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REGULARITY OF SEMIGROUPS IN TERMS OF PYTHAGOREAN FUZZY BI-IDEALS[†]

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Abstract. In this paper, the concept of Pythagorean fuzzy sets are used to describe in semigroups. Then, some characterizations of regular (resp., intra-regular) semigroups by means of Pythagorean fuzzy left (resp., right) ideals and Pythagorean fuzzy (resp., generalized) bi-ideals of semigroups are investigated. Furthermore, the class of both regular and intra-regular semigroups by the properties of many kinds of their Pythagorean fuzzy ideals also being studied.

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1. Introduction

In 1965, Zadeh [25] presented the idea of fuzzy sets as a function from a nonempty set X to the unit interval $[0, 1]$. When solving problems in the actual world, this notion is helpful in addressing uncertainty. The fuzzy set theory has been used by several writers to generalize the fundamental algebraic structures. The notion of fuzzy groups was first creatively applied to several structures of algebra by Rosenfeld [18]. Then, Kuroki [11, 12] also proposed the concept of fuzzy subsemigroups. As an extension of the idea of fuzzy sets, Atanassov [1] developed the concept of intuitionistic fuzzy sets. In other words, the intuitionistic fuzzy sets provide both membership and non-membership degrees, while the fuzzy sets establish the degree of membership of an element in that set. In 2002, Kim and Jun [10] used intuitionistic fuzzy sets to study semigroups. Later, the definition of Pythagorean fuzzy sets, as the sum of the squares of membership and that non-membership relates to the unit interval $[0, 1]$, was first introduced by Yager [24] in 2013. This concept is a general concept of intuitionistic fuzzy

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sets. The notion of rough Pythagorean fuzzy ideals were first studied in semigroup in 2019 by Hussain et al. [7].

For many years, the study of regularities in semigroups has been essential and famous. Lajos [14] applied the concepts of left ideals and right ideals of semigroups to characterize the regular semigroups in 1968. Additionally, Lajos and Szasz [15, 16] considered the class of intra-regular semigroups using the left and right ideals of semigroups. The intra-regular ordered semigroup was also described by Kehayopulu et al. [9] utilizing the left and right ideals of ordered semigroups. However, the concept of fuzzy sets has been used to investigate the characterizations of semigroups in parallel with the use of properties of different types of their ideals. In 2010, Shabir et al. [21] characterized regular semigroups by the properties of the lower part of $(\in, \in\vee q)$-fuzzy left ideals, $(\in, \in\vee q)$-fuzzy quasi-ideals and $(\in, \in\vee q)$-fuzzy generalized bi-ideals. Moreover, Xie and Tang [23] provided the characterizations of intra-regular and regular ordered semigroups in terms of their many types of fuzzy ideals. Subsequently, the concept of fuzzy bi-quasi-ideals, as a generalization of fuzzy bi-ideals in Γ-semigroups, were defined by Rao [17] to describe the regular Γ-semigroups. After that, Gatetem and Khamrot [4] deployed bipolar fuzzy weakly interior ideals to investigate the characterizations of regular, left (resp., right) regular, intra-regular, weakly regular and quasi-regular semigroups. By via intuitionistic fuzzy left ideals, intuitionistic fuzzy right ideals, and intuitionistic fuzzy bi-ideals, Hong and Fang [5] presented results characterizing intra-regular semigroups. Next, Hur et al. [6] examined intuitionistic fuzzy left, right, two-sided ideals and intuitionistic fuzzy bi-ideals of semigroups to characterize the class of regular semigroups. In addition, the class of intra-regular ordered semigroups have been identified by Shabir and Khan [20] using types of their intuitionistic fuzzy interior ideals.

In 2020, Chinram and Panityakul [3] introduced the notion of rough Pythagorean fuzzy ideals in ternary semigroups and gave some properties. Afterwards, Subha et al. [22] discussed rough interval valued Pythagorean fuzzy sets in semigroups. Meanwhile, Chinnadurai and Arulselvam [2] considered the notion of rough cubic Pythagorean fuzzy sets in the semigroups and investigated some of its related properties. Recently, in 2023, Julatha and Iampan [8] investigated the regularity of semigroups by properties of $(\inf, \sup)$-hesitant fuzzy bi-ideals of semigroups. In this paper, the classes of regular (resp., intra-regular) semigroups were characterized by means of Pythagorean fuzzy left (resp., right) ideals and Pythagorean fuzzy (resp., generalized) bi-ideals of semigroups. Finally, some characterizations of both regular and intra-regular semigroups by using the properties of many kinds of their Pythagorean fuzzy ideals are presented.

2. Preliminaries

A *semigroup* is the structure $(S, \cdot)$ consisting of a nonempty set S together with a binary associative operation $\cdot$ on S, that is, the condition $(xy)z = x(yz)$ for all $x, y, z \in S$ holds. For any nonempty subsets A and B of S, we denote that

$AB = \{ab \mid a \in A, b \in B\}$. A nonempty subset A of a semigroup S is called a *subsemigroup* of S if $AA \subseteq A$, A is called a *left* (resp., *right*) *ideal* of S if $SA \subseteq A$ (resp., $AS \subseteq A$), and if A is both a left and a right ideal of S, then A is called an *ideal* of S. A subsemigroup A of S is said to be a *bi-ideal* of S if $ASA \subseteq A$. A nonempty subset A of S is called a *generalized bi-ideal* of S if $ASA \subseteq A$.

A *fuzzy set* (briefly, FS) [25] of a nonempty set X is a function $\mu : X \to [0,1]$. For every two fuzzy sets μ and λ of a nonempty set X, the fuzzy sets $\mu \cap \lambda$ and $\mu \cup \lambda$ of S are defined by $(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\}$ and $(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$ for all $x \in X$.

Definition 2.1. [1] Let X be an universe set. An *intuitionistic fuzzy set* (briefly, IFS) $\mathcal{A}$ is defined as the form

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where $\mu_{\mathcal{A}} : X \to [0,1]$ and $\lambda_{\mathcal{A}} : X \to [0,1]$ mean the degree of membership and the degree of non-membership of an element $x \in X$ to $\mathcal{A}$, respectively, and the condition $0 \leq \mu_{\mathcal{A}}(x) + \lambda_{\mathcal{A}}(x) \leq 1$ holds.

Definition 2.2. [24] Let X be an universe set. A *Pythagorean fuzzy set* (briefly, PFS) $\mathcal{A}$ is an object having the form

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where the mapping $\mu_{\mathcal{A}} : X \to [0,1]$ and $\lambda_{\mathcal{A}} : X \to [0,1]$ define the degree of membership and the degree non-membership of the element $x \in X$ to a set $\mathcal{A}$, respectively, and also $0 \leq (\mu_{\mathcal{A}}(x))^2 + (\lambda_{\mathcal{A}}(x))^2 \leq 1$.

It is not difficult to see that the concept of PFSs generalizes the concepts of FSs and IFSs. Throughout this paper, we use the symbol PFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ instead of the PFS $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)\rangle \mid x \in X\}$.

Definition 2.3. [24] Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be PFSs on a nonempty set X. Then:

(i) $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$ and $\lambda_{\mathcal{A}}(x) \geq \lambda_{\mathcal{B}}(x)$ for all $x \in X$;
(ii) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$;
(iii) $\mathcal{A} \cap \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \cap \lambda_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \cup \lambda_{\mathcal{B}})(x)\rangle \mid x \in X\}$;
(iv) $\mathcal{A} \cup \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \cup \lambda_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \cap \lambda_{\mathcal{B}})(x)\rangle \mid x \in X\}$.

We note that $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ are PFSs of X if $\mathcal{A}$ and $\mathcal{B}$ are PFSs of X.

Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be any two PFSs on a semigroup S. Then the *product* [7] of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} \odot \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(x)\rangle \mid x \in X\},$$

where

$$(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x) = \begin{cases} \sup\limits_{x=ab} [\min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(b)\}] & \text{if } x \in S^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(x) = \begin{cases} \inf\limits_{x=ab}[\max\{\lambda_{\mathcal{A}}(a), \lambda_{\mathcal{B}}(b)\}] & \text{if } x \in S^2, \\ 1 & \text{otherwise.} \end{cases}$$

The *Pythagorean characteristic function* of a subset A of a nonempty set X, as a PFS of X, defined by $\mathcal{C}_A = \{\langle x, \mu_{\mathcal{C}_A}(x), \lambda_{\mathcal{C}_A}(x)\rangle \mid x \in X\}$, where

$$\mu_{\mathcal{C}_A}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases} \text{ and } \lambda_{\mathcal{C}_A}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise.} \end{cases}$$

For any semigroup S, we denote by $\mathcal{PFS}(S)$ the collection of Pythagorean fuzzy sets on S with $\mathcal{S} = \{\langle x, 1, 0\rangle \mid x \in X\}$ and $\mathbf{0} = \{\langle x, 0, 1\rangle \mid x \in X\}$. If $A = S$ (resp., $A = \emptyset$), then $\mathcal{C}_A = \mathcal{S}$ (resp., $\mathcal{C}_A = \mathbf{0}$).

Definition 2.4. [7] Let S be a semigroup. A PFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ on S is called:

(i) a *Pythagorean fuzzy subsemigroup* (briefly, PFSub) of S if for every $x, y \in S$, it satisfies

$$\mu_{\mathcal{A}}(xy) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \text{ and } \lambda_{\mathcal{A}}(xy) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\};$$

(ii) a *Pythagorean fuzzy left ideal* (briefly, PFL) of S if for every $x, y \in S$, it satisfies

$$\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(y) \text{ and } \lambda_{\mathcal{A}}(xy) \leq \lambda_{\mathcal{A}}(y);$$

(iii) a *Pythagorean fuzzy right ideal* (briefly, PFR) of S if for every $x, y \in S$, it satisfies

$$\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(x) \text{ and } \lambda_{\mathcal{A}}(xy) \leq \lambda_{\mathcal{A}}(x);$$

(iv) a *Pythagorean fuzzy ideal* (briefly, PFI) of S if it is both a PFL and a PFR of S.

Definition 2.5. [7] A PFSub $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ of a semigroup S is known to be a *Pythagorean fuzzy bi-ideal* (briefly, PFB) of S if for every $x, y, z \in S$, it holds $\mu_{\mathcal{A}}(xyz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}$ and $\lambda_{\mathcal{A}}(xyz) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(z)\}$.

Next, we introduce the concept of Pythagorean fuzzy generalized bi-ideals in semigroups which is a generalization of the PFBs.

Definition 2.6. A PFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ of a semigroup S is said to be a *Pythagorean fuzzy generalized bi-ideal* (briefly, PFGB) of S if for every $x, y, z \in S$, it holds $\mu_{\mathcal{A}}(xyz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}$ and $\lambda_{\mathcal{A}}(xyz) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(z)\}$.

From the concepts mentioned above, we can see that every PFL (resp., PFR) is also a PFB, and any PFB is also a PFGB of a semigroup.

Example 2.7. Let $S = \{a, b, c, d\}$ with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Then, $(S, \cdot)$ is a semigroup, see [15]. Define the PFS $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ on S by

$$\mu_\mathcal{A}(a) = 0.9, \mu_\mathcal{A}(b) = 0.5, \mu_\mathcal{A}(c) = 0.7, \mu_\mathcal{A}(d) = 0.3,$$
$$\lambda_\mathcal{A}(a) = 0.2, \lambda_\mathcal{A}(b) = 0.7, \lambda_\mathcal{A}(c) = 0.6, \lambda_\mathcal{A}(d) = 0.8.$$

It turns out that $\mathcal{A}$ is a PFB of S. We consider, $\mu_\mathcal{A}(dc) = 0.3 < 0.6 = \mu_\mathcal{A}(c)$ and $\lambda_\mathcal{A}(dc) = 0.7 > 0.6 = \lambda_\mathcal{A}(c)$. This shows that $\mathcal{A}$ is not a PFL of S. Furthermore, $\mathcal{A}$ is also not a PFR of S.

Example 2.8. Let $S = \{a, b, c, d\}$. Define the binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Thus, $(S, \cdot)$ is a semigroup, see [19]. Let a PFS $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ on S be defined by

$$\mu_\mathcal{A}(a) = 0.8, \mu_\mathcal{A}(b) = 0.2, \mu_\mathcal{A}(c) = 0.7, \mu_\mathcal{A}(d) = 0.4,$$
$$\lambda_\mathcal{A}(a) = 0.4, \lambda_\mathcal{A}(b) = 0.9, \lambda_\mathcal{A}(c) = 0.5, \lambda_\mathcal{A}(d) = 0.8.$$

We can see that $\mathcal{A}$ is a PFGB of S. However, $\mathcal{A}$ is not a PFSub of S, because $\mu_\mathcal{A}(cc) = 0.2 < 0.7 = \min\{\mu_\mathcal{A}(c), \mu_\mathcal{A}(c)\}$ and $\lambda_\mathcal{A}(cc) = 0.9 > 0.5 = \min\{\lambda_\mathcal{A}(c), \lambda_\mathcal{A}(c)\}$. This means that $\mathcal{A}$ is not a PFB of S.

The following lemmas can be proved straightforward.

Lemma 2.9. *Let A and B be any nonempty subsets of a semigroup S. The following statements hold:*

(i) $\mathcal{C}_{A\cap B} = \mathcal{C}_A \cap \mathcal{C}_B$;
(ii) $\mathcal{C}_{AB} = \mathcal{C}_A \odot \mathcal{C}_B$.

Lemma 2.10. *Let $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A}), \mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B}), \mathcal{C} = (\mu_\mathcal{C}, \lambda_\mathcal{C})$ and $\mathcal{D} = (\mu_\mathcal{D}, \lambda_\mathcal{D})$ be any PFSs on a semigroup S. If $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{A} \odot \mathcal{C} \subseteq \mathcal{B} \odot \mathcal{D}$.*

Lemma 2.11. *Let $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ be a PFS on a semigroup S. Then the following properties hold:*

(i) *$\mathcal{A}$ is a PFSub of S if and only if $\mathcal{A} \odot \mathcal{A} \subseteq \mathcal{A}$;*
(ii) *$\mathcal{A}$ is a PFL of S if and only if $\mathcal{S} \odot \mathcal{A} \subseteq \mathcal{A}$;*
(iii) *$\mathcal{A}$ is a PFR of S if and only if $\mathcal{A} \odot \mathcal{S} \subseteq \mathcal{A}$;*
(iv) *$\mathcal{A}$ is a PFB of S if and only if $\mathcal{A} \odot \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A} \odot \mathcal{S} \odot \mathcal{A} \subseteq \mathcal{A}$;*
(v) *$\mathcal{A}$ is a PFGB of S if and only if $\mathcal{A} \odot \mathcal{S} \odot \mathcal{A} \subseteq \mathcal{A}$.*

Proof. (i) Assume that $\mathcal{A}$ is a PFSub of S. Let $a \in S$. If $a \neq pq$ for all $p, q \in S$, then it is well done. Suppose that there exist $x, y \in S$ such that $a = xy$. So, we have

$$(\mu_\mathcal{A} \circ \mu_\mathcal{A})(a) = \sup_{a=xy} [\min\{\mu_\mathcal{A}(x), \mu_\mathcal{A}(y)\}] \leq \sup_{a=xy} [\mu_\mathcal{A}(xy)] = \mu_\mathcal{A}(a),$$

$$(\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{A}})(a) = \inf_{a=xy}[\max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\}] \geq \inf_{a=xy}[\lambda_{\mathcal{A}}(xy)] = \lambda_{\mathcal{A}}(a).$$

Hence, $\mathcal{A} \odot \mathcal{A} \subseteq \mathcal{A}$.

Conversely, let $x, y \in S$. Thus, we have

$$\mu_{\mathcal{A}}(xy) \geq (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(xy) = \sup_{xy=mn}[\min\{\mu_{\mathcal{A}}(m), \mu_{\mathcal{A}}(n)\}] \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\},$$

$$\lambda_{\mathcal{A}}(xy) \leq (\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{A}})(xy) = \inf_{xy=mn}[\max\{\lambda_{\mathcal{A}}(m), \lambda_{\mathcal{A}}(n)\}] \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\}.$$

This means that $\mathcal{A}$ is a PFSub of S.

In the other cases, we can proved in a similar way. □

Lemma 2.12. *For any nonempty subset A of a semigroup S, then:*

(i) *A is a subsemigroup of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFSub of S;*
(ii) *A is a left ideal of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFL of S;*
(iii) *A is a right ideal of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFR of S;*
(iv) *A is an ideal of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFI of S;*
(v) *A is a bi-ideal of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFB of S;*
(vi) *A is a generalized bi-ideal of S if and only if $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFGB of S.*

Proof. (i) Assume that A is a subsemigroup of S. Suppose that there are $a, b \in S$ such that $\mu_{\mathcal{C}_A}(ab) < \min\{\mu_{\mathcal{C}_A}(a), \mu_{\mathcal{C}_A}(b)\}$. It implies that $\mu_{\mathcal{C}_A}(ab) = 0$ and $\min\{\mu_{\mathcal{C}_A}(a), \mu_{\mathcal{C}_A}(b)\} = 1$, and then $ab \notin A$ where $a, b \in A$. But by the hypothesis, we obtain that $ab \in A$ as a contradiction. Hence,

$$\mu_{\mathcal{C}_A}(xy) \geq \min\{\mu_{\mathcal{C}_A}(x), \mu_{\mathcal{C}_A}(y)\} \text{ for all } x, y \in S.$$

Similarly, suppose that $\lambda_{\mathcal{C}_A}(ab) > \max\{\lambda_{\mathcal{C}_A}(a), \lambda_{\mathcal{C}_A}(b)\}$ for some $a, b \in S$. Then, $\lambda_{\mathcal{C}_A}(ab) = 1$ and $\max\{\lambda_{\mathcal{C}_A}(a), \lambda_{\mathcal{C}_A}(b)\} = 0$. It turns out that $ab \notin A$ and $a, b \in A$. Since A is a subsemigroup of S, it implies that $ab \in A$. This is a contradiction. So,

$$\lambda_{\mathcal{C}_A}(xy) \leq \max\{\lambda_{\mathcal{C}_A}(x), \lambda_{\mathcal{C}_A}(y)\} \text{ for all } x, y \in S.$$

Therefore, $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFSub of S.

Conversely, assume that $\mathcal{C}_A = (\mu_{\mathcal{C}_A}, \lambda_{\mathcal{C}_A})$ is a PFSub of S. Let $x, y \in A$. Then,

$$\mu_{\mathcal{C}_A}(xy) \geq \min\{\mu_{\mathcal{C}_A}(x), \mu_{\mathcal{C}_A}(y)\} = 1.$$

We obtain that $\mu_{\mathcal{C}_A}(xy) = 1$, that is, $xy \in A$. Consequently, A is a subsemigroup of S.

Other conditions can be shown similarly to the proof of (i). □

3. Regular Semigroups

In this section, we study the characterizations of regular semigroups by the properties of PFLs, PFRs, PFBs and PFGBs of semigroups.

A semigroup S is said to be *regular* [14] if for each element a in S, there exists an element x in S such that $a = axa$.

Theorem 3.1. *In a regular semigroup S, every PFGB of S is also a PFB of S.*

Proof. Let $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFGB of S and let $a, b \in S$. Then, there exists $x \in S$ such that $b = bxb$. Thus, $\mu_{\mathcal{G}}(ab) = \mu_{\mathcal{G}}(a(bx)b) \geq \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(b)\}$ and $\lambda_{\mathcal{G}}(ab) = \lambda_{\mathcal{G}}(a(bx)b) \leq \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(b)\}$. This shows that $\mathcal{G}$ is a PFSub of S. Hence, $\mathcal{G}$ is a PFB of S. □

Lemma 3.2. [14] *For a semigroup S, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$R \cap L = RL$, for every left ideal L and every right ideal R of S.*

Theorem 3.3. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \odot \mathcal{L}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and every PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ of S.*

Proof. Assume that S is regular. Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFL and PFR of S, respectively. Also, by Lemma 2.10 and Lemma 2.11, we have that $\mathcal{R} \odot \mathcal{L} \subseteq \mathcal{S} \odot \mathcal{L} \subseteq \mathcal{L}$ and $\mathcal{R} \odot \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{S} \subseteq \mathcal{R}$. Thus, $\mathcal{R} \odot \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. Let $a \in S$. Then, there exists $x \in S$ such that $a = axa$. So, we have

$$\begin{aligned}(\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a=pq} [\min\{\mu_{\mathcal{R}}(p), \mu_{\mathcal{L}}(q)\}] \geq \min\{\mu_{\mathcal{R}}(ax), \mu_{\mathcal{L}}(a)\} \\ &\geq \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\} = (\mu_{\mathcal{R}} \cap \mu_{\mathcal{L}})(a), \\ (\lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(a) &= \inf_{a=pq} [\max\{\lambda_{\mathcal{R}}(p), \lambda_{\mathcal{L}}(q)\}] \leq \max\{\lambda_{\mathcal{R}}(ax), \lambda_{\mathcal{L}}(a)\} \\ &\leq \max\{\lambda_{\mathcal{R}}(a), \lambda_{\mathcal{L}}(a)\} = (\lambda_{\mathcal{R}} \cup \lambda_{\mathcal{L}})(a).\end{aligned}$$

This shows that $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{L}$. Hence, $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \odot \mathcal{L}$.

Conversely, let L and R be any left ideal and any right ideal of S, respectively. Then, $RL \subseteq R \cap L$. Next, let $x \in R \cap L$. By Lemma 2.12, we have that $\mathcal{C}_L = (\mu_{\mathcal{C}_L}, \lambda_{\mathcal{C}_L})$ and $\mathcal{C}_R = (\mu_{\mathcal{C}_R}, \lambda_{\mathcal{C}_R})$ are a PFL and a PFR of S, respectively. By the given assumption and Lemma 2.9, it follows that $\mathcal{C}_{RL} = \mathcal{C}_R \odot \mathcal{C}_L = \mathcal{C}_R \cap \mathcal{C}_L = \mathcal{C}_{R\cap L}$. So, $\mu_{\mathcal{C}_{RL}}(x) = \mu_{\mathcal{C}_{R\cap L}}(x) = 1$. That is, $x \in RL$. Hence, $R \cap L \subseteq RL$. We obtain that $R \cap L = RL$. Therefore, S is regular by Lemma 3.2. □

Lemma 3.4. [15] *Let S be a semigroup. Then, S is regular if and only if $B = BSB$, for each bi-ideal B of S.*

Theorem 3.5. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{B} = \mathcal{B} \odot \mathcal{S} \odot \mathcal{B}$, for every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. Assume that S is regular. Let $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be a PFB of S. By Lemma 2.11, $\mathcal{B} \odot \mathcal{S} \odot \mathcal{B} \subseteq \mathcal{B}$. On the other hand, let $a \in S$. Then, there exists $x \in S$ such that $a = axa$. Thus, we have

$$\begin{aligned}(\mu_{\mathcal{B}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=pq} [\min\{\mu_{\mathcal{B}}(p), (\mu_{\mathcal{S}} \circ \mu_{\mathcal{B}})(q)\}] \\ &\geq \min\{\mu_{\mathcal{B}}(a), (\mu_{\mathcal{S}} \circ \mu_{\mathcal{B}})(xa)\}\end{aligned}$$

$$
\begin{aligned}
&= \min\left\{\mu_{\mathcal{B}}(a), \sup_{xa=mn}[\min\{\mu_{\mathcal{S}}(m), \mu_{\mathcal{B}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{B}}(a), \min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{B}}(a)\}\} \\
&= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{B}}(a)\} \\
&= \mu_{\mathcal{B}}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_{\mathcal{B}} \circ \lambda_{\mathcal{S}} \circ \lambda_{\mathcal{B}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{B}}(p), (\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{B}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{B}}(a), (\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{B}})(xa)\} \\
&= \max\left\{\lambda_{\mathcal{B}}(a), \inf_{xa=mn}[\max\{\lambda_{\mathcal{S}}(m), \lambda_{\mathcal{B}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{B}}(a), \max\{\lambda_{\mathcal{S}}(x), \lambda_{\mathcal{B}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{B}}(a), \lambda_{\mathcal{B}}(a)\} \\
&= \lambda_{\mathcal{B}}(a).
\end{aligned}
$$

This implies that $\mathcal{B} \subseteq \mathcal{B} \odot \mathcal{S} \odot \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{B} \odot \mathcal{S} \odot \mathcal{B}$.

Conversely, let B be any bi-ideal of S. Also, $BSB \subseteq B$. By Lemma 2.12, $\mathcal{C}_B = (\mu_{\mathcal{C}_B}, \lambda_{\mathcal{C}_B})$ is a PFB of S. Let $a \in B$. By the hypothesis and Lemma 2.9, we have that $\mathcal{C}_B = \mathcal{C}_B \odot \mathcal{S} \odot \mathcal{C}_B = \mathcal{C}_{BSB}$. Thus, $\mu_{\mathcal{C}_{BSB}}(a) = \mu_{\mathcal{C}_B}(a) = 1$, implies that $a \in BSB$. Hence, $B \subseteq BSB$. That is, $B = BSB$. By Lemma 3.4, we conclude that S is regular. □

The following theorem can be presented by Theorem 3.1 and Theorem 3.5.

Theorem 3.6. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{G} = \mathcal{G} \odot \mathcal{S} \odot \mathcal{G}$, for every PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S.*

Theorem 3.7. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{B} \cap \mathcal{A} = \mathcal{B} \odot \mathcal{A} \odot \mathcal{B}$, for every PFI $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. Assume that S is regular. Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be a PFI and a PFB of S, respectively. By Lemma 2.10 and Lemma 2.11, we obtain that $\mathcal{B} \odot \mathcal{A} \odot \mathcal{B} \subseteq \mathcal{B} \odot \mathcal{S} \odot \mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{B} \odot \mathcal{A} \odot \mathcal{B} \subseteq (\mathcal{S} \odot \mathcal{A}) \odot \mathcal{S} \subseteq \mathcal{A} \odot \mathcal{S} \subseteq \mathcal{A}$. Also, $\mathcal{B} \odot \mathcal{A} \odot \mathcal{B} \subseteq \mathcal{B} \cap \mathcal{A}$. Now, let $a \in S$. Then, there exists $x \in S$ such that $a = axa = a(xax)a$. Thus, we have

$$
\begin{aligned}
(\mu_{\mathcal{B}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{B}}(p), (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(q)\}] \\
&\geq \min\{\mu_{\mathcal{B}}(a), (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(xaxa)\} \\
&= \min\left\{\mu_{\mathcal{B}}(a), \sup_{xaxa=mn}[\min\{\mu_{\mathcal{A}}(m), \mu_{\mathcal{B}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{B}}(a), \min\{\mu_{\mathcal{A}}(xax), \mu_{\mathcal{B}}(a)\}\} \\
&\geq \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(a)\} \\
&= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{A}}(a)\}
\end{aligned}
$$

$$= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{A}})(a)$$

and

$$\begin{aligned}
(\lambda_{\mathcal{B}} \circ \lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{B}}(p), (\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{B}}(a), (\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(xaxa)\} \\
&= \max\left\{\lambda_{\mathcal{B}}(a), \inf_{xaxa=mn}[\max\{\lambda_{\mathcal{A}}(m), \lambda_{\mathcal{B}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{B}}(a), \max\{\lambda_{\mathcal{A}}(xax), \lambda_{\mathcal{B}}(a)\}\} \\
&\leq \max\{\lambda_{\mathcal{B}}(a), \lambda_{\mathcal{A}}(a), \lambda_{\mathcal{B}}(a)\} \\
&= \max\{\lambda_{\mathcal{B}}(a), \lambda_{\mathcal{A}}(a)\} \\
&= (\lambda_{\mathcal{B}} \cup \lambda_{\mathcal{A}})(a).
\end{aligned}$$

Hence, $\mathcal{B} \cap \mathcal{A} \subseteq \mathcal{B} \odot \mathcal{A} \odot \mathcal{B}$. Therefore, $\mathcal{B} \cap \mathcal{A} = \mathcal{B} \odot \mathcal{A} \odot \mathcal{B}$.

Conversely, let $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be a PFB of S. By the given assumption, $\mathcal{B} = \mathcal{B} \cap \mathcal{S} = \mathcal{B} \odot \mathcal{S} \odot \mathcal{B}$. Consequently, S is regular by Theorem 3.5. □

The following theorem follows by Theorem 3.1 and Theorem 3.7.

Theorem 3.8. *Let S be a semigroup. Then, S is regular if and only if $\mathcal{G} \cap \mathcal{A} = \mathcal{G} \odot \mathcal{A} \odot \mathcal{G}$, for every PFI $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and every PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S.*

Theorem 3.9. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*
(ii) *$\mathcal{G} \cap \mathcal{L} \subseteq \mathcal{G} \odot \mathcal{L}$, for each PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;*
(iii) *$\mathcal{B} \cap \mathcal{L} \subseteq \mathcal{B} \odot \mathcal{L}$, for each PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. (i) ⇒ (ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFL and a PFGB of S, respectively. Let $a \in S$. By assumption, there exists $x \in S$ such that $a = axa$. So, we have

$$\begin{aligned}
(\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{G}}(p), \mu_{\mathcal{L}}(q)\}] \geq \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(xa)\} \\
&\geq \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\} = (\mu_{\mathcal{G}} \cap \mu_{\mathcal{L}})(a), \\
(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{G}}(p), \lambda_{\mathcal{L}}(q)\}] \leq \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{L}}(xa)\} \\
&\leq \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{L}}(a)\} = (\lambda_{\mathcal{G}} \cup \lambda_{\mathcal{L}})(a).
\end{aligned}$$

It turns out that $\mathcal{G} \cap \mathcal{L} \subseteq \mathcal{G} \odot \mathcal{L}$.

(ii) ⇒ (iii) Since every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S is also a PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S, it follows that (iii) holds.

(iii) ⇒ (i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFL and a PFR of S, respectively. We obtain that $\mathcal{R}$ is also a PFB of S. Then, by the given assumption, we have that $\mathcal{R} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{L}$. Otherwise, $\mathcal{R} \odot \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. Therefore, $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \odot \mathcal{L}$. By Theorem 3.3, we get that S is regular. □

The following result can be proved similar to Theorem 3.9.

Theorem 3.10. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*
(ii) *$\mathcal{R}\cap\mathcal{G} \subseteq \mathcal{R}\odot\mathcal{G}$, for each PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;*
(iii) *$\mathcal{R}\cap\mathcal{B} \subseteq \mathcal{R}\odot\mathcal{B}$, for each PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Next, we give a characterization of regular semigroups by the properties of PFLs, PFRs, PFBs and PFGBs of semigroups.

Theorem 3.11. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is regular;*
(ii) *$\mathcal{R} \cap \mathcal{G} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{G} \odot \mathcal{L}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;*
(iii) *$\mathcal{R} \cap \mathcal{B} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{B} \odot \mathcal{L}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}}), \mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFL, a PFR and a PFGB of S, respectively. Let $a \in S$. Then, there exists $x \in S$ such that $a = axa = (ax)(axa)(xa)$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{R}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{R}}(p), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(q)\}] \\
&\geq \min\{\mu_{\mathcal{R}}(ax), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(axaxa)\} \\
&= \min\left\{\mu_{\mathcal{R}}(ax), \sup_{axaxa=mn}[\min\{\mu_{\mathcal{G}}(m), \mu_{\mathcal{L}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{R}}(ax), \min\{\mu_{\mathcal{G}}(axa), \mu_{\mathcal{L}}(xa)\}\} \\
&\geq \min\{\mu_{\mathcal{R}}(a), \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \mu_{\mathcal{L}}(a)\}\} \\
&= \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{G}} \cap \mu_{\mathcal{L}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{R}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{R}}(p), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{R}}(ax), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}})(axaxa)\} \\
&= \max\left\{\lambda_{\mathcal{R}}(ax), \inf_{axaxa=mn}[\max\{\lambda_{\mathcal{G}}(m), \lambda_{\mathcal{L}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{R}}(ax), \max\{\lambda_{\mathcal{G}}(axa), \lambda_{\mathcal{L}}(xa)\}\} \\
&\leq \max\{\lambda_{\mathcal{R}}(a), \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \lambda_{\mathcal{L}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{R}}(a), \lambda_{\mathcal{G}}(a), \lambda_{\mathcal{L}}(a)\}
\end{aligned}$$

$$= (\lambda_{\mathcal{R}} \cup \lambda_{\mathcal{G}} \cup \lambda_{\mathcal{L}})(a).$$

This means that $\mathcal{R} \cap \mathcal{G} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{G} \odot \mathcal{L}$.

(ii) $\Rightarrow$ (iii) Since every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S is a PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S, we obtain that (iii) holds.

(iii) $\Rightarrow$ (i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFL and a PFR of S, respectively. So, we have that $\mathcal{S}$ itself is a PFB of S. Then, using the assumption, implies that $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \cap \mathcal{S} \cap \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{S} \odot \mathcal{L} \subseteq \mathcal{R} \odot \mathcal{L}$. On the other hand, $\mathcal{R} \odot \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. Therefore, $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \odot \mathcal{L}$. Consequently, S is regular by Theorem 3.3. $\square$

4. Intra-regular Semigroups

In this section, we discuss characterizations of intra-regular semigropus by menas of their PFLs, PFRs, PFBs and PFGBs.

A semigroup S is known to be *intra-regular* [13] if for each element a of S, there exist elements x, y in S such that $a = xa^2y$.

Lemma 4.1. *(cf.* [23]*) Let S be a semigroup. Then, S is intra-regular if and only if $L \cap R \subseteq LR$, for every left ideal L and every right ideal R of S.*

Theorem 4.2. *Let S be a semigroup. Then, S is intra-regular if and only if $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \odot \mathcal{R}$, for each PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ of S.*

Proof. Assume that S is intra-regular. Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be any PFL and any PFR of S, respectively. Let $a \in S$. Then, there exist $x, y \in S$ such that $a = xa^2y$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{L}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a=pq} [\min\{\mu_{\mathcal{L}}(p), \mu_{\mathcal{R}}(q)\}] \geq \min\{\mu_{\mathcal{L}}(xa), \mu_{\mathcal{R}}(ay)\} \\
&\geq \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{R}}(a)\} = (\mu_{\mathcal{L}} \cap \mu_{\mathcal{R}})(a), \\
(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a=pq} [\max\{\lambda_{\mathcal{L}}(p), \lambda_{\mathcal{R}}(q)\}] \leq \max\{\lambda_{\mathcal{L}}(xa), \lambda_{\mathcal{R}}(ay)\} \\
&\leq \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{R}}(a)\} = (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{R}})(a).
\end{aligned}$$

This shows that $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \odot \mathcal{R}$.

Conversely, let L be any left ideal and and R be any right ideal of S. Let $x \in L \cap R$. By Lemma 2.12, $\mathcal{C}_L$ and $\mathcal{C}_B$ are a PFL and a PFR of S, respectively. Then, using the hypothesis and Lemma 2.9, we have that $\mathcal{C}_{L \cap R} = \mathcal{C}_L \cap \mathcal{C}_R \subseteq \mathcal{C}_L \odot \mathcal{C}_R = \mathcal{C}_{LR}$. So, $\mu_{\mathcal{C}_{LR}}(x) \geq \mu_{\mathcal{C}_{L \cap R}}(x) = 1$. That is, $\mu_{\mathcal{C}_{LR}}(x) = 1$, and then $x \in LR$. Hence, $L \cap R \subseteq LR$. By Lemma 4.1, we obtain that S is intra-regular. $\square$

Theorem 4.3. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is intra-regular;*

(ii) *$\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \odot \mathcal{G} \odot \mathcal{S}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and every PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;*

(iii) $\mathcal{L} \cap \mathcal{B} \subseteq \mathcal{L} \odot \mathcal{B} \odot \mathcal{S}$, *for every PFL* $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ *and every PFB* $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ *of* S.

Proof. (i) $\Rightarrow$ (ii) Assume that S is intra-regular. Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFL and a PFGB of S, respectively. For each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$. Then, $a = xa^2y = x(xa^2y)ay = (x^2a)(ayay)$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{L}}(p), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(q)\}] \\
&\geq \min\{\mu_{\mathcal{L}}(x^2a), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(ayay)\} \\
&= \min\left\{\mu_{\mathcal{L}}(x^2a), \sup_{ayay=mn}[\min\{\mu_{\mathcal{G}}(m), \mu_{\mathcal{S}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{L}}(x^2a), \min\{\mu_{\mathcal{G}}(aya), \mu_{\mathcal{S}}(y)\}\} \\
&= \min\{\mu_{\mathcal{L}}(x^2a), \mu_{\mathcal{G}}(aya)\} \\
&\geq \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\} \\
&= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a)\} \\
&= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{S}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{L}}(p), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{S}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{L}}(x^2a), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{S}})(ayay)\} \\
&= \max\left\{\lambda_{\mathcal{L}}(x^2a), \inf_{ayay=mn}[\max\{\lambda_{\mathcal{G}}(m), \lambda_{\mathcal{S}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{L}}(x^2a), \max\{\lambda_{\mathcal{G}}(aya), \lambda_{\mathcal{S}}(y)\}\} \\
&= \max\{\lambda_{\mathcal{L}}(x^2a), \lambda_{\mathcal{G}}(aya)\} \\
&\leq \max\{\lambda_{\mathcal{L}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a)\} \\
&= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}})(a).
\end{aligned}$$

It turns out that $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \odot \mathcal{G} \odot \mathcal{S}$.

(ii) $\Rightarrow$ (iii) In the fact that every PFB of S is also a PFGB of S. Hence, (iii) holds.

(iii) $\Rightarrow$ (i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFL and a PFR of S, respectively. Then, we get that $\mathcal{R}$ is also a PFB of S. By the hypothesis, $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \odot (\mathcal{R} \odot \mathcal{S}) \subseteq \mathcal{L} \odot \mathcal{R}$. Therefore, S is intra-regular by Theorem 4.2. □

Theorem 4.4. *Let* S *be a semigroup. Then, the following statements are equivalent:*

(i) S *is intra-regular;*

(ii) $\mathcal{G} \cap \mathcal{R} \subseteq \mathcal{S} \odot \mathcal{G} \odot \mathcal{R}$, *for every PFR* $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ *and every PFGB* $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ *of* S;

(iii) $\mathcal{B} \cap \mathcal{R} \subseteq \mathcal{S} \odot \mathcal{B} \odot \mathcal{R}$, *for every PFR* $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ *and every PFB* $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ *of* S.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFR and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFGB of S. Consider any element $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$. Also, $a = xa^2y = xa(xa^2y)y = (xaxa)(ay^2)$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a=pq}[\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(p), \mu_{\mathcal{R}}(q)\}] \\
&\geq \min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(xaxa), \mu_{\mathcal{R}}(ay^2)\} \\
&= \min\left\{\sup_{xaxa=mn}[\min\{\mu_{\mathcal{S}}(m), \mu_{\mathcal{G}}(n)\}], \mu_{\mathcal{R}}(ay^2)\right\} \\
&\geq \min\left\{\min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{G}}(axa)\}, \mu_{\mathcal{R}}(ay^2)\right\} \\
&= \min\{\mu_{\mathcal{G}}(axa), \mu_{\mathcal{R}}(ay^2)\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \mu_{\mathcal{R}}(a)\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\} \\
&= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{R}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a=pq}[\max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}})(p), \lambda_{\mathcal{R}}(q)\}] \\
&\leq \max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}})(xaxa), \lambda_{\mathcal{R}}(ay^2)\} \\
&= \max\left\{\inf_{xaxa=mn}[\max\{\lambda_{\mathcal{S}}(m), \lambda_{\mathcal{G}}(n)\}], \lambda_{\mathcal{R}}(ay^2)\right\} \\
&\leq \max\left\{\max\{\lambda_{\mathcal{S}}(x), \lambda_{\mathcal{G}}(axa)\}, \lambda_{\mathcal{R}}(ay^2)\right\} \\
&= \max\{\lambda_{\mathcal{G}}(axa), \lambda_{\mathcal{R}}(ay^2)\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \lambda_{\mathcal{R}}(a)\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a)\} \\
&= (\lambda_{\mathcal{G}} \cup \lambda_{\mathcal{R}})(a).
\end{aligned}$$

We obtain that $\mathcal{G} \cap \mathcal{R} \subseteq \mathcal{S} \odot \mathcal{G} \odot \mathcal{R}$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be a PFL and a PFR of S, respectively. Then, $\mathcal{L}$ is also a PFB of S. By the given assumption, it follows that $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{S} \odot \mathcal{L}) \odot \mathcal{R} \subseteq \mathcal{L} \odot \mathcal{R}$. Consequently, S is intra-regular by Theorem 4.2. $\square$

5. Regular and Intra-regular Semigroups

In the last section, we characterize the both regular and intra-regular semigroups by using the concepts of PFLs, PFRs, PFBs and PFGBs of semigroups.

Lemma 5.1. *(cf. [23]) For any semigroup S, it is both regular and intra-regular if and only if $B = B^2$ for every bi-ideal B of S.*

Theorem 5.2. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is both regular and intra-regular;*
(ii) *$\mathcal{B} = \mathcal{B} \odot \mathcal{B}$, for any PFB $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ of S;*
(iii) *$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \odot \mathcal{B}$, for any PFBs $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ and $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ of S.*

Proof. (i) ⇒ (iii) Assume that S is both regular and intra-regular. Let $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ and $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ be PFBs of S and let $a \in S$. Then, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. So,

$$a = axa = axaxa = ax(ya^2z)xa = (axya)(azxa).$$

Thus, we have

$$\begin{aligned}(\mu_\mathcal{A} \circ \mu_\mathcal{B})(a) &= \sup_{a=pq}[\min\{\mu_\mathcal{A}(p), \mu_\mathcal{B}(q)\}]\\ &\geq \min\{\mu_\mathcal{A}(axya), \mu_\mathcal{B}(azxa)\}\\ &\geq \min\{\min\{\mu_\mathcal{A}(a), \mu_\mathcal{A}(a)\}, \min\{\mu_\mathcal{B}(a), \mu_\mathcal{B}(a)\}\}\\ &= \min\{\mu_\mathcal{A}(a), \mu_\mathcal{B}(a)\}\\ &= (\mu_\mathcal{A} \cap \mu_\mathcal{B})(a)\end{aligned}$$

and

$$\begin{aligned}(\lambda_\mathcal{A} \circ \lambda_\mathcal{B})(a) &= \inf_{a=pq}[\max\{\lambda_\mathcal{A}(p), \lambda_\mathcal{B}(q)\}]\\ &\leq \max\{\lambda_\mathcal{A}(axya), \lambda_\mathcal{B}(azxa)\}\\ &\leq \max\{\max\{\lambda_\mathcal{A}(a), \lambda_\mathcal{A}(a)\}, \max\{\lambda_\mathcal{B}(a), \lambda_\mathcal{B}(a)\}\}\\ &= \max\{\lambda_\mathcal{A}(a), \lambda_\mathcal{B}(a)\}\\ &= (\lambda_\mathcal{A} \cup \lambda_\mathcal{B})(a).\end{aligned}$$

It follows that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \odot \mathcal{B}$.

(iii) ⇒ (ii) It is obvious.

(ii) ⇒ (i) Let B be any bi-ideal of S and $a \in B$. By Lemma 2.12, $\mathcal{C}_B$ is a PFB of S. Then, using the given assumption and Lemma 2.9, we have that $\mathcal{C}_B = \mathcal{C}_B \odot \mathcal{C}_B = \mathcal{C}_{BB}$. So, $\mu_{\mathcal{C}_{BB}}(a) = \mu_{\mathcal{C}_B}(a) = 1$. This implies that $a \in BB$. Hence, $B \subseteq BB$. Otherwise, $BB \subseteq B$. It turns out that $B = BB$. By Lemma 5.1, we conclude that S is both regular and intra-regular. □

The proof of the following theorem is similar to Theorem 5.2.

Theorem 5.3. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is both regular and intra-regular;*
(ii) *$\mathcal{B} \cap \mathcal{G} \subseteq \mathcal{B} \odot \mathcal{G}$, for any PFB $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ and any PFGB $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ of S;*

(iii) $\mathcal{B}\cap\mathcal{G} \subseteq \mathcal{G}\odot\mathcal{B}$, *for any PFB* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *and every PFGB* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *of* S;

(iv) $\mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} \odot \mathcal{H}$, *for any PFGBs* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *and* $\mathcal{H} = (\mu_\mathcal{H}, \lambda_\mathcal{H})$ *of* S.

The next theorem follows directly by Theorem 3.3 and Theorem 4.2.

Theorem 5.4. *Let* S *be a semigroup. Then,* S *is both regular and intra-regular if and only if* $\mathcal{R}\cap\mathcal{L} \subseteq (\mathcal{R}\odot\mathcal{L})\cap(\mathcal{L}\odot\mathcal{R})$, *for every PFL* $\mathcal{L} = (\mu_\mathcal{L}, \lambda_\mathcal{L})$ *and every PFR* $\mathcal{R} = (\mu_\mathcal{R}, \lambda_\mathcal{R})$ *of* S.

Theorem 5.5. *For a semigroup* S, *the following statements are equivalent:*

(i) S *is both regular and intra-regular;*

(ii) $\mathcal{L}\cap\mathcal{G} \subseteq (\mathcal{L}\odot\mathcal{G})\cap(\mathcal{G}\odot\mathcal{L})$, *for every PFL* $\mathcal{L} = (\mu_\mathcal{L}, \lambda_\mathcal{L})$ *and every PFGB* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *of* S;

(iii) $\mathcal{L}\cap\mathcal{B} \subseteq (\mathcal{L}\odot\mathcal{B})\cap(\mathcal{B}\odot\mathcal{L})$, *for every PFL* $\mathcal{L} = (\mu_\mathcal{L}, \lambda_\mathcal{L})$ *and every PFB* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *of* S;

(iv) $\mathcal{R}\cap\mathcal{G} \subseteq (\mathcal{R}\odot\mathcal{G})\cap(\mathcal{G}\odot\mathcal{R})$, *for each PFR* $\mathcal{R} = (\mu_\mathcal{R}, \lambda_\mathcal{R})$ *and each PFGB* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *of* S;

(v) $\mathcal{R}\cap\mathcal{B} \subseteq (\mathcal{R}\odot\mathcal{B})\cap(\mathcal{B}\odot\mathcal{R})$, *for each PFR* $\mathcal{R} = (\mu_\mathcal{R}, \lambda_\mathcal{R})$ *and each PFB* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *of* S;

(vi) $\mathcal{A}\cap\mathcal{B} \subseteq (\mathcal{A}\odot\mathcal{B})\cap(\mathcal{B}\odot\mathcal{A})$, *for any PFBs* $\mathcal{A} = (\mu_\mathcal{A}, \lambda_\mathcal{A})$ *and* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *of* S;

(vii) $\mathcal{B}\cap\mathcal{G} \subseteq (\mathcal{B}\odot\mathcal{G})\cap(\mathcal{G}\odot\mathcal{B})$, *for any PFB* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *and any PFGB* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *of* S;

(viii) $\mathcal{G}\cap\mathcal{H} \subseteq (\mathcal{G}\odot\mathcal{H})\cap(\mathcal{H}\odot\mathcal{G})$, *for any PFGBs* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *and* $\mathcal{H} = (\mu_\mathcal{H}, \lambda_\mathcal{H})$ *of* S.

Proof. (i) $\Rightarrow$ (iv) It follows by Theorem 5.3.

Since every PFL (resp., PFR) of S is also a PFB of S, and every PFB of S is also a PFGB of S, it implies that (vi) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (vi) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) are clear.

(viii) $\Rightarrow$ (i) It follows from Theorem 5.2.

(iii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) It obtains by Theorem 5.4. $\square$

Lemma 5.6. [11] *For a semigroup* S, *the following conditions are equivalent:*

(i) S *is both regular and intra-regular;*

(ii) $B \cap L \subseteq BLB$, *for each left ideal* L *and each bi-ideal* B *of* S;

(iii) $B \cap R \subseteq BRB$, *for each right ideal* R *and each bi-ideal* B *of* S.

Theorem 5.7. *Let* S *be a semigroup. Then, the following statements are equivalent:*

(i) S *is both regular and intra-regular;*

(ii) $\mathcal{L}\cap\mathcal{G} \subseteq \mathcal{G}\odot\mathcal{L}\odot\mathcal{G}$, *for any PFL* $\mathcal{L} = (\mu_\mathcal{L}, \lambda_\mathcal{L})$ *and any PFGB* $\mathcal{G} = (\mu_\mathcal{G}, \lambda_\mathcal{G})$ *of* S;

(iii) $\mathcal{L}\cap\mathcal{B} \subseteq \mathcal{B}\odot\mathcal{L}\odot\mathcal{B}$, *for any PFL* $\mathcal{L} = (\mu_\mathcal{L}, \lambda_\mathcal{L})$ *and any PFB* $\mathcal{B} = (\mu_\mathcal{B}, \lambda_\mathcal{B})$ *of* S;

(iv) $\mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \odot \mathcal{R} \odot \mathcal{G}$, *for any PFR* $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ *and any PFGB* $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ *of* S*;*
(v) $\mathcal{R} \cap \mathcal{B} \subseteq \mathcal{B} \odot \mathcal{R} \odot \mathcal{B}$, *for any PFR* $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ *and any PFB* $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ *of* S.

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFL and a PFGB of S, respectively. Let $a \in S$. Then, by assumption, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. It follows that

$$a = axa = (axa)x(axa) = ax(ya^2z)x(ya^2z)xa = (axya)(azxya)(azxa).$$

Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{G}}(p), (\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(q)\}] \\
&\geq \min\{\mu_{\mathcal{G}}(axya), (\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(azxyaazxa)\} \\
&= \min\left\{\mu_{\mathcal{G}}(axya), \sup_{azxyaazxa=mn}[\min\{\mu_{\mathcal{L}}(m), \mu_{\mathcal{G}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{G}}(axya), \min\{\mu_{\mathcal{L}}(azxya), \mu_{\mathcal{G}}(azxa)\}\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\}\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{G}}(p), (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{G}}(axya), (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(azxyaazxa)\} \\
&= \max\left\{\lambda_{\mathcal{G}}(axya), \inf_{azxyaazxa=mn}[\max\{\lambda_{\mathcal{L}}(m), \lambda_{\mathcal{G}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{G}}(axya), \max\{\lambda_{\mathcal{L}}(azxya), \lambda_{\mathcal{G}}(azxa)\}\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \max\{\lambda_{\mathcal{L}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}\}\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{L}}(a)\} \\
&= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}})(a).
\end{aligned}$$

This means that $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{G} \odot \mathcal{L} \odot \mathcal{G}$.

(ii) $\Rightarrow$ (iii) It is true by every PFB of S is also a PFGB of S.

(iii) $\Rightarrow$ (i) Let L be any left ideal and B be any bi-ideal of S. Then, $\mathcal{C}_L$ and $\mathcal{C}_B$ are a PFL and a PFB of S, respectively. Let $x \in B \cap L$. By using the hypothesis and Lemma 2.9, so

$$\mathcal{C}_{B \cap L} = \mathcal{C}_B \cap \mathcal{C}_L \subseteq \mathcal{C}_B \odot \mathcal{C}_L \odot \mathcal{C}_B = \mathcal{C}_{BLB}.$$

This means that $\mu_{\mathcal{C}_{BLB}}(x) \geq \mu_{\mathcal{C}_{B \cap L}}(x) = 1$, that is, $x \in BLB$. Hence, $B \cap L \subseteq BLB$. Therefore, S is both regular and intra-regular by Lemma 5.6.

In the similar way, we can show that (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) holds. $\square$

Theorem 5.8. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is both regular and intra-regular;*

(ii) *$\mathcal{L} \cap \mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \odot \mathcal{R} \odot \mathcal{L}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every PFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;*

(iii) *$\mathcal{L} \cap \mathcal{R} \cap \mathcal{B} \subseteq \mathcal{B} \odot \mathcal{R} \odot \mathcal{L}$, for every PFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every PFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every PFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be a PFL, a PFR and a PFGB of S, respectively. For any $a \in S$, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Then,

$$a = axa = axaxaxa = ax(ya^2z)x(ya^2z)xa = (axya)(azxya)(azxa).$$

Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{G}} \circ \mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{G}}(p), (\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(q)\}] \\
&\geq \min\{\mu_{\mathcal{G}}(axya), (\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(azxyaazxa)\} \\
&= \min\left\{\mu_{\mathcal{G}}(axya), \sup_{azxyaazxa=mn}[\min\{\mu_{\mathcal{R}}(m), \mu_{\mathcal{L}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{G}}(axya), \min\{\mu_{\mathcal{R}}(azxya), \mu_{\mathcal{L}}(azxa)\}\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\}\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{R}} \cap \mu_{\mathcal{G}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(a) &= \inf_{a=pq}[\max\{\lambda_{\mathcal{G}}(p), (\lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(q)\}] \\
&\leq \max\{\lambda_{\mathcal{G}}(axya), (\lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(azxyaazxa)\} \\
&= \max\left\{\lambda_{\mathcal{G}}(axya), \inf_{azxyaazxa=mn}[\max\{\lambda_{\mathcal{R}}(m), \lambda_{\mathcal{L}}(n)\}]\right\} \\
&\leq \max\{\lambda_{\mathcal{G}}(axya), \max\{\lambda_{\mathcal{R}}(azxya), \lambda_{\mathcal{L}}(azxa)\}\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \max\{\lambda_{\mathcal{R}}(a), \lambda_{\mathcal{L}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a), \lambda_{\mathcal{L}}(a)\} \\
&= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{R}} \cup \lambda_{\mathcal{G}})(a).
\end{aligned}$$

This shows that $\mathcal{L} \cap \mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \odot \mathcal{R} \odot \mathcal{L}$.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) Let $a \in S$. We can show that $a \cup Sa, a \cup aS$ and $a \cup aa \cup aSa$ are a left ideal, a right ideal and a bi-ideal of S with containing a, respectively. Also, we obtain that $\mathcal{C}_{a \cup Sa}, \mathcal{C}_{a \cup aS}$ and $\mathcal{C}_{a \cup aa \cup aSa}$ are a PFL, a PFR and a PFB of S, respectively. Then, using the given assumption and Lemma 2.9, we have

$$\mathcal{C}_{(a \cup Sa) \cap (a \cup aS) \cap (a \cup aa \cup aSa)} = \mathcal{C}_{a \cup Sa} \cap \mathcal{C}_{a \cup aS} \cap \mathcal{C}_{a \cup aa \cup aSa}$$

$$\subseteq \mathcal{C}_{a\cup aa\cup aSa} \odot \mathcal{C}_{a\cup Sa} \odot \mathcal{C}_{a\cup aS}$$
$$= \mathcal{C}_{(a\cup aa\cup aSa)(a\cup Sa)(a\cup aS)}.$$

It turns out that

$$\mu_{\mathcal{C}_{(a\cup aa\cup aSa)(a\cup Sa)(a\cup aS)}}(a) \geq \mu_{\mathcal{C}_{(a\cup Sa)\cap(a\cup aS)\cap(a\cup aa\cup aSa)}}(a) = 1.$$

This means that $a \in (a\cup aa\cup aSa)(a\cup Sa)(a\cup aS)$. Hence, $a \in (aSa)\cap(Sa^2S)$. That is, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Consequently, S is both regular and intra-regular. □

6. Conclusion

The concept of Pythagorean fuzzy sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. In this paper, we examined on characterizations of regularities in semigroups using the concept of Pythagorean fuzzy sets. In section 3, we have characterized regular semigroups by the concept of Pythagorean fuzzy left (resp., right) ideals and Pythagorean fuzzy (resp., generalized) bi-ideals of semigroups. Moreover, in section 4, some characterizations of intra-regular semigroups in terms of Pythagorean fuzzy left (resp., right) ideals and Pythagorean fuzzy (resp., generalized) bi-ideals of semigroups are presented. Finally, the both regular and intra-regular semigroups characterized by the properties of Pythagorean fuzzy left (resp., right) ideals and Pythagorean fuzzy (resp., generalized) bi-ideals of semigroups. In our future work, it will be achievable to characterize many classes of regularities in semigroups or other algebraic structures by the concepts of their Pythagorean fuzzy sets.

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