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A NOTE ON VERTEX PAIR SUM k-ZERO RING LABELING

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Abstract. Let $G = (V, E)$ be a graph with p-vertices and q-edges and let R° be a finite zero ring of order n. An injective function $f : V(G) \rightarrow \{r_1, r_2, \ldots, r_k\}$, where $r_i \in R^\circ$ is called vertex pair sum k-zero ring labeling, if it is possible to label the vertices $x \in V$ with distinct labels from R° such that each edge $e = uv$ is labeled with $f(e = uv) = [f(u) + f(v)]$ (mod n) and the edge labels are distinct. A graph admits such labeling is called vertex pair sum k-zero ring graph. The minimum value of positive integer k for a graph G which admits a vertex pair sum k-zero ring labeling is called the vertex pair sum k-zero ring index denoted by $\psi_{pz}(G)$. In this paper, we defined the vertex pair sum k-zero ring labeling and applied to some graphs.

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1. Introduction

Numerous fields rely heavily on the field of graph theory. One of the key areas of graph theory is Graph labeling. Graph labeling is an assignment of integers to vertices or edges, or both, under certain conditions. Labeled graphs are effective mathematical models for a variety of applications. In 1977, Bloom and Golomb studied the applications of graph labeling. They discussed the detailed applications of graph labeling in [1]. A study on Group S_3 Cordial remainder labeling for path and cycle related graphs were done by A. Lourdusami,S. Jenifer Wency and F. Patrick [5]. For standard terminology of Graph theory, we used [4] and for all terminology regarding graph labeling, we follow [3].

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In 2014, Acharya et al. [7] introduced zero ring labeling. Dela Rosa-Reynera [6] constructed optimal zero ring labelings for some classes of graphs. In [2],discussed the efficient zero ring labeling of graphs. R. Ponraj et al. [8] discussed about pair sum graphs. Motivated from these studies we introduced a new notion of vertex labeling for graphs, called vertex pair sum k-zero ring labeling, is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the edges are labeled by the sum of the labels of the corresponding end vertices and are distinct. There are many interesting properties of zero ring mentioned in [7, 9]. In this paper, we discussed the vertex pair sum k-zero ring labeling of path, cycle, star graph, comb and bull graph.

2. Preliminaries

Definition 2.1. Let R be a ring with additive identity 0. If $xy = 0$ for any $x, y \in R$ then R is a zero ring.

Here we used the zero ring $M_2^{\circ}(R)$. Let R be a finite ring of order n with additive identity 0. We denote by $M_2^{\circ}(R)$ the set of all 2×2 matrices of the form $\begin{bmatrix} p & -p \\ p & -p \end{bmatrix}, p \in R$.

It can be verified that $M_2^{\circ}(R)$ is a ring under matrix addition and matrix multiplication with additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Since for any $p, q \in R$, $\begin{bmatrix} p & -p \\ p & -p \end{bmatrix} \begin{bmatrix} q & -q \\ q & -q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2^{\circ}(R)$ is a zero ring.

Since Z_n is a finite ring, it follows that $M_2^{\circ}(Z_n)$ is a zero ring. We use W_i to denote the matrix $\begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, i \in Z_n$.

Definition 2.2. Let $G = (V, E)$ be a graph with p vertices and q edges, and let R° be a finite zero ring of order n. An injective function f is called vertex pair sum k-zero ring labeling, if it is possible to label the vertices $x \in V$ with distinct labels from R° such that each edge $e = uv$ is labeled with $f(e = uv) = [f(u) + f(v)] \pmod{n}$ and the edge labels are distinct. A graph admits such labeling is called vertex pair sum k-zero ring graph.

The minimum value of positive integer k for a graph G which admits a vertex pair sum k-zero ring labeling is called the vertex pair sum k-zero ring index denoted by $\psi_{pz}(G)$.

Definition 2.3. The bull graph is a planar undirected graph with 5 vertices and 5 edges, in the form of a triangle with two disjoint pendant edges.

3. Main Results

Lemma 3.1. $\psi_{pz}(G) \geq |E|$, for any graph.

Proof. Let $\psi_{pz}(G) = n$ and $|E| = m$. If possible, let us assume that $\psi_{pz}(G) < m$. Take $W_{a_1}, W_{a_2}, \ldots, W_{a_n}$ as the vertex pair sum n-zero ring labeling.

Then the sums $W_{a_i} + W_{a_j}$, $i \neq j$, must be the distinct labels of edges.

If $\psi_{pz}(G) < m$, then there must be two same labels for distinct edges. So $\psi_{pz}(G) \geq |E|$. □

Lemma 3.2. $\psi_{pz}(G) \geq |V|$, for any graph.

In this study, the zero ring that will be used in the vertex labelings is the zero ring $M_2^{\circ}(Z_n)$, the set of all 2×2 matrices of the form $\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$, $a \in Z_n$.

Theorem 3.1. Any path $P_n, n \geq 2$ is a vertex pair sum n-zero ring graph.

Proof. Let P_n be the path $u_1, u_2, \ldots, u_n$. Edges are $u_i u_{i+1}$ for $i = 1, 2, 3, \ldots, (n-1)$.

Case (1): When n is even.

Define a function $f : V(P_n) \to M_2^{\circ}(Z_n)$ by

$$f(u_i) = \begin{cases} W_{\frac{i-2}{2}} & i \quad \text{is even,} \\ W_{\frac{n+i-1}{2}} & i \quad \text{is odd.} \end{cases}$$

The edge labels are as follows. $f(u_i u_{i+1}) = W_{\frac{n+2i-2}{2}}$. Then the edge labels are distinct elements of $M_2^{\circ}(Z_n)$.

Case (2): When n is odd.

Define a function $f : V(P_n) \to M_2^{\circ}(Z_n)$ by $f(u_i) = W_{i-1}$, $1 \leq i \leq n$.

$$f(u_i u_{i+1}) = W_{2i-1}, \quad 1 \leq i \leq (n-1).$$

Then the edge labels are distinct elements of $M_2^{\circ}(Z_n)$. Here f is a vertex pair sum n-zero ring labeling. i.e. $\psi_{pz}(P_n) \leq n$.

By Lemma (3.2)$\psi_{pz}(P_n) \geq n$.

$\therefore \psi_{pz}(P_n) = n$.

Hence we can conclude that for all integers $n \geq 2$, P_n is a vertex pair sum n-zero ring graph. □

Illustration 3.1. A vertex pair sum 5-zero ring labeling of P_5.

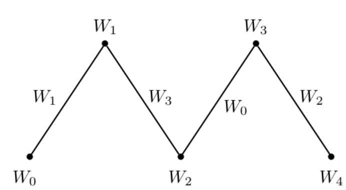
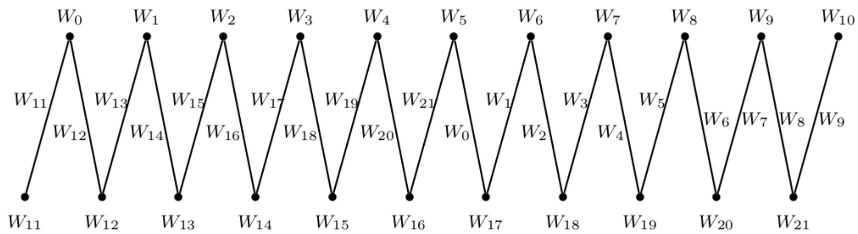


Illustration 3.2. A vertex pair sum 22-zero ring labeling of P_{22}.



Corollary 3.1. The vertex pair sum k-zero ring index of path graph P_n is n. That is $\psi_{pz}(P_n) = n$.

Theorem 3.2. The cycles $C_n, n = 2m + 1$, $m \in N$ are vertex pair sum k-zero ring graphs having $\psi_{pz}(C_n) = n$.

Proof. Let C_n be a cycle of length n such that $n = 2m + 1$, $m \in N$.

Let the cycles be $u_1u_2u_3 \ldots u_nu_1$. The edges are u_iu_{i+1}, $1 \leq i \leq n$ and u_nu_1.

Define a function $f : V(C_n) \to M_2^\circ(Z_n)$ by $f(u_i) = W_{i-1}$, $1 \leq i \leq n$.

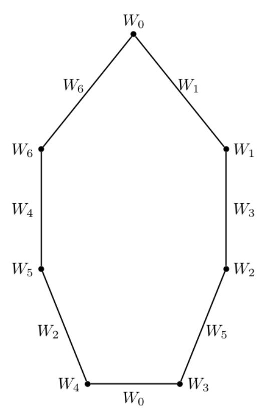
The edge labels are

$$f(u_iu_{i+1}) = W_{i-1} + W_i = W_{2i-1}, \quad 1 \leq i \leq n - 1.$$

and $f(u_nu_1) = W_{n-1} + W_0 = W_{n-1}$.

Then we get the edge labels as $W_0, W_1, W_2, \ldots, W_{n-1}$. Therefore the cycles C_n, $n = 2m + 1$, $m \in N$ are vertex pair sum k-zero ring graphs and $\psi_{pz}(C_n) = n$. □

Illustration 3.3. A vertex pair sum 7-zero ring labeling of C_7 is shown below.



Corollary 3.2. A vertex pair sum k-zero ring index of cycle C_n is n for odd n.

Theorem 3.3. The cycle C_n is not a vertex pair sum n-zero ring graph when n is even.

Proof. Since n is even, take $n = 2m$. Suppose C_{2m} admits vertex pair sum $2m$-zero ring labeling and let $W_{x_0}, W_{x_1}, \ldots, W_{x_{2m-1}}$ be the vertex pair sum $2m$-zero ring labeling. Note that in W_{x_i}, $i = 0, 1, 2, \ldots, 2m-1$, where x_i are members of Z_{2m}.

Also, $W_{x_i} + W_{x_j} = W_{x_i + x_j}$. Therefore, the sums $x_0 + x_1, x_1 + x_2, \ldots, x_{2m-1} + x_0$ congruent modulo $2m$.

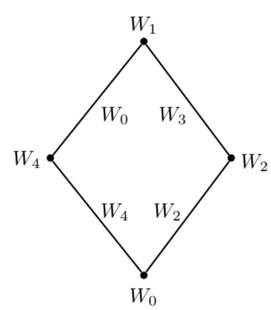
Then we get

$$
\begin{aligned}
& x_0 + x_1 \equiv 0 \pmod{2m} \\
& x_1 + x_2 \equiv 1 \pmod{2m} \\
& \vdots \\
& x_{2m-1} + x_0 \equiv (2m-1) \pmod{2m} \\
& \therefore\ x_0 + x_1 + x_2 + \cdots + x_{2m-1} + x_0 \equiv (0 + 1 + 2 + \cdots + (2m-1)) \pmod{2m} \\
& \Rightarrow 2(x_0 + x_1 + x_2 + \cdots + x_{2m-1}) \equiv (0 + 1 + 2 + \cdots + (2m-1)) \pmod{2m} \\
& \Rightarrow 2 \equiv 0 \pmod{2m}
\end{aligned}
$$

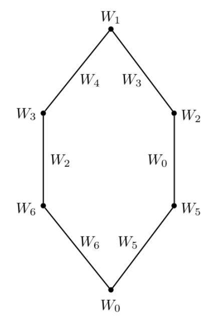
which is a contradiction. Hence the theorem. □

Theorem 3.4. $\psi_{pz}(C_n) = n + 1$, for n even.

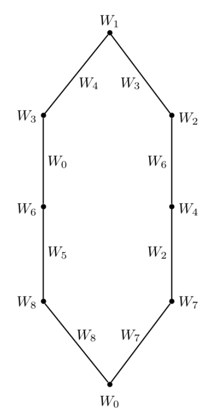
Proof. Let $v_1 v_2 v_3 \cdots v_n v_1$ be the cycle C_n of length n, where $n = 2m$, $m \in N$. The edges are $v_i v_{i+1}$, $1 \leq i \leq n$ and $v_n v_1$. There are five cases.
Case (i): When $n = 4$



Case (ii): When $n = 6$.



Case (iii): When $n = 8$



Case (iv): When $n = 4k$, $k \geq 3$. Define a function $f : V(C_n) \to M_2^{\circ}(Z_{n+1})$ as follows.

Set $f(v_1) = W_1, \quad f(v_2) = W_2$

$$f(v_{i+1}) = W_{2i}, \quad i = 2, 3, \ldots, \frac{n}{4}$$
$$f\left(v_{\frac{n+8}{4}}\right) = W_{\frac{n+6}{2}}$$
$$f\left(v_{\frac{n+8+4i}{4}}\right) = W_{\frac{n+6+4i}{2}}, \quad i = 1, 2, \ldots, \frac{n-8}{4}$$
$$f\left(v_{\frac{n+2}{2}}\right) = W_0$$
$$f\left(v_{\frac{n+2+2i}{2}}\right) = W_{n-2i+2}, \quad i = 1, 2, \ldots, \frac{n}{4}$$
$$f\left(v_{\frac{3n+8}{4}}\right) = W_{\frac{n-2}{2}}$$
$$f\left(v_{\frac{3n+8+4i}{4}}\right) = W_{\frac{n-2}{2}-2i}, \quad i = 1, 2, \ldots, \frac{n-8}{4}.$$

The we get the vertex labels as $W_0, W_1, W_2, \ldots, W_{n-1}, W_n$ except the member $W_{\frac{n+2}{2}}$.

Case (v): When $n = 4k + 2$, $k \geq 2$. Define a function $f : V(C_n) \to M_2^{\circ}(Z_{n+1})$ as follows:

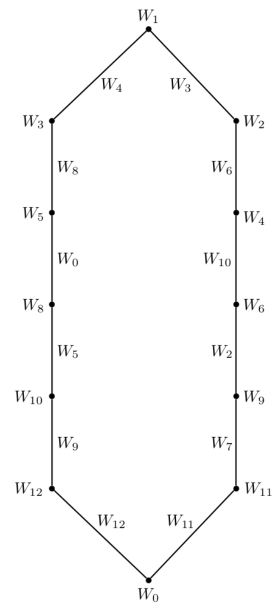
Set $\quad f(v_1) = W_1, \quad f(v_2) = W_2, \quad f(v_{i+1}) = W_{2i}, \quad i = 2, 3, \ldots, \left\lceil \frac{n}{4} \right\rceil - 1$

$$f\left(v_{\left\lceil \frac{n}{4} \right\rceil + 1}\right) = W_{2\left\lceil \frac{n}{4} \right\rceil + 1}$$
$$f\left(v_{\left\lceil \frac{n}{4} \right\rceil + 1 + i}\right) = W_{2\left\lceil \frac{n}{4} \right\rceil + 1 + 2i}, \quad i = 1, 2, \ldots, \frac{n}{2} - \left\lceil \frac{n}{4} \right\rceil - 1$$

$$f\left(\frac{v_{n+2}}{2}\right) = W_0$$
$$f\left(\frac{v_{n+2+2i}}{2}\right) = W_{n-2i+2}, \quad i = 1, 2, \ldots, \left\lfloor \frac{n}{4} \right\rfloor$$
$$f\left(v_{\frac{n}{2} + \left\lfloor \frac{n}{4} \right\rfloor + 2}\right) = W_{n-2\left\lfloor \frac{n}{4} \right\rfloor - 1}$$
$$f\left(v_{\frac{n}{2} + \left\lfloor \frac{n}{4} \right\rfloor + 2 + i}\right) = W_{n-2\left\lfloor \frac{n}{4} \right\rfloor - 1 - 2i}, \quad i = 1, 2, \ldots, \frac{n-6}{4}.$$

Then we get the vertex labels as $W_0, W_1, W_2, \ldots, W_{n-1}, W_n$ except the member $W_{\frac{n+2}{2}}$. From the above two cases f allows the required labeling. □

Illustration 3.4. Vertex pair sum 13-zero ring labeling of C_{12} using $M_2^{\circ}(Z_{13})$.



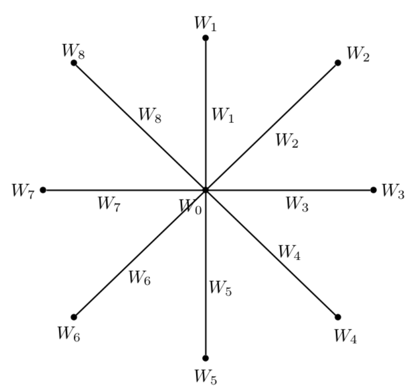
Theorem 3.5. A star graph S_n has a vertex pair sum k-zero ring labeling with vertex pair sum k-zero ring index n.

Proof. Let G be the star graph S_n. Label the central vertex u_1 as W_o. The other $(n-1)$ vertices $u_2, u_3, \dots u_n$ are labeled as $W_1, W_2, \dots, W_{n-1}$.

i.e., $f(u_i) = W_{i-1}, 1 \leq i \leq n$. Then the edge labels are $f(u_1 u_i) = W_0 + W_{i-1} = W_{i-1}$, where $2 \leq i \leq n$.

Then we get the distinct edge labels as $W_1, W_2, W_3, \dots, W_{n-1}$. Hence star graph admits pair sum zero ring labeling and $\psi_{pz}(S_n) = n$. □

Illustration 3.5. Vertex pair sum 9-zero ring labeling of S_9 using $M_2^\circ(Z_9)$.



Theorem 3.6. Any comb graph $P_n \odot K_1$ is a vertex pair sum k-zero ring graph having vertex pair sum k-zero index $2n$.

Proof. Let $P_n \odot K_1$ be a comb obtained from a path $P_n = u_1u_2 \ldots u_n$ by joining a vertex u_i to $v_i (1 \leq i \leq n)$.

Case (1): When n is even.

Define a function $f : V(P_n \odot K_1) \to M_2^\circ(Z_{2n})$ by

$$\begin{aligned} f(u_i) &= W_{i-1}, 1 \leq i \leq n \\ f(v_i) &= W_{n+i-1}, 1 \leq i \leq n. \end{aligned}$$

Edges are labeled by $f(u_iu_{i+1}) = W_{i-1} + W_i$, $1 \leq i \leq n-1$.

$$f(u_iv_i) = W_{i-1} + W_{n+i-1}, 1 \leq i \leq n.$$

Then we get edge labels from all the distinct elements of $M_2^\circ(Z_{2n})$ except W_{2n-1}.

Case (2): When n is odd

Define a function $f : V(P_n \odot K_1) \to M_2^\circ(Z_{2n})$ by

$$\begin{aligned} f(u_i) &= W_{2i-2}, \quad 1 \leq i \leq n \\ f(v_i) &= W_{2i-1}, \quad 1 \leq i \leq n. \end{aligned}$$

Edges are labeled by

$$\begin{aligned} f(u_iu_{i+1}) &= W_{2i-2}, +W_{2i} \quad 1 \leq i \leq (n-1) \\ f(u_iv_i) &= W_{2i-2} + W_{2i-1}, \quad 1 \leq i \leq n. \end{aligned}$$

Then we get edge labels from all the distinct elements of $M_2^\circ(Z_{2n})$ except W_{2n-2}.

$\therefore P_n \odot K_1$ is pair sum zero ring graph and $\psi_{pz}(P_n \odot K_1) = 2n$. □

Illustration 3.6. The vertex pair sum 12-zero ring labeling of comb obtained from $P_6 \cdot K_1$ is given below.

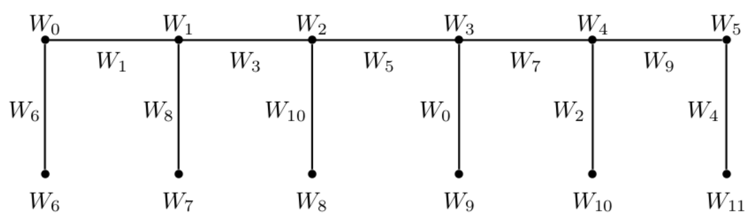
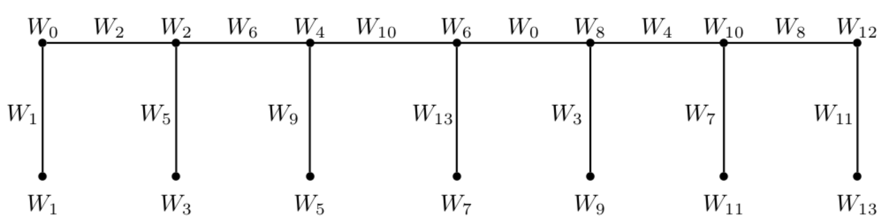
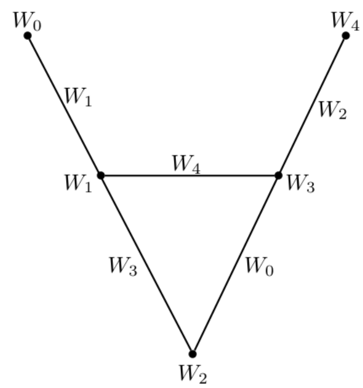


Illustration 3.7. The vertex pair sum 14-zero ring labeling of comb obtained from $P_7 \cdot K_1$ is given below.



Theorem 3.7. The bull graph admits vertex pair sum 5-zero ring labeling.

Proof. Define a function $f : V \to M_2^{\circ}(Z_5)$ by $f(v_i) = W_{i-1}, 1 \leq i \leq 5$.



From the definition it is clear that vertex pair sum k-zero ring index of bull graph is 5. □

4. Conclusion

In this paper, we defined vertex pair sum k-zero ring index and determined the same for the graphs such as path, cycle, star graph, comb graph and bull graph. Similar results can be obtained for other classes of graphs also. Here we are taking the finite zero ring ${M_2}^0(Z_n)$. In future this study can be extended to other finite zero rings too.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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