In addition, we find the relationship with the generating function of the q-Stirling numbers of the second kind.

2. New fully modified q-poly-Euler polynomials

In this section, we construct a new fully modified q-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,q}(x)$ of the first type. Using the generating functions of the polynomials, we derive some identities that are related with the q-analogue of ordinary Euler polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we define fully modified q-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,q}(x)$ of the first type by

$$\frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt) = \sum_{n=0}^{\infty} \widetilde{E}^{(k)}_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{2.1}$$

When $x = 0$, $\widetilde{E}^{(k)}_{n,q} = \widetilde{E}^{(k)}_{n,q}(0)$ are called fully modified q-poly-Euler numbers of the first type.

Corollary 2.2. *If we set $q \to 1$ in definition 2.1, then we get the poly-Euler polynomials $E^{(k)}_n(x)$ and*

$$\frac{2Li_k(1-e^{-t})}{t(e^t+1)} e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_n(x) \frac{t^n}{n!}.$$

Theorem 2.3. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$. Then we have*

$$\widetilde{E}^{(k)}_{n,q}(x) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \widetilde{E}^{(k)}_{l,q}\, x^{n-l}. \tag{2.2}$$

Using the q-exponential functions, we introduce the following fully modified q-poly-Euler polynomials of the first type with two variables.

Definition 2.4. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$. We define the fully modified q-poly-Euler polynomials $\widetilde{E}^{(k)}_{n,q}(x, y)$ of the first type with two variables as below

$$\frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)-1)} e_q(xt) E_q(yt) = \sum_{n=0}^{\infty} \widetilde{E}^{(k)}_{n,q}(x, y) \frac{t^n}{[n]_q!}. \tag{2.3}$$

Theorem 2.5. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, then we have*

$$\widetilde{E}^{(k)}_{n,q}(x, y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} \widetilde{E}^{(k)}_{l,q}(x) y^{n-l}.$$

Proof. Let n be a nonnegative integer, $k \in \mathbb{Z}$ and $0 < q < 1$. Then we get

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x,y)\frac{t^n}{[n]_q!} &= \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt)E_q(yt)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l\end{bmatrix}_{p,q} \widetilde{E}_{l,q}^{(k)}(x) q^{\binom{n-l}{2}} y^{n-l}\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

Therefore, we have

$$\widetilde{E}_{n,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l\end{bmatrix}_{q} q^{\binom{n-l}{2}} \widetilde{E}_{l,q}^{(k)}(x) y^{n-l}.$$

□

The fully modified q-poly Euler polynomials of the first type with two variables is represented by the following formula.

Theorem 2.6. *For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $0 < q < 1$, we derive*

$$\widetilde{E}_{n,q}^{(k)}(x,y) - \widetilde{E}_{n,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n \\ l\end{bmatrix}_{q} q^{\binom{n-l}{2}} y^{n-l}\ \widetilde{E}_{l,q}^{(k)}(x).$$

Proof. Let $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$ and $0 < q < 1$. Using (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x,y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!}\\ &\quad= \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt)(E_q(yt)-1)\\ &\quad= \sum_{n=0}^{\infty} \widetilde{E}_{l,q}^{(k)}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} y^{n+1}\frac{t^{n+1}}{[n+1]_q!}\\ &\quad= \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\begin{bmatrix} n \\ l\end{bmatrix}_{q} q^{\binom{n-l}{2}} y^{n-l}\ \widetilde{E}_{l,q}^{(k)}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, it is obvious the above result for $n \in \mathbb{N}$. □

Specially, in the case $y = 1$, we get

$$\widetilde{E}_{n,q}^{(k)}(x,1) - \widetilde{E}_{n,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n \\ l\end{bmatrix}_{q} q^{\binom{n-l}{2}}\ \widetilde{E}_{l,q}^{(k)}(x).$$

Using (1.5), the modified q-Euler polynomials of the first type of two variables are represented by the q-Euler numbers and the (q, r, w)-Stirling numbers.

Theorem 2.7. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $q \in \mathbb{R}$ such that $0 < q < 1$, we get*

$$\widetilde{E}_{n,q}^{(k)}(x,y) = \sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n \\ a\end{bmatrix}_{q}\frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}[a+1]_q} S_{q,x,y}(a+1+x+y, l+x+y)E_{n-a,q}.$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, we have

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x,y)\frac{t^n}{[n]_q!} = \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt)E_q(yt)$$

$$= \sum_{n=0}^{\infty}\sum_{l=1}^{n+1} \frac{(-1)^{l+n+l}[l]_q!}{[l]_q^k} \frac{S_{q,x,y}(n+1+x+y, l+x+y)}{[n+1]_q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_q \frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}} \frac{S_{q,x,y}(a+1+x+y, l+x+y)}{[a+1]_q} E_{n-a,q}\frac{t^n}{[n]_q!}.$$

□

The modified q-polylogarithm function is represented by q-Stirling numbers as below

$$\begin{aligned}\frac{1}{t}Li_{k,q}(1-e_q(-t)) &= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e_q(-t))^l}{[l]_q^k} \\ &= \sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+1}[l-1]_q!}{[l]_q^{k-1}[n+1]_q}S_q(n+1,l)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.4}$$

Using (2.4), we can find the equation that contain the q-Euler polynomials and the q-Stirling numbers of the second kind.

Theorem 2.8. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$. Then we have*

$$\widetilde{E}_{n,q}^{(k)}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_q \frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}} \frac{S_q(a+1,l)}{[a+1]_q} E_{n-a,q}(x).$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < 1$, the modified q-polylogarithm function is expressed with the q-Stirling numbers of the second kind. Then we get

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{E}_{n,q}^{(k)}(x)\frac{t^n}{[n]_q!} &= \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(t)+1)} e_q(xt) \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n+1} \frac{(-1)^{l+n+l}[l]_q!}{[l]_{p,q}^k[n+1]_q} S_q(n+1,l)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_q \frac{(-1)^{l+a+1}[l-1]_q!}{[l]_q^{k-1}} \frac{S_q(a+1,l)}{[a+1]_q} E_{n-a,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

□

3. Zeros of the fully modified q-poly-Euler polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$. The fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ can be determined explicitly.

A few of them are

$$\widetilde{E}_{1,q}^{(k)}(x) = -\frac{3}{4} - \frac{q}{4} + \frac{1}{2[2]_q^{k-1}} + \frac{x}{2} + \frac{qx}{2},$$

$$\begin{aligned}\widetilde{E}_{2,q}^{(k)}(x) = {} & \frac{1}{8(1-q)} + \frac{q}{8(1-q)} - \frac{q^2}{8(1-q)} - \frac{q^3}{8(1-q)} - \frac{1}{4(1-q)[2]_q^{k-1}} \\ & + \frac{q^2}{4(1-q)[2]_q^{k-1}} - \frac{1}{(1-q)(1+q)[2]_q^{k-1}} + \frac{q^2}{(1-q)(1+q)[2]_q^{k-1}} \\ & + \frac{1}{2(1-q)(1+q+q^2)} - \frac{q^2}{2(1-q)(1+q+q^2)} + \frac{1}{2(1-q)^2(1+q+q^2)[3]_{p,q}^{k-1}} \\ & - \frac{q^2}{(1-q)^2(1+q+q^2)[3]_{p,q}^{k-1}} + \frac{q^4}{2(1-q)^2(1+q+q^2)[3]_{p,q}^{k-1}} - \frac{3x}{4(1-q)} \\ & - \frac{qx}{4(1-q)} + \frac{3q^2x}{4(1-q)} + \frac{q^3x}{4(1-q)} + \frac{x}{2(1-q)[2]_q^{k-1}} \\ & - \frac{q^2x}{2(1-q)[2]_q^{k-1}} + \frac{x^2}{2(1-q)} - \frac{q^2x^2}{2(1-q)}.\end{aligned}$$

We investigate the zeros of the fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ by using a computer. We plot the zeros of the q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ for $n = 20$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 20, q =$

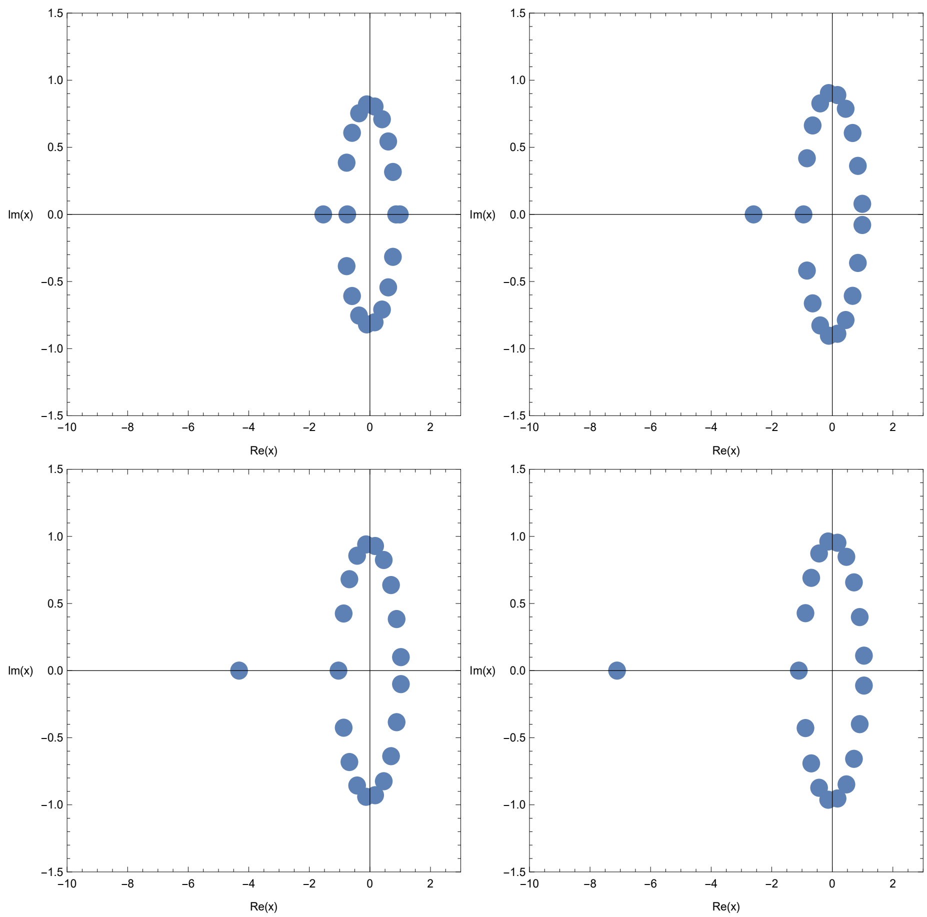

FIGURE 1. Zeros of $\widetilde{E}_{n,q}^{(k)}(x) = 0$

$1/10$, and $k = -5$. In Figure 1(top-right), we choose $n = 20, q = 1/10$, and $k = -10$. In Figure 1(bottom-left), we choose $n = 20, q = 1/10$, and $k = -15$. In Figure 1(bottom-right), we choose $n = 20, q = 1/10$, and $k = -20$.

We investigate the zeros of the fully modified q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ by using a computer. We plot the zeros of the (p,q)-poly-tangent polynomials $\widetilde{E}_{n,q}^{(k)}(x)$ for $n = 20$ and $x \in \mathbb{C}$(Figure 1). In Figure 2(top-left), we choose

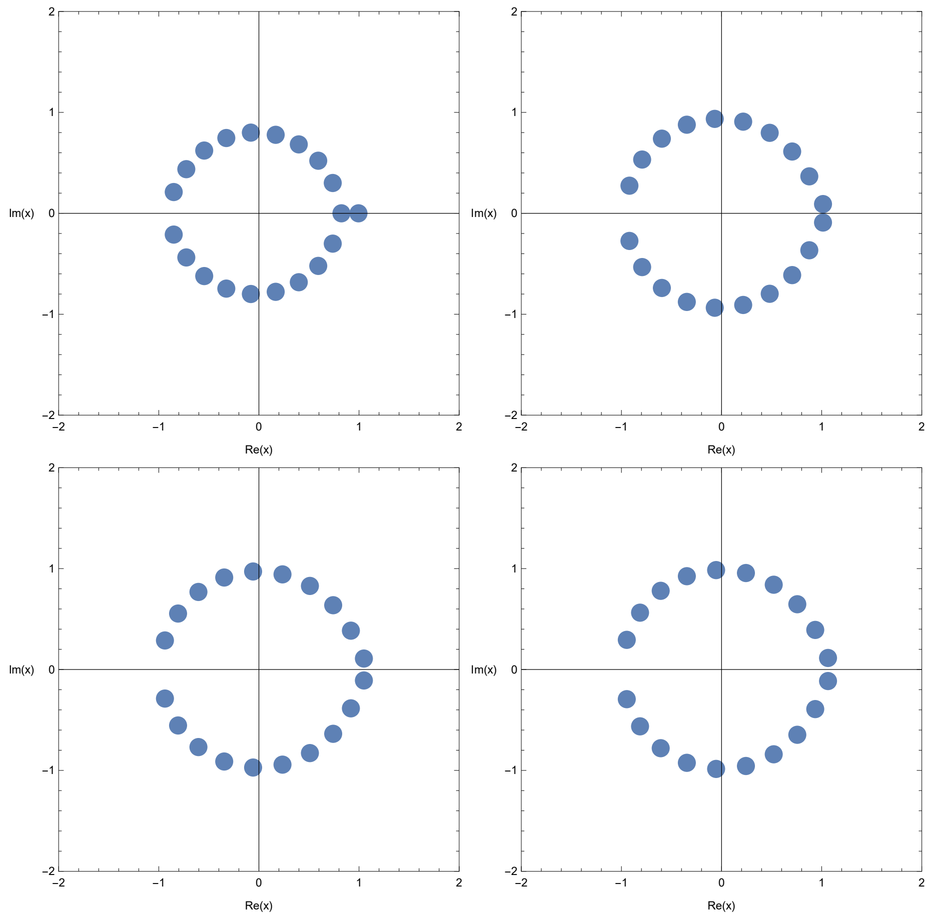

FIGURE 2. Zeros of $\widetilde{E}_{n,q}^{(k)}(x) = 0$

$n = 20, q = 7/10$, and $k = 1$. In Figure 2(top-right), we choose $n = 20, q = 7/10$, and $k = 10$. In Figure 2(bottom-left), we choose $n = 20, q = 7/10$, and $k = 20$. In Figure 2(bottom-right), we choose $n = 20, q = 7/10$, and $k = 30$.

The plot of real zeros of $\widetilde{E}_{n,q}^{(k)}(x) = 0$ for $1 \leq n \leq 20$ structure are presented(Figure 3).

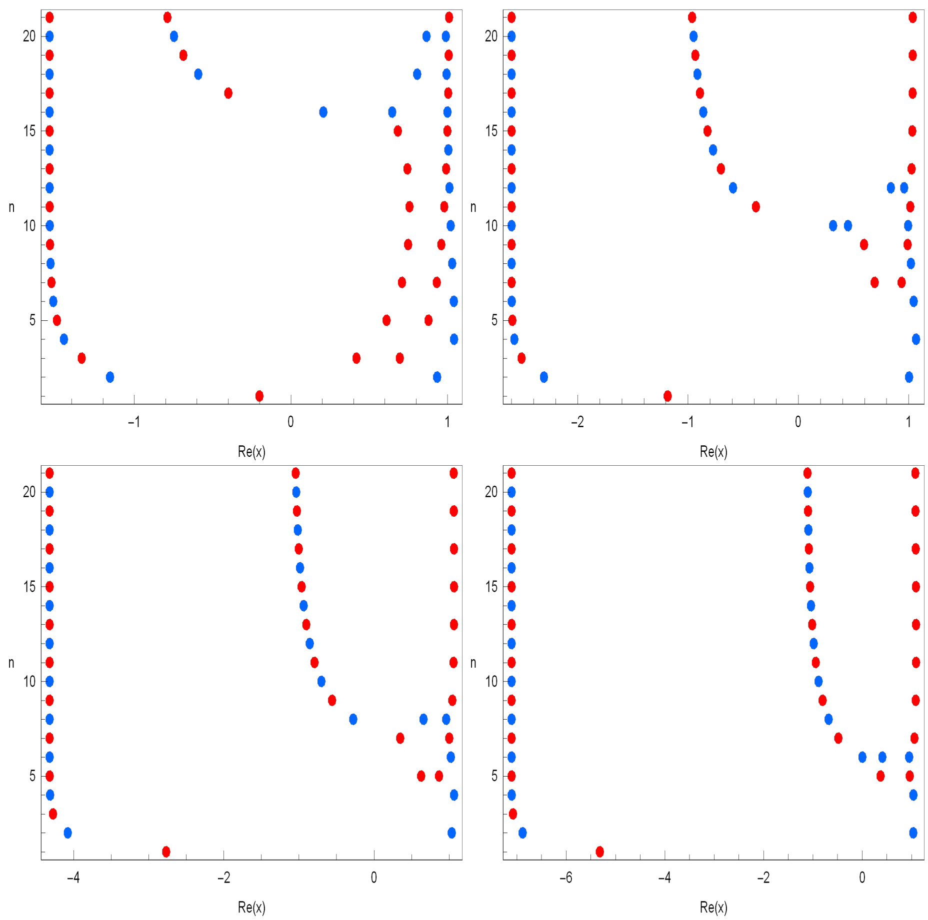

FIGURE 3. Real zeros of $\widetilde{E}_{n,q}^{(k)}(x) = 0$ for $1 \leq n \leq 20$

In Figure 3(top-left), we choose $q = 7/10$, and $k = -5$. In Figure 3(top-right), we choose $q = 7/10$, and $k = -10$. In Figure 3(bottom-left), we choose $q = 7/10$, and $k = -15$. In Figure 3(bottom-right), we choose $q = 7/10$, and $k = -20$.

Next, we calculated an approximate solution satisfying q-poly-Euler polynomials $\widetilde{E}_{n,q}^{(k)}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $\widetilde{E}_{n,q}^{(-5)}(x) = 0, q = 1/10$

degree n	x
1	0.78817
2	−1.1546, 0.93306
3	−1.3359, 0.41789, 0.69447
4	−1.4488, 1.0410
5	−1.4936, 0.61005, 0.87766
6	−1.51721.0396
7	−1.5286, 0.70853, 0.93062
8	−1.5346, 1.0290
9	−1.5377, 0.74775, 0.95948
10	−1.5392, 1.0193

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

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C.S. Ryoo received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing, p-adic functional analysis, and analytic number theory. More recently, he has been working with differential equations, dynamical systems, quantum calculus, and special functions.

Department of Mathematics, Hannam University, Daejeon 306-791, Korea.
e-mail: ryoocs@hnu.kr