J. Appl. Math. & Informatics Vol. 42(2024), No. 2, pp. 283 - 290
https://doi.org/10.14317/jami.2024.283

$\beta-$PRODUCT OF PRODUCT FUZZY GRAPHS

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Abstract. In this article, a new operation on product fuzzy graphs (PFGs) is provide; namely $\beta-$product. We give sufficient conditions for the $\beta-$product of two PFGs to be strong and we prove if the $\beta-$product of two PFGs is complete, then one of them is strong. We also study the unbiased notion of the class of PFGs and necessary and sufficient conditions for the $\beta-$product to be unbiased are given.

AMS Mathematics Subject Classification : 05C72.
Key words and phrases : PFG, unbiased PFG, complete PFG, *$\beta-$product*.

1. Background

Graph theory has many applications in mathematics and economics. Since most problems of graphs are undetermined, it is necessary to handle these facets via the method of fuzzy logic. Fuzzy relations were introduced by Zadeh [25] in 1965. Rosenfeld [22] in 1975, introduced fuzzy graphs (simply, FG) and some ideas that are generalizations of those of graph's. Now days, this theory is having more and more applications in which the information level immanent in the system differ with various levels of accuracy. Fuzzy models are convenient as they reduce differences between long-established numerical models of expert systems and symbolic models. Peng and Mordeson [16] defnined the conceptualization of FG's complement and conscious FG's operations. In [24], improved complement's defninition in order to guarantee the original FG is isomorphic to complement of the complement, which concur with the case of crisp graphs. In addition, self-complementary FGs properties and the complement under FG's join, union and composition (introduced in [16]) were explored. Al-Hawary [2] introduced the concept of balanced in the class of FGs and Al-Hawary and others have deeply explored this ides for many types of

Received December 9, 2022. Revised August 2, 2023. Accepted December 18, 2023.
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FGs. For more on the foregoing concepts and those coming after ones, one can see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 21, 24].

A mapping ţ:Ű→ $[0,1]$ is a fuzzy subset of a non-empty set Ű and a fuzzy subset of Ű×Ű is called a fuzzy relation ς on ţ. We assume that ţ is reflexive, Ű is finite and ς is symmetric.

Definition 1.1. [22] A *fuzzy graph (simply, FG),* with Ű as the underlying set, is a pair Ģ: (ţ, ς) where ţ:Ű→ $[0,1]$ is a fuzzy subset and ς :Ű×Ű→ $[0,1]$ is a fuzzy relation on ţ such that ς(c,s) ≤ţ(c)∧ţ(s) for all c, s ∈Ű, where ∧ stands for minimum. The crisp graph of Ģ is denoted by Ģ* : (ţ*, ς*) where ţ$^* = \sup c($ţ$) =$ $\{c$ ∈Ű:ţ$(c) > 0\}$ and ς$^* = \sup c($ς$) = \{(c,s)$ ∈Ű×Ű: ς$(c,s) > 0\}$.$H = ($ţ$', $ς$')$ is a *fuzzy subgraph* of Ģ if there exists c ∈Ű such that ţ$' : c \to [0,1]$ is a fuzzy subset and ς$' : c \times c \to [0,1]$ is a fuzzy relation on ţ$'$ such that ς(c,s) ≤ţ(c)∧ţ(s) for all $c, s \in c$.

Definition 1.2. [22] *Two FGs* Ģ$_1$: (ţ$_1$, ς$_1$) *are* isomorphic *if there exists a bijection* h :Ű$_1$ →Ű$_2$ *such that* ţ$_1(c)$ =ţ$_2(h(c))$ for all c ∈Ű$_1$*and* ς$_1(c,s) =$ ς$_2(h(c), h(s))$ for all (c,s) ∈Ĕ$_1$. We then write Ģ$_1$ ≃Ģ$_2$ and h is called an *isomorphism.*

Using the operation of product instead of minimum, Ramaswamy and Poornima in [23] established PFGs.

Definition 1.3. [23] *Let* Ģ* : (Ű,Ĕ) *be a graph,* ţ *be a fuzzy subset of* Ű *and* ς *be a fuzzy subset of* Ű×Ű.*We call* Ģ: (ţ, ς) a *product fuzzy graph* (simply, PFG) *if* ς(c,s) ≤ţ(c)ţ(s) for all c, s ∈Ű.

The following result is immediate:

Lemma 1.4. *Every PFG is a FG, but the converse need not be true.*

Definition 1.5. [23] *A PFG* Ģ: (ţ, ς) *is called* complete *if* ς(c,s) =ţ(c)ţ(s) for all c, s ∈Ű.

Definition 1.6. [23] *A PFG* Ģ: (ţ, ς) *is called* strong *if* ς(c,s) =ţ(c)ţ(s) for all (c,s) ∈Ĕ.

Definition 1.7. [23] *The* complement *of a PFG* Ģ: (ţ, ς) *is* Ģ^c : (ţ^c, ς^c) *where* ţ^c =ţ *and*

$$\begin{aligned} ς^c(c,s) &= ţ^c(c)ţ^c(s) - ς(c,s) \\ &= ţ(c)ţ(s) - ς(c,s). \end{aligned}$$

Lemma 1.8. [7]*If* Ģ: (ţ, ς) *is a self-complementary PFG, then*

$$\sum_{(c,s)\in \breve{E}} ς(c,s) = \frac{1}{2} \sum_{(c,s)\in \breve{E}} ţ(c)ţ(s).$$

Lemma 1.9. [7]*Let* $\c{G}$: $(\c{t},\varsigma)$ *be a PFG such that* $\varsigma(c,s) = \frac{1}{2}\c{t}(c)\c{t}(s)$ *for all* $c,s \in \H{U}$. *Then* $\c{G}$ *is self-complementary.*

Several types of products of two FGs were explored. The notion of $\beta-$product of FGs was introduced and studied in [17] where the regularity property for this product was the main idea. In Section 2 of this paper, we launch the conception of $\beta-$product of PFGs. We give sufficient conditions for the $\beta-$product of two PFGs to be strong (complete) and we show that to have at least one factor is a complete PFG, the $\beta-$product should be complete. Section 3 is devoted to give necessary and sufficient conditions for the $\beta-$product of two unbiased PFGs to be unbiased.

2. $\beta-$product of PFGs

We begin this section by defnining the rooted product of PFGs.

Definition 2.1. The $\beta-$*product* of two PFGs $\c{G}_1 : (\c{t}_1, \varsigma_1)$ is defnined to be the PFG $\c{G}_1 \boxplus_\beta \c{G}_2 : (\c{t}_1 \boxplus_\beta \c{t}_2, \varsigma_1 \boxplus_\beta \varsigma_2)$ on the vertwx set $\H{U}_1 \times \H{U}_2$, where

$$(\c{t}_1 \boxplus_\beta \c{t}_2)(\H{u},\ddot{y}) = \c{t}_1(\H{u})\c{t}_2(\ddot{y}), \text{ for all } (\H{u},\ddot{y}) \in \H{U}_1 \times \H{U}_2 \text{ and}$$

$$(\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1,\ddot{y}_1)(\H{u}_2,\ddot{y}_2)) = \begin{cases} \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2) & \H{u}_1\H{u}_2 \in \breve{E}_1, \ddot{y}_1\ddot{y}_2 \in \breve{E}_2 \\ \varsigma_1(\H{u}_1\H{u}_2)\c{t}_2(\ddot{y}_1)\c{t}_2(\ddot{y}_2) & \H{u}_1\H{u}_2 \in \breve{E}_1, \ddot{y}_1 \neq \ddot{y}_2 \\ \c{t}_1(\H{u}_1)\c{t}_1(\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2) & \H{u}_1 \neq \H{u}_2, \ddot{y}_1\ddot{y}_2 \in \breve{E}_2 \end{cases}.$$

Next, we show that the above defninition is well-defnined.

Theorem 2.2. *The* $\beta-$*product of two PFGs is a PFG.*

Proof. Let $\c{G}_1 : (\c{t}_1, \varsigma_1)$ and $\c{G}_2 : (\c{t}_2, \varsigma_2)$ be two PFGs.

Case 1: If $\H{u}_1\H{u}_2 \in \breve{E}_1, \ddot{y}_1\ddot{y}_2 \in \breve{E}_2$, then

$$\begin{aligned}(\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1,\ddot{y}_1)(\H{u}_2,\ddot{y}_2)) &= \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2) \\ &\leq \c{t}_1(\H{u}_1)\c{t}_1(\H{u}_2)\c{t}_2(\ddot{y}_1)\c{t}_2(\ddot{y}_2) \\ &= ((\c{t}_1 \boxplus_\beta \c{t}_2)(\H{u}_1,\ddot{y}_1))((\c{t}_1 \boxplus_\beta \c{t}_2)(\H{u}_2,\ddot{y}_2)).\end{aligned}$$

Case 2: If $\H{u}_1\H{u}_2 \in \breve{E}_1, \ddot{y}_1 \neq \ddot{y}_2$, then

$$\begin{aligned}(\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1,\ddot{y}_1)(\H{u}_2,\ddot{y}_2)) &= \varsigma_1(\H{u}_1\H{u}_2)\c{t}_2(\ddot{y}_1)\c{t}_2(\ddot{y}_2) \\ &\leq \c{t}_1(\H{u}_1)\c{t}_1(\H{u}_2)\c{t}_2(\ddot{y}_1)\c{t}_2(\ddot{y}_2) \\ &= ((\c{t}_1 \boxplus_\beta \c{t}_2)(\H{u}_1,\ddot{y}_1))((\c{t}_1 \boxplus_\beta \c{t}_2)(\H{u}_2,\ddot{y}_2)).\end{aligned}$$

Case 3: If $\H{u}_1 \neq \H{u}_2, \ddot{y}_1\ddot{y}_2 \in \breve{E}_2$, this case is similar to Case 2. □

Theorem 2.3. *If* $\c{G}_1 : (\c{t}_1, \varsigma_1)$ *and* $\c{G}_2 : (\c{t}_2, \varsigma_2)$ *are strong PFGs, then* $\c{G}_1 \boxplus_\beta \c{G}_2$ *is a strong PFG.*

Proof. Let $Ģ_1 : (ţ_1, ς_1)$ and $Ģ_2 : (ţ_2, ς_2)$ be two strong PFGs.

Case 1: If $ű_1ű_2 \in Ĕ_1, ÿ_1ÿ_2 \in Ĕ_2$, then as $Ģ_1$ and $Ģ_2$ are strong,

$$\begin{aligned}(ς_1 \boxplus_\beta ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ς_1(ű_1ű_2)ς_2(ÿ_1ÿ_2)\\ &= ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\beta ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\beta ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 2: If $ű_1ű_2 \in Ĕ_1, ÿ_1 \neq ÿ_2$, then

$$\begin{aligned}(ς_1 \boxplus_\beta ς_2)((ű_1, ÿ_1)(ű_2, ÿ_2)) &= ς_1(ű_1ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)\\ &= ((ţ_1 \boxplus_\beta ţ_2)(ű_1, ÿ_1))((ţ_1 \boxplus_\beta ţ_2)(ű_2, ÿ_2)).\end{aligned}$$

Case 3: If $ű_1 \neq ű_2, ÿ_1ÿ_2 \in Ĕ_2$, this case is similar to Case 2.

Thus, $Ģ_1 \boxplus_\beta Ģ_2$ is a strong PFG. □

Corollary 2.4. *If $Ģ_1 : (ţ_1, ς_1)$ and $Ģ_2 : (ţ_2, ς_2)$ are fuzzy complete (strong) FGs, then $Ģ_1 \boxplus_\beta Ģ_2$ is a strong FG.*

We remark that if $Ģ_1$ and $Ģ_2$ are complete PFGs, then $Ģ_1 \boxplus_\beta Ģ_2$ need not be a complete PFG.

Example 2.5. Consider $Ģ_1 : (ţ_1, ς_1)$ where $ţ_1(ű) = 1, ţ_1(\mathrm{w}) = 1, ς_1(ű,\mathrm{w}) = 1$ and $Ģ_2 : (ţ_2, ς_2)$ where $ţ_2(c) = 1 = ţ_2(s)$ and $ς_2(c, s) = 1$. Then both are complete PFGs while $Ģ_1 \boxplus_\beta Ģ_2$ is not a complete PFG since $(ς_1 \boxplus_\beta ς_2)((ű, c)(\mathrm{w}, c)) = 0 \neq (1)(1) = (ţ_1 \boxplus_\beta ţ_2)(ű, c)(ţ_1 \boxplus_\beta ţ_2)(\mathrm{w}, c)$.

An interesting property of complement is given next.

Theorem 2.6. *If $Ģ_1 : (ţ_1, ς_1)$ and $Ģ_2 : (ţ_2, ς_2)$ are fuzzy complete graphs, then $\overline{Ģ_1 \boxplus_\beta Ģ_2} \simeq \overline{Ģ_1} \boxplus_\beta \overline{Ģ_2}$.*

Proof. Let Ģ: $(ţ, \overline{ς}) = \overline{Ģ_1 \boxplus_\beta Ģ_2}$, $\overline{ς} = \overline{ς_1 \boxplus_\beta ς_2}$, $\overline{Ģ^*} = (Ű, \overline{Ĕ})$, $\overline{Ģ_1} : (ţ_1, \overline{ς_1})$, $\overline{Ģ_1^*} = (Ű_1, \overline{Ĕ_1})$, $\overline{Ģ_2} : (ţ_2, \overline{ς_2})$, $\overline{Ģ_2^*} = (Ű_2, \overline{Ĕ_2})$ and $\overline{Ģ_1} \boxplus_\beta \overline{Ģ_2} : (ţ_1 \boxplus_\beta ţ_2, \overline{ς_1} \boxplus_\beta \overline{ς_2})$. We only need to show $\overline{ς_1 \boxplus_\beta ς_2} = \overline{ς_1} \boxplus_\beta \overline{ς_2}$. For any arc e in Ĕ joining nodes of Ű,we have the following cases:

The cases $ű_1ű_2 \in Ĕ_1, ý_1ý_2 \in Ĕ_2$, then as $Ģ_1$ is complete, $\overline{ς_1}(\mathrm{e}) = 0$. On the other hand $\overline{ς_1 \boxplus_\beta ς_2}(\mathrm{e}) = 0$ since $ű_1ű_2 \notin \overline{Ĕ_1}$.

The case $ű_1ű_2 \in Ĕ_1, ý_1 \neq ý_2$ and the case $ű_1 \neq ű_2, ÿ_1ÿ_2 \in Ĕ_2$ are not possible to occur as both $Ģ_1$ and $Ģ_2$ are complete.

In all cases $\overline{ς_1 \boxplus_\beta ς_2} = \overline{ς_1} \boxplus_\beta \overline{ς_2}$ and therefore, $\overline{Ģ_1 \boxplus_\beta Ģ_2} \simeq \overline{Ģ_1} \boxplus_\beta \overline{Ģ_2}$. □

Next, we show that if the $\beta-$product of two PFGs is complete, then at least one of the two PFGs must be complete.

Theorem 2.7. *If $Ģ_1 : (ţ_1, ς_1)$ and $Ģ_2 : (ţ_2, ς_2)$ are PFGs such that $Ģ_1 \boxplus_\beta Ģ_2$ is complete, then at least $Ģ_1$ or $Ģ_2$ must be complete.*

Proof. Suppose to the contrary that both Ģ$_1$ and Ģ$_2$ are not complete. Then there exists at least one $\H{u}_1,\H{u}_2 \in \H{U}_1$ and $\acute{y}_1, \acute{y}_2 \in \H{U}_2$ such that

$$\begin{aligned} \varsigma_1(\H{u}_1\H{u}_2) &< ţ_1(\H{u}_1)ţ_1(\H{u}_2)) \text{ and} \\ \varsigma_2(\acute{y}_1\acute{y}_2) &< ţ_2(\acute{y}_1)ţ_2(\acute{y}_2)). \end{aligned}$$

Then we have the following cases:

Case 1: If $\H{u}_1\H{u}_2 \in \breve{E}_1, \acute{y}_1\acute{y}_2 \in \breve{E}_2$, then $(\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) = \varsigma_1(\H{u}_1\H{u}_2)$ $\varsigma_2(\ddot{y}_1\ddot{y}_2)$ and as Ģ$_1\boxplus_\beta$Ģ$_2$ is complete,

$$\begin{aligned} (\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) &= (ţ_1 \boxplus_\beta ţ_2)((\H{u}_1, \ddot{y}_1))(ţ_1 \boxplus_\beta ţ_2)((\H{u}_2, \ddot{y}_2)) \\ &> ţ_1(\H{u}_1)ţ_1(\H{u}_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2) \\ &= \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2), \end{aligned}$$

which is a contradiction.

Case2: $\H{u}_1 \neq \H{u}_2, \ddot{y}_{1\ddot{y}2} \in \breve{E}_2$, then

$$\begin{aligned} (\varsigma_1 \boxplus_\beta \varsigma_2)((\H{u}_1, \ddot{y}_1)(\H{u}_2, \ddot{y}_2)) &= ţ_1(\H{u}_1)ţ_1(\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2) \\ &> \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2), \end{aligned}$$

which is a contradiction.

Case 3: If $\H{u}_1\H{u}_2 \in \breve{E}_1, \ddot{y}_1 \neq \ddot{y}_2$ is similar to Case 2. □

3. Unbiased FGs

We begin this section by recalling the defninition of unbiased (balanced) PFGs from [7] and then proving the following lemma that we shall use to give necessary and sufficient conditions for the $\beta-$product of two unbiased PFGs to be unbiased.

Definition 3.1. [7]. *The* density *of a PFG is* $d(Ģ) = \dfrac{2 \sum\limits_{\H{u}\ddot{y} \in \breve{E}} (\varsigma(\H{u}\ddot{y}))}{\sum\limits_{\H{u},\ddot{y} \in \H{U}} (ţ(\H{u}) \wedge ţ(\ddot{y}))}$. Ģ *is* unbiased (balanced) *if* $d(H) \leq d(Ģ)$ *for any non-empty product fuzzy subgraphs* H *of* Ģ.

Lemma 3.2. *Let* $Ģ_1$ *and* $Ģ_2$ *be PFGs. Then* $d(Ģ_i) \leq d(Ģ_1 \boxplus_\beta Ģ_2)$ *for* $i = 1, 2$ *if and only if* $d(Ģ_1) = d(Ģ_2) = d(Ģ_1 \boxplus_\beta Ģ_2)$.

Proof. If $d(Ģ_i) \leq d(Ģ_1 \boxplus_\beta Ģ_2)$ for $i = 1, 2$, then

$$\begin{aligned} d(Ģ_1) &= 2(\sum_{\H{u}_1,\H{u}_2 \in \H{U}_1} \varsigma_1(\H{u}_1\H{u}_2))/(\sum_{\H{u}_1,\H{u}_2 \in \H{U}_1} (ţ_1(\H{u}_1) \wedge ţ_1(\H{u}_2))) \\ &\geq 2(\sum_{\substack{\H{u}_1,\H{u}_2 \in \H{U}_1 \\ \ddot{y}_1,\ddot{y}_2 \in \H{U}_2}} \varsigma_1(\H{u}_1\H{u}_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))/(\sum_{\substack{\H{u}_1,\H{u}_2 \in \H{U}_1 \\ \ddot{y}_1,\ddot{y}_2 \in \H{U}_2}} (ţ_1(\H{u}_1)ţ_1(\H{u}_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))) \\ &\geq 2(\sum_{\substack{\H{u}_1,\H{u}_2 \in \H{U}_1 \\ \ddot{y}_1,\ddot{y}_2 \in \H{U}_2}} \varsigma_1(\H{u}_1\H{u}_2)\varsigma_2(\ddot{y}_1\ddot{y}_2))/(\sum_{\substack{\H{u}_1,\H{u}_2 \in \H{U}_1 \\ \ddot{y}_1,\ddot{y}_2 \in \H{U}_2}} (ţ_1(\H{u}_1)ţ_1(\H{u}_2)ţ_2(\ddot{y}_1)ţ_2(\ddot{y}_2))) \end{aligned}$$

$$\geq \quad 2(\sum_{\substack{\H{u}_1,\H{u}_2\in\H{U}_1\\ \ddot{y}_1,\ddot{y}_2\in\H{U}_2}} \varsigma_1 \boxplus_\beta \varsigma_2((\H{u}_1\ddot{y}_1)(\H{u}_2\ddot{y}_2))/(\sum_{\substack{\H{u}_1,\H{u}_2\in\H{U}_1\\ \ddot{y}_1,\ddot{y}_2\in\H{U}_2}} (\c{t}_1 \boxplus_\beta \c{t}_2((\H{u}_1,\ddot{y}_1)(\H{u}_2,\ddot{y}_2)))$$

$$= \quad d(\c{G}_1 \boxplus_\beta \c{G}_2).$$

Hence in all cases $d(\c{G}_1) \geq d(\c{G}_1 \boxplus_\beta \c{G}_2)$ and thus $d(\c{G}_1) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$. Similarly, $d(\c{G}_2) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$. Therefore, $d(\c{G}_1) = d(\c{G}_2) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$. The converse is trivial. □

Theorem 3.3. *Let $\c{G}_1$ and $\c{G}_2$ be unbiased PFGs. Then $\c{G}_1 \boxplus_\beta \c{G}_2$ is unbiased if and only if $d(\c{G}_1) = d(\c{G}_2) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$.*

Proof. If $\c{G}_1 \boxplus_\beta \c{G}_2$ is unbiased, then $d(\c{G}_i) \leq d(\c{G}_1 \boxplus_\beta \c{G}_2)$ for $i = 1, 2$ and by Lemma 3.2, $d(\c{G}_1) = d(\c{G}_2) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$.

Conversely, if $d(\c{G}_1) = d(\c{G}_2) = d(\c{G}_1 \boxplus_\beta \c{G}_2)$ and H is a product fuzzy subgraph of $\c{G}_1 \boxplus_\beta \c{G}_2$, then there exist product fuzzy subgraphs H_1 of $\c{G}_1$ and H_2 of $\c{G}_2$. As $\c{G}_1$ and $\c{G}_2$ are unbiased and $d(\c{G}_1) = d(\c{G}_2) = m_1/k_1$, then $d(H_1) = a_1/b_1 \leq m_1/k_1$ and $d(H_2) = a_2/b_2 \leq m_1/k_1$. Thus $a_1k_1 + a_2k_1 \leq b_1m_1 + b_2m_1$ and hence $d(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq m_1/k_1 = d(\c{G}_1 \boxplus_\beta \c{G}_2)$. Therefore, $\c{G}_1 \boxplus_\beta \c{G}_2$ is unbiased. □

We end this section with the following result which states that unbiased notion is preserved under isomorphism:

Theorem 3.4. *Let $\c{G}_1$ and $\c{G}_2$ be isomorphic PFGs. If one of them is unbiased, then the other is unbiased.*

Proof. Suppose $\c{G}_2$ is unbiased and let $h : \H{U}_1 \to \H{U}_2$ be a bijection such that $\c{t}_1(\H{u}) = \c{t}_2(h(\H{u}))$ and $\varsigma_1(\H{u}\ddot{y}) = \varsigma_2(h(\H{u})h(\ddot{y}))$ for all $\H{u}, \ddot{y} \in \H{U}_1$. Now $\sum_{\H{u}\in\H{U}_1} \c{t}_1(\H{u}) = \sum_{\H{u}\in\H{U}_2} \c{t}_2(\H{u})$ and $\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u}\ddot{y}) = \sum_{\H{u}\ddot{y}\in\breve{E}_2} \varsigma_2(\H{u}\ddot{y})$. If $H_1 = (\c{t}_1, \varsigma_1)$ is a product fuzzy subgraph of $\c{G}_1$ with underlying set W, then $H_2 = (\c{t}_2, \varsigma_2)$ is a product fuzzy subgraph of $\c{G}_2$ with underlying set $h(W)$ where $\c{t}_2(h(\H{u})) = \c{t}_1(\H{u})$ and $\varsigma_2(h(\H{u})h(\ddot{y})) = \varsigma_1(\H{u}\ddot{y})$ for all $\H{u}, \ddot{y} \in W$. Since $\c{G}_2$ is unbiased, $d(H_1) \leq d(\c{G}_2)$ and so $2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_2(h(\H{u})h(\ddot{y}))}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge \c{t}_2(\ddot{y}))} \leq 2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_2(\H{u}\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge \c{t}_2(\ddot{y}))}$. Hence

$$2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u}\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge \c{t}_2(\ddot{y}))} \leq 2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u}\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge \c{t}_2(\ddot{y}))}.$$

Therefore, $\c{G}_1$ is unbiased. □

4. Discussion

Several types of products of two FGs were explored. We launch the conception of $\beta-$product of PFGs and give sufficient conditions for the $\beta-$product of two PFGs to be strong (complete). We also show that to have at least one factor is a complete PFG, the $\beta-$product should be complete. Finally, necessary and sufficient conditions for the $\beta-$product of two unbiased PFGs to be unbiased are provided. As a future research, we might study $\beta-$product of certain types of fuzzy graphs.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors thanks the referees for useful comments and suggestions that improved the paper.

References

1. T. Al-Hawary, *Characterizations of matroid via OFR-sets*, Turkish Journal of Mathematics **25** (2001), 445-455.
2. T. Al-Hawary, *Complete fuzzy graphs*, J. Math. Combinatorics **4** (2011), 26-34.
3. T. Al-Hawary, *Certain classes of fuzzy graphs*, European J. Pure and Appl. Math. **10** (2012), 552-560.
4. T. Al-Hawary, S. Al-Shalaldeh and M. Akram, *Certain Matrices and Energies of fuzzy graphs*, TWMS J. Pure Appl. and Math. **14** (2023), 50-68.
5. T. Al-Hawary and L. Al-Momani, **-balanced fuzzy graphs*, arXiv preprint **arXiv:1804.08677**, (2018), 26-34.
6. T. Al-Hawary and B. Hourani, *On intuitionistic product fuzzy graphs*, Ital. J. Pure Appl. Math (2017), 113-126.
7. T. Al-Hawary and B. Hourani, *On product fuzzy graphs*, Annals Fuzzy Math. Inform. **12** (2016), 279-294.
8. T. Al-Hawary, *Density Results for Perfectly Regular and Perfectly Edge-regular fuzzy graphs*, Disc. Math. Scie. and Cryptography **2** (2022), 1-10.
9. T. Al-Hawary, *Maximal strong product and balanced fuzzy graphs*, To appear in J. Appl. Math. and Informatics.
10. T. Al-Hawary and M. Hashim, *Semi-fuzzy graphs*, To appear in Boletim da Sociedade Paranaense de Matematica.
11. T. Al-Hawary, *Strong Modular Product and Complete Fuzzy Graphs*, To appear in Ital. J. Pure Appl. Math.
12. T. Al-Hawary, *Complete Hamacher fuzzy graphs*, J. Appl. Math. and Informatics **40** (2022), 1043-1052.
13. M. Akram, D. Saleem, T. Al-Hawary, *Spherical fuzzy graphs with application to decision-making*, Math. and Comp. Appl. **25** (2020), 8-40.
14. K.R. Bhutani, *On automorphism of fuzzy graphs*, Pattern Recognition Letter **9** (1969), 159-162.
15. S. Dogra, *Different types of product of fuzzy graphs*, Prog. Nonlin. Dyn. Chaos **3** (2015), 41-56.
16. J.N. Mordeson and C.S. Peng, *Operations on fuzzy graphs*, Information Sciences (1979), 381-3
17. A. Nagoor Gani and B. Fathima Gani, *Beta and Gamma product of fuzzy graphs*, Inter. J. Fuzzy mathematical Archive **4** (2014), 20-36.
18. S. Dogra, *Different types of product of fuzzy graphs*, Prog. Nonlin. Dyn. Chaos **3** (2015), 41-56.
19. A. Nagoor Gani and J. Malarvizhi, *Isomorphism on fuzzy graphs*, Int. J. Comp. and Math. Sci. **2** (2008), 190-196.

20. A. Nagoor Gani and J. Malarvizhi, *Isomorphism properties on strong fuzzy graphs*, Int. J. Algorithms, Comp. and Math. **2** (2009), 39-47.
21. A. Nagoor Gani and K. Radha, *On regular fuzzy graphs*, J. Physical Sciences **12** (2008), 33-40.
22. A. Rosenfeld, FGs, in Zadeh. L.A.K.S. Fu, K. Tanaka and M. Shirmura (Eds), *Fuzzy sets and their applications to cognitive and processes*, Academic Press, New York, 2008, 77-95.
23. V. Ramaswamy and B. Poornima, *Picture fuzzy graphs*, Inter. J. Computer Sci. and Network Security **9** (2009), 114-124.
24. M.S. Sunitha and A.V. Kumar, *Complements of fuzzy graphs*, Indian J. Pure Appl. Math. **33** (2002), 1451-1464.
25. L.A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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