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MATHEMATICAL ANALYSIS OF CONTACT PROBLEM WITH DAMPED RESPONSE OF AN ELECTRO-VISCOELASTIC ROD

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Abstract. We consider a mathematical model which describes the quasistatic contact of electro-viscoelastic rod with an obstacle. We use a modified Kelvin-Voigt viscoelastic constitutive law in which the elasticity operator is nonlinear and locally Lipschitz continuous, taking into account the piezoelectric effect of the material. We model the contact with a general damped response condition. We establish a local existence and uniqueness result of the solution by using arguments of time-dependent nonlinear equations and Schauder's fixed-point theorem and obtain a global existence for small enough data.

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1. Introduction

A piezoelectric material possesses the ability to transform mechanical energy into electrical energy, known as the direct piezoelectric effect, and the reverse process as well. These characteristics have led to a diverse array of applications for these materials, making them the subject of extensive research and advancement. Within the realm of structural mechanics, many scenarios involve the interaction of a deformable piezoelectric material with other bodies. In a medical context, accurately modeling the interaction between surgical instruments and bodily organs is of paramount significance to enable realistic simulations.

In this paper, we present a comprehensive model that addresses the quasistatic

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contact between an electro-elastic-viscoplastic rod and an obstacle, incorporating a general damped response condition. Our approach is particularly novel due to the incorporation of a nonlinear electro-viscoelastic constitutive law into the model, introducing nonstandard elements that enhance its applicability and relevance in real-world scenarios. Furthermore, our work sheds light on the intricate interplay between material properties and contact mechanics in electro-elastic-viscoplastic systems, paving the way for deeper insights into their behavior and potential engineering applications.

The referenced studies primarily focused on problems of contact, both dynamic and quasistatic, involving beams and rods, with a predominant emphasis on materials exhibiting linear elastic or viscoelastic behavior, as documented in [1, 2, 3, 4, 5, 6], among others in the cited literature. The studies conducted in [7, 8] delved into the intricacies of initial and boundary frictional problems concerning nonlinear Kelvin-Voigt viscoelastic bodies, shedding light on their complex behavior and response. In all these papers the elasticity operator was assumed to be a Lipschitz continuous operator and the weak solutions of the corresponding mechanical problems were global in time.

The distinctive aspect of our paper lies in the incorporation of a modified Kelvin-Voigt model, where the elasticity operator is locally Lipschitz continuous, and it takes into account the influence of the piezoelectric effect. This novel approach introduces a nonstandard mathematical problem, for which we establish a global existence and uniqueness result, marking a significant contribution to the field of electro-elastic modeling. Furthermore, our findings hold the potential to expand the scope of applications in this domain, enriching our understanding of complex material behaviors.

The paper is organized as follows. In Section 2, we describe the model for the process. In Section 3, we list the assumptions on the problem data, present the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1 and Theorem 3.2 as well as the proof of local existence and uniqueness result, and it is based on the theory of a time-dependent nonlinear equations and Schauder's fixed-point theorem. In Section 4, we prove the global existence.

2. Problem statement and its variational formulation

In this section, we construct a mathematical model for the process of contact with a damped response between an electro-viscoelastic rod and an obstacle or foundation, and provide its variational formulation. The physical setting and the process are as follows: An electro-viscoelastic rod occupies, in its reference configuration, the interval $\Omega = (0, L)$, and it moves along the x-axis. It is clamped at its left ($x = 0$), where the displacement and electrical potential vanish. The right end ($x = L$) is in contact with the obstacle. The rod is subjected to body forces, leading to the evolution of its state (Figure 1).

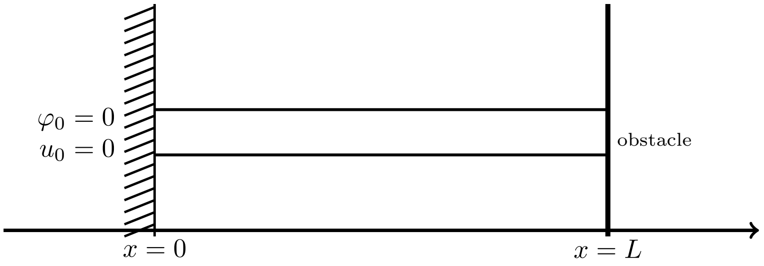

FIGURE 1. The rod in a contact process.

Let $[0,T]$ represent the time interval of interest, where $T > 0$, and consider $Q_T = (0,L) \times [0,T]$. In the following, for $(x,t) \in Q_T$, we will use the following notations: $u = u(x,t)$ for the displacement field, $\sigma = \sigma(x,t)$ for the stress tensor, $D = D(x,t)$ for the electric displacement field, and $E(\varphi) = -\partial_x \varphi$ for the electric field. Here, $\varphi = \varphi(x,t)$ represents the electric potential.

Thus, the contact problem with damped response of an electro-viscoelastic rod is described as follows.

Problem P. *Find a displacement function* $u : Q_T \to \mathbb{R}$*, a stress function* $\sigma : Q_T \to \mathbb{R}$*, an electric potential* $\varphi : Q_T \to \mathbb{R}$ *and an electric displacement* $D : Q_T \to \mathbb{R}$ *such that:*

$$\sigma = \eta \partial_x \dot{u} + \mu(\partial_x u - \Gamma(\partial_x u)^2) - \varrho E(\varphi), \tag{2.1}$$

$$D = \varrho \partial_x u + \beta E(\varphi), \tag{2.2}$$

$$\partial_x \sigma + f = 0, \tag{2.3}$$

$$\partial_x D - q = 0, \tag{2.4}$$

$$u(0,t) = 0 \text{ for } t \in [0,T], \tag{2.5}$$

$$-\sigma(L,t) = p(\dot{u}(L,t)) \text{ for } t \in (0,T), \tag{2.6}$$

$$\varphi(0,t) = 0 \text{ for } t \in (0,T), \tag{2.7}$$

$$D(L,t) = 0 \text{ for } t \in (0,T), \tag{2.8}$$

$$u(x,0) = 0 \text{ for } x \in (0,L). \tag{2.9}$$

A quadriplet of functions (u, σ, φ, D) which satisfies (2.1)-(2.9) for all $t \in [0,T]$ is called a *global* solution of the mathematical **Problem** P. A quadriplet of functions (u, σ, φ, D) which satisfies (2.1)-(2.9) for all $t \in [0,T^*]$ where $T^* < T$ is called a *local* solution of the mathematical **Problem** P.

Here and below, for simplicity, we do not explicitly indicate the dependence of various functions on the spatial variables $x \in [0,L]$ and $t \in [0,T]$, and the symbol $\dot{u}$ denotes the time derivative, i.e., $\dot{u} = \partial_t u$.

We now provide a brief description of equations and conditions.

First, the equations (2.1) and (2.2) represent the electro-viscoelastic behavior of the rod. Here, η, μ and Γ are functions on $x \in \Omega$ which describe the viscosity, the elasticity and the nonlinearity of the material, respectively. ϱ represents a piezoelectric coefficient and β denotes the electric permittivity constant. Notice that (2.1) represents an electro-viscoelastic constitutive law; indeed, this equality shows that the mechanical properties of the materials are described by a viscoelastic Kelvin-Voigt constitutive relation (see [9] for details) and, moreover, it takes into account the dependence of the stress field on the electric field. Relation (2.2) describes a linear dependence of the electric displacement field on the strain and electric fields; such kind of relations have been frequently considered in the literature, see for instance [10] and the references therein. Existence and uniqueness results for quasistatic displacement-tractions problems involving Kelvin-Voigt constitutive law were recently obtained in [11].

Equations (2.3) and (2.4) represent the equilibrium equations for the stress and the electric displacement fields, respectively, where $f = f(x,t)$ denotes the (linear) density of the applied forces and q denote a uniform linear electrical charge density.

Condition (2.5) is the displacement boundary conditions which means that the rod is attached at its left end.

Condition (2.6) represent a general damped response contact condition which state that the reaction of the obstacle at $x = L$ depends on the velocity. Here p is a real valued prescribed function such that $p(r) = 0$ for $r \leq 0$ which means that the obstacle reacts only in compression.

Condition (2.7) and (2.8) represent the electric boundary conditions which state that the electrical potential vanishes on $x = 0$ and no free electrical charges prescribed on $x = L$ where the electric displacement field vanishes.

Finally, condition (2.9) represent the initial condition of displacement.

To derive a variational formulation of problem (2.1)-(2.9), we need some additional notations. to this end, let V be the closed subspace of $H^1(\Omega)$, defined as:

$$V = \{\ v \in H^1(\Omega)\,;\ v(0) = 0\ \}.$$

On V, we consider the inner product given by

$$\langle u, v\rangle_V = \int_0^L \partial_x u\, \partial_x v\, dx. \tag{2.10}$$

and let $\|\cdot\|_V$ be the associated norm, *i.e.*, $\|v\|_V = \|\partial_x v\|_{L^2(\Omega)}$. By observing that:

$$\|v\|_{L^2(\Omega)} \leq L\|\partial_x v\|_{L^2(\Omega)}, \qquad \text{for all } v \in V, \tag{2.11}$$

it follows that the usual norm $\|\cdot\|_{H^1}$ and the associated norm $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \langle\cdot,\cdot\rangle_V)$ is a real Hilbert space.

Let us now introduce the set K_M:

$$K_M = \left\{\theta \in \mathcal{C}([0,T^*];V) \cap L^\infty(0,T^*;H^2(\Omega)) \text{ such that } \|\theta\|_{L^\infty(0,T^*;H^2(\Omega))} \leq 2M\right\},$$

where the constant M is given by:

$$M = \frac{|\eta|_{W^{1,\infty}}}{\eta_1}\left(2 + L + \sqrt{\frac{\eta_2}{\eta_1}}\right)\|u_0\|_{H^2(\Omega)}, \tag{2.12}$$

and T^*, $0 < T^* \leq T$ will be chosen later.
We use also the standard notation for L^p and Sobolev spaces (see e.g. [12, 13]). Furthermore, $\mathcal{C}^1(\overline{\Omega})$ represent the space of real valued continuous differentiable functions defined on $[0, L]$. If $(X, \|\cdot\|_X)$ is a real Hilbert space, we shall denote by $\mathcal{C}([0,T];X)$ and $\mathcal{C}^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, , respectively, with the norms:

$$\|x\|_{\mathcal{C}([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X, \ \|x\|_{\mathcal{C}^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X,$$

In the study of the our mechanical problem (2.1)-(2.9), we assume the following on the data:
The force density f, the charge density q and the initial condition u_0 satisfy:

$$f \in \mathcal{C}([0,T];L^2(0,L)), \quad q \in \mathcal{C}([0,T];L^2(0,L)), \quad u_0 \in H^2(0,L) \cap V. \tag{2.13}$$

The elasticity function μ, the nonlinearity function Γ and the viscosity function η satisfy the following assumptions:

$$\mu \in W^{1,\infty}(0,L), \quad \mu\Gamma \in W^{1,\infty}(0,L), \quad \eta \in W^{1,\infty}(0,L), \tag{2.14}$$

and that there exist μ_1, μ_2, γ_1, γ_2, η_1 and η_2 non negatives numbers such that:

$$\mu_1 \leq \mu \leq \mu_2 \ \text{ on } \ (0,L), \tag{2.15}$$

$$\gamma_1 \leq \mu\Gamma \leq \gamma_2 \ \text{ on } \ (0,L), \tag{2.16}$$

$$\eta_1 \leq \eta \leq \eta_2 \ \text{ on } \ (0,L). \tag{2.17}$$

We also assume that the electric permittivity coefficient and the piezoelectric coefficient satisfy:

$$\beta \in L^\infty(0,L), \text{ and there exists } \beta^* > 0 \text{ such that } \beta(x) \geq \beta^* \text{ a.e. } x \in (0,L), \tag{2.18}$$

$$\varrho \in L^\infty(0,L). \tag{2.19}$$

Finally,, the damped response function $p : \mathbb{R} \to \mathbb{R}_+$ verifies the following.

$$\left\{\begin{array}{l} \text{(a) There exists a constant } c_{1,p} > 0 \text{ such that for all } r_1,\, r_2 \in \mathbb{R}, \\ \qquad |p(r_1) - p(r_2)| \leq c_{1,p}\, |r_1 - r_2|, \\ \text{(b) For any } r_1,\, r_2 \in \mathbb{R},\ (p(r_1) - p(r_2))(r_1 - r_2) \geq 0, \\ \text{(c) For all } r \leq 0,\ p(r) = 0, \\ \text{(d) There exists } \ c_{2,p} > 0 \text{ such that } \ |p(r)| \leq c_{2,p}, \ \ \forall r \in \mathbb{R}. \end{array}\right. \tag{2.20}$$

Next, it is straightforward to prove that if (u, σ, φ, D) are regular enough satisfying (2.1)-(2.6) then for all $t \in [0,T]$ and $(w, \psi) \in V \times V$, we have:

$$\begin{aligned} &\int_0^L \eta \partial_x \dot{u}\, \partial_x w\, dx + \int_0^L \mathcal{G}(\partial_x u)\, \partial_x w\, dx + \int_0^L \varrho \partial_x \varphi\, \partial_x w\, dx + j(\dot{u}, w) \\ &= \int_0^L f w\, dx, \end{aligned} \tag{2.21}$$

$$\int_0^L \beta\partial_x\varphi\,\partial_x\psi\,dx - \int_0^L \varrho\partial_x u\,\partial_x\psi\,dx = \int_0^L q\psi\,dx,$$

where

$$\mathcal{G}(\partial_x u) = \mu\big(\partial_x u - \Gamma(\partial_x u)^2\big). \tag{2.22}$$

Thus, from (2.1)-(2.9) and (2.22), we obtain the following variational formulation of mechanical **Problem** P:

Problem PV. Find $u : [0,T] \to V \cap H^2(\Omega)$ and $\varphi : [0,T] \to H^1(\Omega)$ such that:

$$\begin{aligned}&\int_0^L \eta\,\partial_x\dot{u}\,\partial_x w\,dx + \int_0^L \mathcal{G}(\partial_x u)\,\partial_x w\,dx + \int_0^L \varrho\partial_x\varphi\partial_x w\,dx + j(\dot{u}, w)\\ &= \int_0^L f\,w\,dx, \quad \forall w \in V,\end{aligned} \tag{2.23}$$

$$\int_0^L \beta\partial_x\varphi\partial_x\psi\,dx - \int_0^L \varrho\partial_x u\partial_x\psi\,dx = \int_0^L q\psi\,dx, \quad \forall\psi \in V, \tag{2.24}$$

$$u(0) = u_0, \tag{2.25}$$

for all $t \in [0,T]$, (u,φ) which satisfies (2.23)-(2.25)is called a weak solution of problem (2.1)-(2.9). The well-posedness of variational **Problem** PV is discussed in the next section, where an existence and uniqueness result in the study of this problem is established.

3. Existence and uniqueness results

The unique solvability of **Problem** P follows from the following result.

Theorem 3.1. *Assume that* (2.13))-(2.20) *hold. Then there exists* $T^* > 0$, $0 < T^* \le T$ *such that the problem* (2.1)-(2.9) *has a unique solution* (u, σ, φ, D) *satisfying the following regularity conditions:*

$$\begin{aligned}&u \in \mathcal{C}^1([0,T^*]; H^2(0,L)), &&\sigma \in \mathcal{C}([0,T^*]; H^1(0,L)),\\ &\varphi \in \mathcal{C}([0,T^*]; H^1(0,L)), &&D \in \mathcal{C}([0,T^*]; H^1(0,L)).\end{aligned}$$

Moreover, we establish the following result:

Theorem 3.2. *Assume that conditions* (2.13)-(2.20) *hold, the data* u_0 *and* f *are sufficiently small, and that*

$$\|\partial_x\mu\|^2_{L^\infty(0,L)} + \|\mu\partial_x\eta/\eta\|^2_{L^\infty(0,L)} \le \mu_1/2. \tag{3.1}$$

Then, we can take $T^* = T$.

We will prove Theorem 3.1 in several steps based on Schauder fixed point arguments and the time-dependent nonlinear equations with strongly monotone operators and the classical Cauchy-Lipschitz theorem. We assume in the sequel that (2.13)-(2.20) hold and c denotes a positive constant that does not depend on the data, and its value may change from place to place. In the following, we need the following notations. We denote by j the functional defined as:

$$\begin{aligned} j : L^\infty(0,T;V) \times V &\to \mathbb{R} \\ (u,w) &\mapsto p(u(L,.))w(L). \end{aligned}$$

We also denote by $b : V \times V \to \mathbb{R}$ the following bilinear and symmetric application:

$$b(\varphi,\psi) = \int_0^L \beta\, \partial_x\varphi\, \partial_x\psi\, dx.$$

Additionally, we denote by $e : V \times V \to \mathbb{R}$ and $e^* : V \times V \to \mathbb{R}$ the following bilinear forms

$$e(u,\varphi) = \int_0^L \varrho\, \partial_x u\, \partial_x\varphi\, dx = \int_0^L \varrho\, \partial_x\varphi\, \partial_x w\, dx = e^*(\varphi,u).$$

It is easy to see that b is continuous an V-elliptic form in the following sense:

$$|b(\varphi,\psi)| \le M_b\|\varphi\|_V\|\psi\|_V \quad \text{and} \quad b(\varphi,\varphi) \ge \beta^*\|\varphi\|_V^2. \tag{3.2}$$

Furthermore, there exists $M_e > 0$ such that for all $(u,\varphi) \in V \times V$, we have:

$$|e(u,\varphi)| \le M_e\|u\|_V\|\varphi\|_V. \tag{3.3}$$

Thus the equation (2.24) will be:

$$b(\varphi,\psi) = e(u,\psi) + \langle q,\psi\rangle_V, \quad \forall\psi \in V, \forall t \in [0,T]. \tag{3.4}$$

To proceed we need the following equivalence result:

Lemma 3.3. *The couple (u,φ) is solution to* **Problem** PV *if only if for all $w \in V$ and $t \in [0,T]$, we have:*

$$\begin{aligned} &\int_0^L \eta\, \partial_x\dot u\, \partial_x w\, dx + \int_0^L \mathcal{G}(\partial_x u)\, \partial_x w\, dx + \langle \mathcal{E}^*\mathcal{B}^{-1}\mathcal{E}u, w\rangle_V + j(\dot u, w) \\ &= \langle f - \mathcal{E}^*\mathcal{B}^{-1}q, w\rangle_V, \end{aligned} \tag{3.5}$$

$$\mathcal{B}\varphi = \mathcal{E}u + q, \tag{3.6}$$

$$u(0) = u_0. \tag{3.7}$$

where $\mathcal{B} : V \to V$, $\mathcal{E} : V \to V$ and $\mathcal{E}^*$ (adjoint of $\mathcal{E}$): $V \to V$ will be defined below.

Proof. Now, let (u,φ) be solution of **Problem** PV. We will solve the equation (3.4) with the electric potential φ, then this variable will be the input data in the equation (2.23). To this end, let $u : [0,T] \to V$ and find $\varphi : [0,T] \to V$. By using the properties of the bilinear forms b, e and the Lax-Milgram lemma we see that there exists a unique element $\varphi \in V$ for all $t \in [0,T]$. Moreover, We use Riesz's representation theorem to define the operators $\mathcal{B} : V \to V$, $\mathcal{E} : V \to V$ and $\mathcal{E}^* : V \to V$ by:

$$\langle \mathcal{B}\varphi,\psi\rangle_V = b(\varphi,\psi), \quad \forall\psi \in V,\ \forall t \in [0,T], \tag{3.8}$$

$$\langle \mathcal{E}u, \psi\rangle_V = e(u,\psi), \quad \forall \psi \in V,\ \forall t \in [0,T], \tag{3.9}$$

$$\langle \mathcal{E}^*\varphi, v\rangle_V = e(v,\varphi), \quad \forall v \in V,\ \forall t \in [0,T]. \tag{3.10}$$

Hence, using (3.8)-(3.10), the equation (3.4) can be write in the form (3.6). By replacing (3.6) in (2.23) we prove that **Problem** PV is equivalent to : Find $u : [0,T] \to V$ and $\varphi : [0,T] \to V$ such that (3.5)-(3.7) are satisfied. □

Next, by using Riesz's representation theorem, we define the operator $\mathcal{C} : V \to V$ and the function f_1 such that:

$$\mathcal{C}(v) = \mathcal{E}^*\mathcal{B}^{-1}\mathcal{E}(v), \quad \forall v \in V, \forall t \in [0,T], \tag{3.11}$$

$$f_1 = f - \mathcal{E}^*\mathcal{B}^{-1}q, \quad \forall t \in [0,T]. \tag{3.12}$$

Keeping in mind the properties of $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{E}^*$ it follows that $\mathcal{C}$ is a linear continuous operator on V.

$$\exists M_{\mathcal{C}} > 0, \quad \|\mathcal{C}(u_1) - \mathcal{C}(u_2)\|_V \leq M_{\mathcal{C}}\|u_1 - u_2\|_V, \quad \forall t \in [0,T]. \tag{3.13}$$

Thus, we investigate the properties of the operators $\mathcal{B}$ and $\mathcal{E}^*$, we remark that:

$$f_1 \in V, \quad \forall t \in [0,T]. \tag{3.14}$$

These results lead us to consider a variational formulation problem in which the unknowns are v_θ, σ_θ for all $\theta \in K_M$.

Problem PV$_\theta$. Find $v_\theta : [0,T] \to V \cap H^2(\Omega)$ and $\sigma_\theta : [0,T] \to H^1(\Omega)$ such that for all $\theta \in K_M$ and $w \in V$, we have:

$$\sigma_\theta = \eta\, \partial_x v_\theta + \mathcal{G}(\partial_x \theta) + \mathcal{C}(\theta), \tag{3.15}$$

$$\begin{aligned}\int_0^L \eta\ \partial_x v_\theta\, \partial_x w\, dx + j(v_\theta, w) = &- \int_0^L \mathcal{G}(\partial_x\theta)\, \partial_x w\, dx \\ &- \langle \mathcal{C}\theta, w\rangle_V + \langle f_1, w\rangle_V.\end{aligned} \tag{3.16}$$

To solve (3.15)-(3.16), we consider the bilinear form $a(\cdot,\cdot)$ on V defined as follows:

$$a(u,v) = \int_0^L \eta\, \partial_x u\, \partial_x v\, dx, \quad \forall u,\ v \in V. \tag{3.17}$$

It follows from (2.16) and (2.11) that $a(\cdot,\cdot)$ is a bilinear continuous and coercive form on V, that is:

$$|a(u,v)| \leq C\|v\|_V\|u\|_V, \quad \forall u,\ v \in V, \tag{3.18}$$

$$|a(u,u)| \geq C\|u\|_V^2, \quad \forall u \in V. \tag{3.19}$$

Furthermore, by using the Riesz's representation theorem there exists $f_\theta \in V$ such that:

$$\langle f_\theta, w\rangle_V = \langle f_1, w\rangle_V - \langle \mathcal{G}(\partial_x\theta), \partial_x w\rangle_{L^2(\Omega)} - \langle \mathcal{C}\theta, w\rangle_V. \tag{3.20}$$

Now, by using (3.16), (3.17) and (3.20), we obtain:

$$a(v_\theta, w) + j(v_\theta, w) = \langle f_\theta, w\rangle_V. \tag{3.21}$$

We can now state the following lemma:

Lemma 3.4. *Let $\theta \in K_M$, and assume that f satisfies (*2.13*). Then there exists a unique solution v_θ to* (3.16) *such that:*

$$v_\theta \in L^\infty(0,T;H^2(\Omega)\cap V) \text{ and } \sigma_\theta \in L^\infty(0,T;H^1(\Omega)).$$

Proof. By using Riesz's representation theorem, we can define the operator $B : V \to V$ by the relation:

$$(Bv_\theta, w)_V = a(v_\theta, w) + j(v_\theta, w), \quad \forall w \in V. \tag{3.22}$$

Combining (3.21) and (3.22) we find

$$(Bv_\theta, w)_V = (f_\theta, w)_V, \quad \forall w \in V. \tag{3.23}$$

Furthermore, based on (2.20), we conclude that:

$$j(u_1, u_1 - u_2) - j(u_2, u_1 - u_2) \geq 0, \quad \forall u_1,\, u_2 \in V, \tag{3.24}$$

$$|j(u_1, v) - j(u_2, v)| \leq C\|u_1 - u_2\|_V \|v\|_V, \quad \forall u_1,\, u_2,\, v \in V. \tag{3.25}$$

We will now demonstrate that the operator B is strongly monotone and Lipschitz continuous on V. For this purpose, let $u_1, , u_2 \in V$, and then from (3.19) and (3.24), we obtain:

$$C\|u_1 - u_2\|_V^2 \leq (Bu_1 - Bu_2, u_1 - u_2)_V. \tag{3.26}$$

Subsequently, from (3.18) and (3.25), we find:

$$\|Bu_1 - Bu_2\|_V \leq C\|u_1 - u_2\|_V. \tag{3.27}$$

Using now (3.26) and (3.27) we deduce that the operator B is strongly monotone and Lipschitz continuous on V. Moreover, It follows from classical results for non linear equations (see [14] Corollary 15) that there exists a unique element $v_\theta \in L^\infty(0,T^*;V)$. Now, let us choose w in $\mathcal{D}(\Omega)$ (the space of test functions, the space $\mathcal{C}^1(\Omega)$ equipped with the inductive limit topology). by using (3.25), we then obtain:

$$-\eta\, \partial_x^2 v_\theta = f_1 - \mathcal{C}\theta + \partial_x \mathcal{G}(\partial_x\theta)\, \partial_x^2\theta + \frac{d\eta}{dx}\partial_x v_\theta. \tag{3.28}$$

Considering that $\theta \in L^\infty(0,T;H^2(\Omega))$ and $\eta \in W^{1,\infty}(\Omega)$, we infer that $\partial_x^2 v_\theta \in L^\infty(0,T;L^2(\Omega))$. Consequently, we can assert that $v_\theta \in L^\infty(0,T;H^2(\Omega))$. Furthermore, we deduce σ_θ from expression (3.15). This completes the demonstration of lemma 3.4. □

Next, we consider the operator Λ defined by:

$$\Lambda\theta = u_\theta, \text{ with } \; u_\theta(t) = \int_0^t v_\theta(s)\, ds + u_0, \quad \forall t \in [0,T]. \tag{3.29}$$

We will show that the operator Λ has a fixed point.

Lemma 3.5. *The map $\Lambda : K_M \longmapsto \mathcal{K}_M$ is continuous for the topology of $\mathcal{C}([0,T^*];\, V)$.*

Then, we turn to prove the following lemma.

Lemma 3.6. *Under assumptions of Theorem* 3.1 *there exists a constant* T^*, $0 < T^* \leq T$ *such that* Λ *maps* K_M *into* $\mathcal{K}_M \subset\subset \mathcal{C}([0,T^*];V)$ *where* $\subset\subset$ *denotes compact embedding and* $\mathcal{K}_M \subset K_M$.

Proof. Let $\theta \in K_M$. Using equation (3.15) yields:

$$\begin{aligned}-\eta\partial_x^2 u_\theta = &-\left(\eta\,\frac{d^2u_0}{dx^2} + \frac{d\eta}{dx}\frac{du_0}{dx}\right) + \frac{d\eta}{dx}\partial_x u_\theta + \int_0^t \frac{d\mu}{dx}\partial_x\theta\, ds\\ &+ \int_0^t \mu\,\partial_x^2\theta\, ds - \frac{d(\mu\Gamma)}{dx}\int_0^t (\partial_x\theta)^2\, ds - 2\mu\Gamma\int_0^t \partial_x\theta\,\partial_x^2\theta\, ds\\ &+ \frac{d\mathcal{C}}{dx}\int_0^t \theta\, ds + \mathcal{C}\int_0^t \partial_x\theta\, ds + \int_0^t f_1(s)\, ds,\end{aligned}$$

and since $\theta \in K_M$, we have:

$$\begin{aligned}\eta_1\|\partial_x^2 u_\theta\|_{L^2(\Omega)} \leq\ &\|\eta\|_{W^{1,\infty}(\Omega)}\ \|u_0\|_{H^2(\Omega)} + \left\|\frac{d\eta}{dx}\right\|_{L^\infty(\Omega)} \|\partial_x u_\theta\|_{L^\infty(0,T^*;L^2(\Omega))}\\ &+ t\left(2\big(\|\mu\|_{W^{1,\infty}(\Omega)} + \left\|\frac{d\mathcal{C}}{dx}\right\|_{L^\infty(\Omega)} + \|\mathcal{C}\|_{L^\infty(\Omega)}\big)M\right.\\ &\left.+ 4\left\|\frac{d(\mu\Gamma)}{dx}\right\|_{L^\infty(\Omega)} M^2 + \|f_1\|^2_{L^\infty(0,T^*;L^2(\Omega))}\right). \end{aligned} \tag{3.30}$$

Moreover, by choosing $w = u_\theta(t)$ in (3.5), we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L \eta\,|\partial_x u_\theta|^2\, dx\\ &= \int_0^L f_1\, u_\theta\, dx - j(\dot{u}_\theta, u_\theta) - \int_0^L \mathcal{G}(\partial_x\theta)\,\partial_x u_\theta\, dx + \int_0^L \mathcal{C}\, u_\theta^2\, dx, \forall\, t \in [0,T].\end{aligned}$$

Therefore, integrating in time from 0 to t, we get:

$$\begin{aligned}\int_0^L |\partial_x u_\theta|^2\, dx \leq \frac{\eta_2}{\eta_1}\int_0^L \left|\frac{du_0}{dx}\right|^2 dx + \frac{t^2}{\eta_1^2}\Big(&\|f_1\|^2_{L^\infty(0,T^*;L^2(\Omega))} + c_{1,p}^2\\ &+ \|\mathcal{G}(\partial_x\theta)\|^2_{L^\infty(0,T^*;L^2(\Omega))} + 2\|\mathcal{C}\|_{L^\infty(\Omega)}\Big).\end{aligned} \tag{3.31}$$

Thus, since $\theta \in K_M$, we obtain:

$$\begin{aligned}\|\partial_x u_\theta\|_{L^\infty(0,T^*;L^2(\Omega))} \leq \sqrt{\frac{\eta_2}{\eta_1}}\left\|\frac{du_0}{dx}\right\|_{L^2(\Omega)} + \frac{t}{\eta_1}\Big(&\|f_1\|_{L^\infty(0,T^*;L^2(\Omega))} + c_{1,p}\\ &+ 2M\|\mu\|_{L^\infty(\Omega)}M + 4M^2\|\mu\Gamma\|_{L^\infty(\Omega)} + \sqrt{2}\|\mathcal{C}\|^{\frac{1}{2}}_{L^\infty(\Omega)}\Big).\end{aligned} \tag{3.32}$$

Exploiting (3.32) and from the following inequality:

$$\|u_\theta\|_{L^\infty(0,T^*;L^2(\Omega))} \leq L\|\partial_x u_\theta\|_{L^\infty(0,T^*;L^2(\Omega))},$$

there exists T^*, $0 < T^* \leq T$ such that $\Lambda\theta \in K_M$.

To proceed, we need the following compactness result, which we recall in this section for the convenience of the reader.

Lemma 3.7. (*cf.* [15]) *Let X, B and Y be three Banach spaces such that $X \subset B \subset Y$ where the embedding $X \subset B$ is compact and let $s > 1$. Then*

$$L^\infty(0, T; X) \cap W^{1,s}([0, T]; Y) \subset \mathcal{C}([0, T]; B),$$

with the corresponding compact embedding.

First, let's prove that K_M endowed with the topology of $\mathcal{C}([0, T]; V)$ is a closed set in $\mathcal{C}([0, T]; V)$. Consider a sequence (θ_k) in K_M such that θ_k strongly converges to θ in $\mathcal{C}([0, T^*]; V)$. As $\theta_k \in K_M$, there exists a subsequence, still denoted as (θ_k), such that:

$$\theta_k \rightharpoonup z \quad \text{in} \quad L^\infty(0, T^*;\ H^2(\Omega)) \text{ weak}^*,$$

with

$$\|z\|_{L^\infty(0,T^*;\ H^2(\Omega))} \leq \lim inf \|\theta_k\|_{L^\infty(0,T^*;H^2(\Omega))}. \tag{3.33}$$

By uniqueness of the limit in $\mathcal{D}'(Q_{T^*})$, we conclude that:

$$\theta \in L^\infty(0, T^*;\ H^2(\Omega)),$$

and

$$\|\theta\|_{L^\infty(0,T^*;\ H^2(\Omega))} \leq 2M. \tag{3.34}$$

Now, let us remark that $\theta \longmapsto u_\theta$ maps K_M into a relative compact set $\mathcal{K}_M$ of $\mathcal{C}([0, T^*];\ V)$. Indeed, from Lemma 3.6 and (3.29), $\mathcal{K}_M$ is bounded in $W^{1,\infty}(0, T^*; H^2(\Omega) \cap V)$ and we conclude with Lemma 3.7. □

We now have all the necessary elements to prove Lemma 3.5. To begin, let's consider a sequence (θ_k) from K_M converging to θ in $\mathcal{C}([0, T^*]; V)$. Utilizing (3.34), we conclude that $\theta \in K_M$. We denote in the sequel by u_{θ_k} and u_θ the solution of **Problem** PV$_\theta$ for θ_k and θ, respectively, we have:

$$\|\Lambda\theta_k(t) - \Lambda\theta(t)\|_V \leq \int_0^t \|v_{\theta_k}(s) - v_\theta(s)\|_V \, ds, \tag{3.35}$$

which leads easily to the existence of a constant $C(T^*) > 0$ such that:

$$\|\Lambda\theta_k - \Lambda\theta\|_{L^\infty(0,T^*;\ V)} \leq C(T^*)\|\theta_k - \theta\|_{L^\infty(0,T^*;V)}. \tag{3.36}$$

which implies Λ is continuous for the topology of $\mathcal{C}([0, T^*];\ V)$. Thus, we have proved that Λ defined from a non empty bounded closed convex set K_M into a non empty bounded relatively compact convex set $\mathcal{K}_M$ in $\mathcal{C}([0, T^*]; V)$ is continuous provided with the topology of $\mathcal{C}([0, T^*]; V)$. Then, we end with the Schauder fixed-point theorem (see [16] Corollary 3.6.2 p. 163), we deduce that the map $\theta \longmapsto u_\theta$ possesses a fixed point denoted by u.

Also, we need to prove that u is more regular than $W^{1,\infty}(0,T^*;H^2(\Omega)) \cap \mathcal{C}^1([0,T^*];V)$. Since u satisfies, in the distribution sense

$$-\eta\,\partial_x^2\dot{u} = f_1 - \mathcal{C}u + \partial_x(\mathcal{G}(\partial_x u)) + \frac{d\eta}{dx}\partial_x\dot{u},$$

and given that $f \in \mathcal{C}([0,T^*];L^2(\Omega))$, we can deduce that:

$$\partial_x^2\dot{u} \in \mathcal{C}([0,T^*];L^2(\Omega)),$$

since $\partial_x(\mathcal{G}(\partial_x u)) \in \mathcal{C}([0,T^*];L^2(\Omega))$ and $\partial_x\eta\partial_x\dot{u} \in \mathcal{C}([0,T^*];L^2(\Omega))$. Therefore (u,σ) satisfy the regularity:

$$(u,\sigma) \in \mathcal{C}^1([0,T^*];H^2(\Omega)) \times \mathcal{C}([0,T^*];H^1(\Omega)).$$

We now have all the ingredients to prove Theorem 3.1.

Existence of solution.

Let $\theta^* \in K_M$ be the fixed point of Λ and let u_{θ^*} and σ_{θ^*} be the functions defined by:

$$u_{\theta^*}(t) = \int_0^t v_{\theta^*}(s)\,ds + u_0, \tag{3.37}$$

$$\sigma_{\theta^*}(t) = \mathcal{G}(\partial_x u_{\theta^*}) + \eta\partial_x v_{\theta^*} + \mathcal{C}(u_{\theta^*}). \tag{3.38}$$

Since $v_{\theta^*} \in \mathcal{C}([0,T^*];\,V)$, using (3.37) and (3.38), we find $u_{\theta^*} \in \mathcal{C}^1([0,T^*];\,V)$ and $\sigma_{\theta^*} \in C([0,T^*];\,L^2(\Omega))$. By using (3.37), we have $u_{\theta^*}(0) = u_0$ and $u_{\theta^*} = 0$ on $\{0\} \times [0,T^*]$. Moreover, since $\theta^* = \Lambda\theta^* = u_\theta^*$ and $v_{\theta^*} = \dot{u}_{\theta^*}$we find:

$$\begin{aligned}\sigma_{\theta^*}(t) &= \mathcal{G}(\partial_x u_{\theta^*}) + \eta\partial_x v_{\theta^*} + \mathcal{C}(u_{\theta^*})\\ &= \mathcal{G}(\partial_x\theta^*) + \eta\partial_x\dot{u}_{\theta^*} + \mathcal{C}(\theta^*),\end{aligned} \tag{3.39}$$

and by (3.16) and (3.39) it follows that:

$$\langle\sigma_{\theta^*}(t),\partial_x w\rangle_{L^2(\Omega)} + j(\dot{u}_{\theta^*},w) = \langle f_1(t),\,w\rangle_V, \quad \forall w \in V.$$

Moreover, we have from (3.5), (3.6), (3.11) and (3.12) that:

$$\int_0^L \eta\,\partial_x\dot{u}_{\theta^*}\,\partial_x w\,dx + \int_0^L \mathcal{G}(\partial_x u_{\theta^*})\,\partial_x w\,dx + \int_0^L \varrho\partial_x\varphi_{\theta^*}\partial_x w\,dx + j(\dot{u},w) = \int_0^L f\,w\,dx, \quad \forall w \in V, \tag{3.40}$$

Taking $w = \psi \in \mathcal{D}(\Omega)$ in the previous equality, we obtain:

$$\langle\sigma(t),\partial_x\psi\rangle_{L^2(\Omega)} = \langle f,\psi\rangle_V.$$

Therefore, we deduce that:

$$\partial_x\sigma + f = 0, \quad \text{in } \mathcal{D}', \tag{3.41}$$

where $\mathcal{D}'$ is the dual space of $\mathcal{D}(\Omega)$ (called the space of (Schwartz) distributions). Then using the following equality:

$$\langle\sigma(t),\partial_x w\rangle_{L^2(\Omega)} = \langle f,w\rangle_{L^2(\Omega)} - p(\dot{u}(L,t))w(L), \quad \forall w \in V, \tag{3.42}$$

we obtain:

$$\langle \sigma(t), \partial_x w\rangle_{L^2(\Omega)} + \langle \partial_x \sigma, w\rangle_{L^2(\Omega)} = -p(\dot{u}(L,t))w(L), \quad \forall w \in V. \tag{3.43}$$

Thus by using (3.41), (3.42) and (3.43), it follows:

$$\sigma(L,t)w(L) = -p(\dot{u}(L,t))w(L), \quad \forall w \in V.$$

which implies:

$$\sigma(L,t) = -p(\dot{u}(L,t)) \text{ on } [0,T^*].$$

To conclude (u,σ) represents a solution to mechanical problem (2.1)-(2.9).

Uniqueness of solution.

Let (u_i,σ_i) be two solutions of (2.1)-(2.9), $i = 1, 2$, having the regularity $\mathcal{C}^1([0,T^*];H^2(\Omega)) \times \mathcal{C}([0,T^*];H^1(\Omega))$. Let us denote $U = u_1 - u_2$. Taking $w = \dot{U}$ in (2.23), we obtain: f or all $t \in [0,T^*]$

$$\int_0^L \eta\, |\partial_x \dot{U}|^2\, dx + \int_0^L (\mathcal{G}(\partial_x u_1) - \mathcal{G}(\partial_x u_2))\partial_x \dot{U}\, dx + j(\dot{u}_1, \dot{U}) - j(\dot{u}_2, \dot{U}) = 0. \tag{3.44}$$

Since $u_i \in \mathcal{C}([0,T^*];H^2(0,L))$ and $\mathcal{G}$ is a locally Lipschitz continuous, and j is monotone, we get:

$$\|\dot{U}\|_V \le c\|U\|_V,$$

and therefore,

$$\frac{d}{dt}\|U\|_V^2 \le c\|U\|_V^2.$$

Using now a Gronwall-type argument, it follows that: $u_1 = u_2$, since $U(0) = 0$. This equality implies $\sigma_1 = \sigma_2$ which concludes the proof.

4. Proof of Theorem 3.2

We proceed to multiply the equations represented in (2.1)-(2.3) by u and integrate over Ω, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_0^L \eta|\partial_x u|^2\, dx + \int_0^L \mu|\partial_x u|^2\, dx$$
$$= \int_0^L f_1\, u\, dx - \langle \mathcal{C}u, u\rangle_V + \int_0^L \mu\Gamma(\partial_x u)^2\, \partial_x u\, dx - j(\dot{u}, u),$$

thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^L \eta|\partial_x u|^2\, dx + \big(\mu_1 - M_{\mathcal{C}} - c\|\mu\Gamma\|_{L^\infty(0,L)}\|u\|_{H^2(\Omega)}\big)\int_0^L |\partial_x u|^2\, dx$$
$$\le (L\|f_1\|_{L^2(\Omega)} + \sqrt{L}c_{2,p})\|u\|_V.$$

This gives us:

$$\frac{d}{dt}\int_0^L \eta|\partial_x u|^2\, dx + \big(\mu_1 - 2M_{\mathcal{C}} - 2c\|\mu\Gamma\|_{L^\infty(0,L)}\|u\|_{H^2(0,L)}\big)\int_0^L |\partial_x u|^2\, dx$$
$$\le 2(L\|f_1\|_{L^2(\Omega)} + \sqrt{L}c_{2,p})\|u\|_V.$$

In a analogous way, we multiply the equations represented in (2.1)-(2.3) by $\partial_x(\eta\partial_x u)$ and then integrate it across the domain Ω, resulting in the following expression:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L |\partial_x(\eta\partial_x u)|^2\,dx = &-\int_0^L f\partial_x(\eta\partial_x u)\,dx - \int_0^L \partial_x\left(\mu\partial_x u\right)\partial_x(\eta\partial_x u)\,dx\\ &+\int_0^L \partial_x\left(\mu\Gamma\left(\partial_x u\right)^2\right)\partial_x(\eta\partial_x u)\,dx - \langle\partial_x\mathcal{C}u,\eta\partial_x u\rangle_V\\ &-\langle\partial_x\mathcal{E}^*\mathcal{B}^{-1}\mathcal{E}q,\eta\partial_x u\rangle_V,\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L |\partial_x(\eta\partial_x u)|^2\,dx = &-\int_0^L f\partial_x(\eta\partial_x u)\,dx - \int_0^L \partial_x\mu\partial_x u\partial_x(\eta\partial_x u)\,dx\\ &-\int_0^L \mu\partial_x^2 u\partial_x(\eta\partial_x u)\,dx + \int_0^L 2\mu\Gamma\partial_x^2 u\partial_x u\partial_x(\eta\partial_x u)\,dx\\ &+\int_0^L \partial_x(\mu\Gamma)(\partial_x u)^2\,\partial_x(\eta\partial_x u)\,dx - \langle\partial_x\mathcal{C}u,\eta\partial_x u\rangle_V\\ &-\langle\partial_x\mathcal{E}^*\mathcal{B}^{-1}\mathcal{E}q,\eta\partial_x u\rangle_V,\end{aligned}$$

since:

$$\int_0^L \mu\partial_x^2\mu\partial_x(\eta\partial_x u)\,dx = \int_0^L \frac{\mu}{\eta}|\partial_x(\eta\partial_x u)|^2\,dx - \int_0^L \frac{\mu}{\eta}\partial_x\eta\partial_x u\partial_x(\eta\partial_x u)\,dx,$$

and

$$\begin{aligned}\int_0^L 2\mu\Gamma\partial_x^2 u\partial_x u\partial_x(\eta\partial_x u)\,dx = &\int_0^L \frac{2\mu\Gamma}{\eta}\partial_x u|\partial_x(\eta\partial_x u)|^2\,dx\\ &-\int_0^L \frac{2\mu\Gamma}{\eta}\partial_x\eta(\partial_x u)^2\partial_x(\eta\partial_x u)\,dx,\end{aligned}$$

it is straightforward that:

$$\begin{aligned}&\frac{d}{dt}\int_0^L |\partial_x(\eta\partial_x u)|^2\,dx + \left(\frac{\mu_1}{\eta_2}\,dx - c_{\mu,\Gamma,\eta}\|u\|_{H^2(\Omega)}\right)\int_0^L |\partial_x(\eta\partial_x u)|^2\,dx\\ &\leq \frac{\eta_2}{\mu_1}\left(\|f\|^2_{L^2(\Omega)} + c\big(\|q\|^2_{L^2(\Omega)} + \|\mathcal{C}\|^2_{L^\infty(\Omega)}\big)\right) + \frac{\eta_2}{\mu_1}\Bigg(\|\partial_x\mu\|^2_{L^\infty(\Omega)}\\ &+\left\|\frac{\mu\partial_x\eta}{\eta}\right\|^2_{L^\infty(\Omega)} + c_{\mu,\Gamma,\eta}\|u\|_{H^2(\Omega)}\Bigg)\int_0^L |\partial_x u|^2\,dx.\end{aligned}$$

Taking into account the assumption (3.1), we derive the subsequent estimation:

$$\begin{aligned}&\frac{d}{dt}\phi + (c_{1,\mu,\Gamma,\eta,L} - c_{2,\mu,\Gamma,\eta,L}\phi^{1/2})\phi\\ &\leq c_{3,\mu,\Gamma,\eta,L}\left(\|f\|^2_{L^2(\Omega)} + c\big(\|q\|^2_{L^2(\Omega)} + \|\mathcal{C}\|^2_{L^\infty(\Omega)}\big)\right) + c_{4,\mu,\Gamma,\eta,L},\end{aligned}$$

where $\phi(t) = \left(\int_0^L |\partial_x u|^2 + |\partial_x^2 u|^2 \, dx \right)(t)$. Let us assume u_0 such that $\phi(0) \leq \frac{c_1}{4c_2}$ and let us suppose that $\phi^{\frac{1}{2}}(t) < \frac{c_1}{2c_2}$ for all $t < t_0$ and $\phi^{\frac{1}{2}}(t_0) = \frac{c_1}{2c_2}$. By the inequality established earlier, we will obtain:

$$\begin{aligned} &\frac{d}{dt}\phi(t_0) + c_{5,\mu,\Gamma,\eta,L} \\ &\leq c_{3,\mu,\Gamma,\eta,L} \left(\|f\|^2_{L^2(\Omega)} + c\big(\|q\|^2_{L^2(\Omega)} + \|\mathcal{C}\|^2_{L^\infty(\Omega)}\big) \right)(t_0) + c_{4,\mu,\Gamma,\eta,L}. \end{aligned}$$

If we assume that:

$$\begin{aligned} &c_{3,\mu,\Gamma,\eta,L} \left(\|f\|_{L^\infty(0,T;L^2(0,L))} + c\big(\|q\|_{L^\infty(0,T;L^2(0,L))} + \|\mathcal{C}\|^2_{L^\infty(\Omega)}\big) \right) + c_{4,\mu,\Gamma,\eta,L} \\ &< c_{5,\mu,\Gamma,\eta,L} \end{aligned}$$

we obtain:

$$\frac{d}{dt}\phi(t_0) < 0,$$

which is not possible. As result, for all periods where u exists,

$$\phi^{\frac{1}{2}}(t) < c_1/2c_2.$$

Now, we can achieve global existence for sufficiently small initial data by consolidating the local solution obtained with:

$$M = \frac{c_1 |\eta|_{W^{1,\infty}}}{2c_2\eta_1} \left(2 + L + \sqrt{\frac{\eta_2}{\eta_1}} \right) (1 + L^2)^{1/2}, \tag{4.1}$$

with the assistance of the uniqueness result, the proof is now concluded.

Conflicts of interest : The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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