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NUMERICAL TREATMENT OF NON-MONOTONIC BLOW-PROBLEMS BASED ON SOME NON-LOCAL TRANSFORMATIONS

BASEM S. ATTILI

Abstract. We consider the numerical treatment of blow-up problems having non-monotonic singular solutions that tend to infinity at some point in the domain. The use of standard numerical methods for solving problems with blow-up solutions can lead to significant errors. The reason being that solutions of such problems have singularities whose positions are unknown in advance. To be able to integrate such non-monotonic blow-up problems, we describe and use a method of non-local transformations. To show the efficiency of the method, we present a comparison of exact and numerical solutions in addition to some comparison with the work of other authors.

AMS Mathematics Subject Classification : 65M20, 65M12.
Key words and phrases : Nonlinear differential equations, blow-up solutions, non-local transformations, numerical integration.

1. Introduction

Integrating nonlinear first and second order Cauchy problems with blow-up solutions using standard numerical methods leads to difficulties and significant errors. This is due to the fact that their solutions tend to infinity at some point x^* in the domain which is not known in advance, see [11].

Many authors treated the blow-up problem numerically. To mention a few, the authors in [1] used an adaptive time step procedure with explicit Runge-Kutta method. An arc length transformation technique was used by [8] that generates a sequence which is linearly convergent. A re-scaling technique was used by [10] with the help of time-series approach that controls the growth of the re-scaled variables. The authors in [12] used a method based on compactification and Lyapunov function validation method. Compactification was also used by [7] to compute critical points at infinity with blow up solutions. The idea of

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transformation was used by [2] to transform the Falkner-Skan boundary value problem on a semi infinite interval with a condition at infinity to a problem defined on a finite interval.

We will consider the numerical treatment of such Cauchy problem by first transforming the system into an equivalent system of equations using some local and non-local transformation. We will consider introducing a new independent variable in two ways; namely, a differential transform as a new variable in one case and a non-local variable in the second. In both cases the solution will be in a parameterized form and is equivalent to the solution of the original Cauchy problem, similar transformations with different details can be found in [13, 14]. One of the most important characteristics of this papraмeterized solution is that it does not have a blow up solution, making it suitable to be solved numerically by any standard initial value solver like 4-th order Runge-Kutta method. The efficiency of the suggested transformation will be shown through solving some examples that admit an exact solution by comparing the numerical solution with the exact. The main advantage of the proposed transformation is that the solution is obtained in a parametric form which, for large values of the introduced new variable, the solution tend to the asymptotic solution exponentially. Other transformations like the hodograph or the arc-length leads to solutions which tend to the asymptotic solution slowly.

The outline of the paper will be as follows. In Section 2 we will consider the first order problem while in Section 3 we will consider the second order problem. In Section 4 we will transform the second order problem using a non-local transformation. Numerical results will are presented in the in each of the sections 2-4. Some conclusions will be given in the final section.

2. Blow-Up Solutions - 1^{st} Order

Consider the general first order problem of the form

$$y' = \beta y^n,\ x > 0,\ y(0) = \alpha \tag{1}$$

with $n > 1,\ \alpha,\ \beta$ are positive constants.

The exact solution of this differential equation can be given in the form

$$y = \frac{C_1}{(x^* - x)^{C_2}}$$

with $C_1 = \left[\beta\,(n-1)^{\frac{-1}{n-1}}\right]$, $C_2 = \frac{1}{n-1}$ and the singular point $x^* = \frac{1}{\alpha^{n-1}\beta(n-1)}$.

Here the solution exists for $0 \leq x < x^*$ and does not exist for $x \geq x^*$. For example, if $\alpha = \beta = 1$ and $n = 2$, then problem (1) will be

$$y' = y^2,\ x > 0,\ y(0) = 1 \tag{2}$$

$C_1 = 1$, $C_2 = 1$ and the singular point $x^* = 1$, hence the solution is $y = (1-x)^{-1}$, the solution exists for $0 \leq x < 1$ and does not exist for $x \geq 1$.

From the this example, see Figure 1, we notice that applying explicit numerical methods will lead to faster growth in the solution leading to an overflow when passing through the singularity which is usually unknown. Again when using implicit methods, the solution goes to the negative region before or close to the singularity.

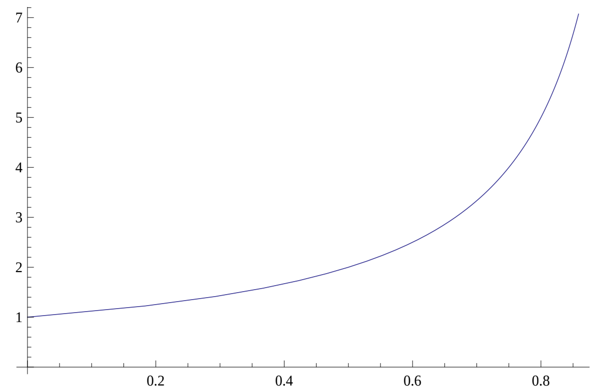


FIGURE 1. Exact solution of the system given in (2).

2.1. Transformation of the 1^{st} Order Problem. Given the general first order initial value problem

$$y' = f(x, y), \ x > 0, \ y(x_0) = y_0 \tag{3}$$

where the prime "$\prime$" represents differentiation with respect to x, $f(x, y) > 0$, $x_0 \geq 0$, $y_0 > 0$ and with $\delta > 0$, we have $\frac{f}{y^{1+\delta}} \to \infty$ as $y \to \infty$.

Now if we let $t = f(x, y)$ then assuming $x = x(t)$ and $y = y(t)$, differentiating with respect to "t", we have

$$1 = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dx} \frac{dx}{dt}.$$

Hence with $t = f(x, y)$ and $y' = t$, we have

$$\frac{dx}{dt} = \frac{1}{f_x + t \ f_y}$$

and similarly

$$\frac{dy}{dt} = \frac{t}{f_x + t \ f_y}.$$

This leads to the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{f_x + t \ f_y}, \ \frac{dy}{dt} = \frac{t}{f_x + t \ f_y} \\ x(t_0) &= x_0, \ y(t_0) = y_0. \end{aligned} \tag{4}$$

with $t_0 = f(x_{0,} \ y_0)$.

Assuming $f_x + t\ f_y > 0$ for $t_0 < t < \infty$, then (4) can be solved numerically using any standard initial value solver without any difficulty since $\frac{dx}{dt} \to 0$ as $t \to \infty$.

If we apply this transformation to the test problem given in (2); that is, $t = y'$ or $t = y^2$, then the system to be solved numerically is given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{f_x + t\ f_y},\ \frac{dy}{dt} = \frac{t}{f_x + t\ f_y} \\ x\,(t_0) &= x_0,\ y(t_0) = y_0. \end{aligned} \tag{5}$$

2.2. Non-Local Transformation. We start by introducing an auxiliary variable η, which is non-local, of the form

$$\eta = \int_{x_0}^{x} g(x, y(x))dx.$$

Then we transform the Cauchy problem into a new system of differential equations as follows. With $\eta > 0$, and from the chain rule, we have

$$\frac{dx}{d\eta}\frac{d\eta}{dx} = 1 \text{ or } \frac{dx}{d\eta} g(x,y) = 1 \text{ or } \frac{dx}{d\eta} = \frac{1}{g(x,y)}$$

and similarly

$$\frac{dy}{d\eta}\frac{d\eta}{dx} = f(x,y) \text{ or } \frac{dy}{d\eta} = \frac{f(x,y)}{g(x,y)}.$$

Here $g(x,y)$ is a regularizing function that depends on the solution of (3) and $g(x,y)$ is such that

$$g > 0 \text{ for } x \geq x_0,\ y \geq y_0 \text{ and } g \to \infty \text{ as } y \to \infty. \tag{6}$$

Notice also that as a result $\frac{f}{g} \to k$ for some constant k as $y \to \infty$ and $\frac{dx}{d\eta} = \frac{1}{g(x,y)} \to 0$ as $y \to \infty$.

There are more than one choice of $g(x,y)$ as long it satisfies the conditions in (6). Some of the choices are as follows:

(1) Arc-length where $g = \sqrt{1+f^2}$ and in such a case, the system to be solved is

$$\begin{aligned} \frac{dx}{d\eta} &= \frac{1}{\sqrt{1+f^2}},\ \frac{dy}{d\eta} = \frac{f}{\sqrt{1+f^2}} \\ x(0) &= x_0,\ y(0) = y_0. \end{aligned}$$

(2) $g = 1 + |f|$ and the system to be solved is

$$\begin{aligned} \frac{dx}{d\eta} &= \frac{1}{1+|f|},\ \frac{dy}{d\eta} = \frac{f}{1+|f|} \\ x(0) &= x_0,\ y(0) = y_0. \end{aligned}$$

(3) $g = \frac{f}{y}$ and the system to be solved is

$$\begin{aligned}\frac{dx}{d\eta} &= \frac{y}{f},\ \frac{dy}{d\eta} = y\\ x(0) &= x_0,\ y(0) = y_0.\end{aligned}$$

whose solution is $y = y_0 e^{\eta}$.

Note that one can combine (1) and (2) and take $g = k_1 + (k_2 + |f|^n)^{\frac{1}{n}}$. We now test some of these choices numerically.

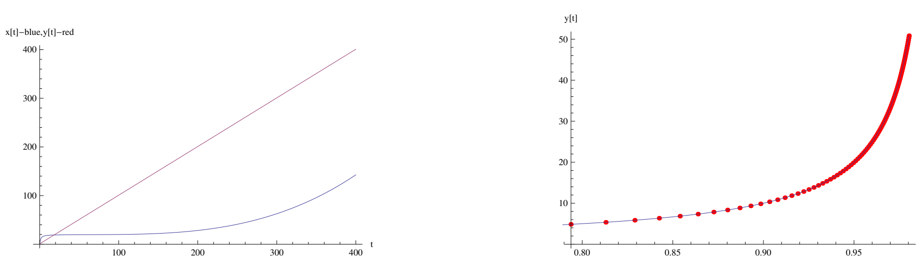
Example 1: Given

$$y' = y^2,\ x(0) = 0,\ y(0) = 1$$

1) If we choose $g = \sqrt{1+f^2}$, then the system to be solved will be

$$\begin{aligned}\frac{dx}{d\eta} &= \frac{1}{\sqrt{1+y^4}},\ \frac{dy}{d\eta} = \frac{y^2}{\sqrt{1+y^4}}\\ x(0) &= 0,\ y(0) = 1.\end{aligned}$$

Then solving using 4th order Runge-Kutta method [4], the results obtained are given in Figure 2. The first part of the figure gives the parametric solutions $x(t)$ and $y(t)$ while the second shows the exact and the computed solution $y(t)$. We observe that the computed solution (dots) agrees very well with the computed one (solid).



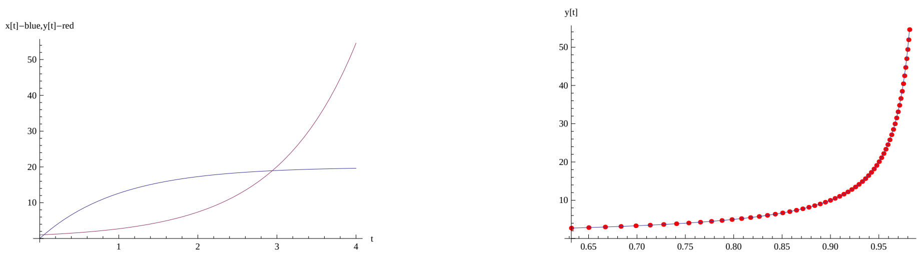
(A) (B)

Figure 2. (A): The solutions $x(t)$ and $y(t)$.
(B): The exact (solid) and computed (dots) solution $y(t)$.

2) If we choose $g = \frac{f}{y}$, then the system to be solved will be

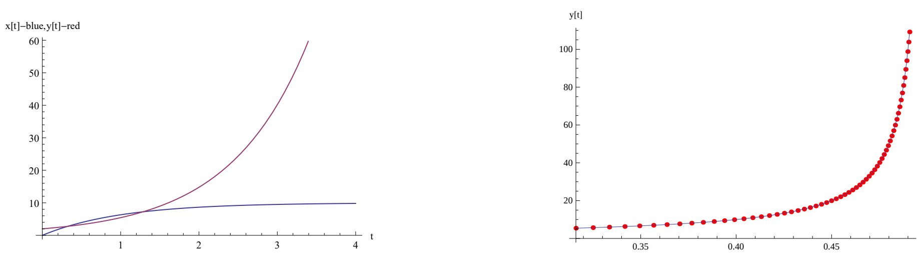
$$\frac{dx}{d\eta} = \frac{1}{y},\ \frac{dy}{d\eta} = y.$$

then with $x(0) = 0,\ y(0) = 1$, the results are given in Figure 3 while with $x(0) = 0,\ y(0) = 2$, the results are given in Figure 4. The same observation again, there is a very well agreement between the exact and the computed solutions.



(A) (B)

FIGURE 3. (A): The solutions $x(t)$ and $y(t)$.
(B): The exact (solid) and computed (dots) solution $y(t)$.



(A) (B)

FIGURE 4. (A): The solutions $x(t)$ and $y(t)$.
(B): The exact (solid) and computed (dots) solution $y(t)$.

3. Blow-Up Problem - 2^{nd} Order

The problem under consideration has the form

$$\begin{aligned} y'' &= f\left(x, y, y'\right), \ x > x_0 \\ y(x_0) &= y_0, \ y'(x_0) = y_1 \end{aligned} \tag{7}$$

where the prime "$'$" denotes differentiation with respect to x with $f(x, y, u) > 0$ if $y > y_0 > 0$ and $u > y_1 > 0$ also f increases fast as $y \to \infty$.

Then using the same logic as that of the first order case, let

$$y' = t, \ y'' = f(x, y, t)$$

with $t = t(x)$ and $y = y(x)$ are the unknown functions.

Now

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = t\frac{dx}{dt}.$$

also since

$$\frac{d}{dt}\left(y'\right) = 1 \text{ implies } 1 = \frac{dy'}{dx}\frac{dx}{dt} = y''\frac{dx}{dt}.$$

or

$$y''\frac{dx}{dt} = 1 \text{ leads to } \frac{dx}{dt} = \frac{1}{f(x,y,t)}$$

and similarly

$$\frac{dy}{dt} = \frac{t}{f(x,y,t)}.$$

This leads to the system to be solved given as

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{f(x,y,t)},\ \frac{dy}{dt} = \frac{t}{f(x,y,t)} \\ x(t_0) &= x_0,\ y(t_0) = y_0,\ t_0 = y_1. \end{aligned}$$

The solution of the system does not have a blow-up singularities and can be solved with a fixed step initial value solver like 4th order Runge-Kutta method, see [4].

To illustrate the transformation above we consider the following general example.

Example 2: Consider the general problem

$$y'' = \beta n y^{n-1} y',\ x > 0,\ y(0) = \alpha,\ y'(0) = \alpha^n \beta. \tag{8}$$

Note that this is the derivative of $y' = \beta y^n,\ y(0) = \alpha$ with $\alpha > 0,\ \beta > 0$ and $n > 1$ whose exact solution is

$$y = A\left(x^* - x\right)^{-B}$$

and no solution for $x \geq x^*,\ A = \left[B\left(n-1\right)\right]^{\frac{1}{1-n}},\ x^* = \frac{1}{\alpha^{n-1}B(n-1)},\ B = \frac{1}{n-1} > 0.$

Now with $\alpha = \beta = 1$ and $n = 2$, the problem becomes

$$y'' = 2y^3 y',\ x > 0,\ y(0) = 1,\ y'(0) = 1.$$

If we let $t = y'$, then the system to be solved is

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2y^3},\ \frac{dy}{dt} = \frac{t}{2y^3} \\ t_0 &= 1,\ x(1) = 0,\ y(1) = 1. \end{aligned}$$

Solving the system using Runge-Kutta of order 4, the results obtained for the exact and the numerical solutions are given in Figure 5.

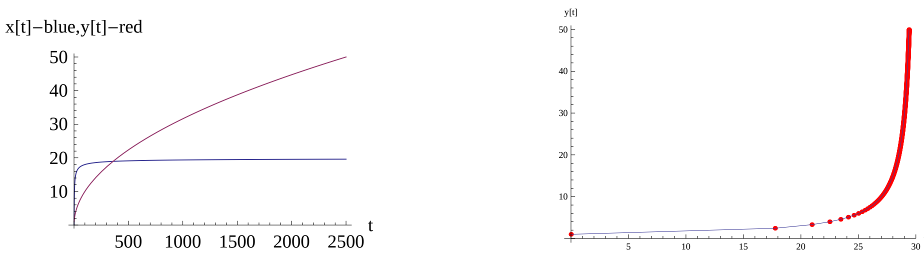
Example 3: Consider the problem

$$y'' = e^{3y},\ x > 0,\ y(0) = 0,\ y'(0) = 1.$$

The resulting system is

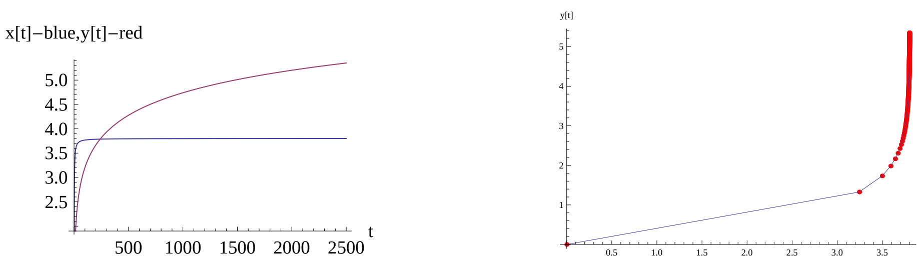
$$\begin{aligned} \frac{dx}{dt} &= e^{-3y},\ \frac{dy}{dt} = te^{-3y} \\ t_0 &= 1,\ x(1) = 0,\ y(1) = 0. \end{aligned}$$

The results obtained are reported in Figure 6.



(A) (B)

FIGURE 5. (A): The solutions $x(t)$ and $y(t)$.
(B): The exact (solid) and computed (dots) solution $y(t)$.



(A) (B)

FIGURE 6. (A): The solutions $x(t)$ and $y(t)$.
(B): The exact (solid) and computed (dots) solution $y(t)$.

4. None-Local Transform for 2^{nd} Order

Consider the second order problem given by (7)

$$\begin{aligned} y'' &= f\left(x, y, y'\right), \ x > x_0 \\ y(x_0) &= y_0, \ y'(x_0) = y_1. \end{aligned}$$

We rewrite the problem in an equivalent first order system

$$\begin{aligned} y' &= u, \ u' = f(x, y, u) \\ y(x_0) &= y_0, \ u(x_0) = y_1. \end{aligned}$$

As we have done before in the first order case, we introduce the variable

$$\eta = \int_{x_0}^{x} g(x, y, u) dx, \ y = y(x), \ u = u(x).$$

This leads to the system (details are omitted since they are similar to the first order case)

$$\begin{aligned} \frac{dx}{d\eta} &= \frac{1}{g(x,y,u)},\ \frac{dy}{d\eta} = \frac{u}{g(x,y,u)}, \frac{du}{d\eta} = \frac{f(x,y,u)}{g(x,y,u)},\ \eta > 0 \\ x(0) &= x_0,\ y(0) = y_0,\ u(0) = y_1. \end{aligned} \tag{9}$$

Then with the proper choice of $g(x, y, u)$, the resulting system in (9) will not have a blow-up singularities. Hence can be solved using any initial value solver like the 4th order Runge-Kutta method with a fixed step.

More than one choice of $g(x, y, u)$ are again possible, to mention some:

(1) $g = f(x, y, u)$: In this case the third equation of (9) implies $u = \eta + y_1$ obtained through simple integration.

(2) $g = \frac{u}{y}$: With this choice the second equation of (9) simplifies and leads to $y = y_0 e^{\eta}$ reducing the system into two equations to be solved for $x(\eta)$ and $u(\eta)$.

(3) $g = (1 + |u|^m + |f|^m)^{\frac{1}{m}}$: If $m = 2$, this transformation corresponds to the arc-length transformation.

We now consider some numerical examples:

Example 4: Consider the example given by (8)

$$y'' = \beta n y^{n-1} y',\ x > 0,\ y(0) = \alpha,\ y'(0) = \alpha^n \beta.$$

Using $g = \frac{u}{y}$ leads to the system

$$\begin{aligned} \frac{dx}{d\eta} &= \frac{y}{u},\ \frac{dy}{d\eta} = y, \frac{du}{d\eta} = \frac{\beta^2 n y^{2n}}{u},\ \eta > 0 \\ x(0) &= 0,\ y(0) = \alpha,\ u(0) = \alpha^n \beta. \end{aligned}$$

Its solution is given as

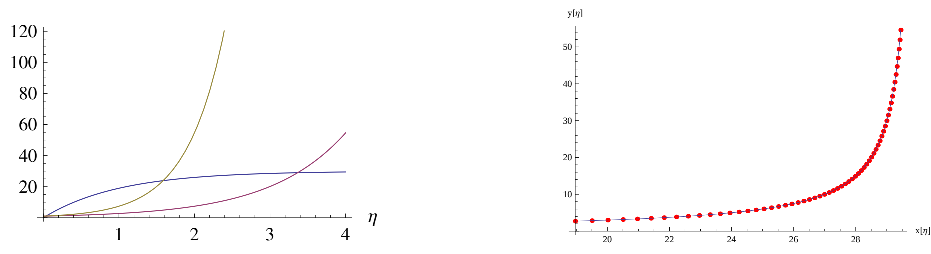
$$x = \frac{1 - e^{-(n-1)\eta}}{\beta\,(n-1)\,\alpha^{n-1}},\ y = \alpha e^{\eta},\ u = \beta \alpha^n e^{n\eta}.$$

The numerical solution for 2 cases with $\alpha = 1, 2,\ \beta = 1,\ n = 2$ is given in Figure 7.

Example 5: For the same previous example but with $g = 1 + |f| + |u|$, the system to be solved is

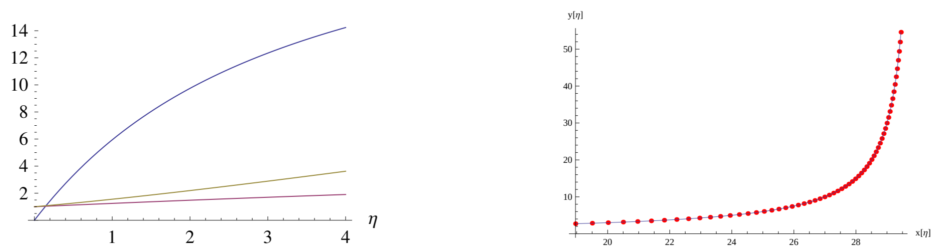
$$\begin{aligned} \frac{dx}{d\eta} &= \frac{1}{1 + |\beta n y^{n-1} u| + |u|}, \\ \frac{dy}{d\eta} &= \frac{u}{1 + |\beta n y^{n-1} u| + |u|}, \\ \frac{du}{d\eta} &= \frac{\beta n y^{n-1} u}{1 + |\beta n y^{n-1} u| + |u|},\ \eta > 0 \\ x(0) &= 0,\ y(0) = \alpha,\ u(0) = \alpha^n \beta. \end{aligned}$$

and the results obtained are given in Figure 8.



(A) (B)

FIGURE 7. (A): The solutions $u(\eta)$, $x(\eta)$ and $y(\eta)$.
(B): The exact (solid) and computed (dots)of the solutions
$x(\eta)$ against $y(\eta)$.



(A) (B)

FIGURE 8. (A): The solutions $u(\eta)$, $x(\eta)$ and $y(\eta)$.
(B): The exact (solid) and computed (dots) of the solutions
$x(\eta)$ against $y(\eta)$.

5. Conclusion

First and second order Cauchy differential equations with blow-up solutions were considered. Through some differential and non-local transformation we were able to produce a parameterized system without the singularity due to the blow-up solution. This allowed us a standard initial value method like the 4th order Runge-Kutta method to compute the solution without difficulties or errors. The suggested transformations compete well with other transformations.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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Basem Attili received M.Sc. and Ph.D. from Southern Methodist University at Dallas-Texas in the US. Since 2008 he has been at Sharjah University. His research interests include numerical analysis and applied mathematics.

Department of Mathematics, College of Sciences, University of Sharjah, Sharjah P.O.Box 27272, United Arab Emirates.
e-mail: b.attili@sharjah.ac.ae