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ON THE LINEAR EQUIVALENCE OF SEQUENCES IN HILBERT SPACES

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ABSTRACT. A similarity transformation of a solution of the Cauchy problem for the linear difference equation in Hilbert space has been studied. In this manuscript, we obtain necessary and sufficient conditions for linear equivalence of the discrete semigroup of operators, generated by the solution of the difference equation utilizing four Canonical semigroups.

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1. Introduction

Similarity transformations of sequences, generated by the discrete semigroup of operators with bounded generators, in this article, Hilbert space is examined. Similarity transformations are used for reduction of some sequence classes to the canonical form. In this connection, the results of work [1–8, 13–18], concerning the similarity transformation of linear operators are used.
Consider the random sequence $Z(n)$ with $MZ(n) = 0$ and the correlation function $K(n, m) = MZ(n)$.
After natural embedding of $Z(n)$ in Hilbert space H, we obtain some sequences in H denoted by $Z(n)$ [7].

In this article,we focus on Evolutionary Representable Sequences ERS in H, which solve the Cauchy problem [7],

$$\begin{cases} Z_{n+1} = AZ_n, \\ Z_{n|_{n=0}} = Z_0, \end{cases} \tag{1}$$

where A is a linear bounded operator in H, $Z_0 \in H$.

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The correlation function here is the following scalar product

$$K(n,m) = \langle Z_n, Z_m \rangle = \langle A^n Z_0, A^m Z_0 \rangle ,$$

where $A^n Z_0$ is the solution of Cauchy problem (1).
Within the correlation theory it is possible to use the unitary transformations of operator A, which don't change the correlation function,but reduce the operator A to definite structure [7].
The following simplification of Z_n is connect with rejection from assumption about the unitary property of the correspondent transformation .
So, there arises a problem about the reduction of the sequence determined by the solution of Cauchy the problem (1), to the simplest form with the similitude transformation.

Definition 1. *Two ERS* $Z_n^k \in H_k, (k = 1, 2)$ *are called linear equivalent if there is such a reversible operator* $B \in [H_1, H_2]$, *so that*

$$A_2 = BA_1B^{-1} \qquad , \qquad Z_{0_2} = BZ_{0_1}.$$

For the solution of the reduction problem for ERS to the simplest form, let us use the results of the articles [9], [12], in which the reduction problem of Volterra operators to the Canonical form in $L^p, 1 < p < \infty$ is considered in full. We consider the case $p = 2$.

Let us introduce the following form of ERS, $\hat{Z}_n(x) = \hat{A}^n \hat{f}_0(x)$ in $L^2_{[0,1]}$, which called Canonical:

(1) $(\hat{A}_1 f)(x) = xf(x)$,

(2) $(\hat{A}_2 f)(x) = i\int\limits_0^x f(y)dy$,

(3) $(\hat{A}_3 f)(x) = \int\limits_0^x (x-y)f(y)dy$,

(4) $(\hat{A}_4 f)(x) = \alpha\int\limits_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)dy$, $(\alpha > 0)$.

2. Mean Results

1. Sakhnovich [10] obtained the sufficient conditions for the linear equivalence of operators $(Af)(x) = \int\limits_0^x G(x,y)f(y)dy$ in $L^2_{[0,1]}$ and operators $\hat{A}_k$, $k = \overline{1,4}$.
Utilizing Sakhnovich's results a series of theorems, related to the reduction problem of the ERS in $L^2_{[0,1]}$ to the Canonical form, are easily obtained.

Theorem 1. *If the ERS* $Z_n(x) = \hat{A}f_0(x)$ *in* $L^2_{[0,1]}$ *has operator* A *of the form* $(Af)(x) = xf(x) \pm i\int\limits_0^x p(x)p(y)f(y)dy$, *where* $p(x)$ *is uniformly bounded nonnegative function over* $[0,1]$ *,then this sequence is linearly equivalent to the sequence* $\hat{Z}_n(x) = x^n \hat{f}_0(x)$, *i.e. the sequence of Form 1.*

Proof. It was proved in [9] that the operator A of the given form may be reduced to the self-adjoint form with the help of similarity transformation operator which is an operator multiplied by independent variable in $L^2_{[0,1]}$ space.
Since $BZ_n(x) = BA^n f_0(x) = (BAB^{-1})^n Bf_0(x) = \hat{A}^n \hat{f}_0(x)$, and $\hat{A}f = xf$, then we obtain the conclusion of the theorem immediately.$\square$

Theorem 2. *If* G is *the kernel of operator* $(Af)(x) = \int\limits_0^x G(x,y)f(y)dy$, *satisfies the following conditions:*

$G(x,y) = 1$ *and* $\left|\frac{\partial^2 G(x,y)}{\partial x \partial y}\right| \prec M$ *, then ERS* $Z_n(x) = A^n f_0(x)$ *in* $L^2_{[0,1]}$ *is linearly equivalent to the sequence* $\hat{Z}_n(x) = \int\limits_0^x \frac{(x-y)^{n-1}}{(n-1)!}\hat{f}(y)dy$, *i.e. a sequence of Form 2.*

Theorem 3. *If* G is the kernel of operator $(Af)(x) = \int\limits_0^x G(x,y)f(y)dy$ *satisfies the following conditions:*

$G(x,y) = 0$, $\frac{\partial}{\partial x}G(x,y) \mid_{x=y} = 1$ *and the derivatives* $\frac{\partial^{k+\ell}G(x,y)}{\partial x^k \partial x^\ell}$, $k,\ell = 0,1,2$, $\frac{d}{dx}(\frac{\partial^2}{\partial x^2}G(x,y) \mid_{x=y})$ *are bounded, then the ERS* $Z_n(x) = A^n f_0(x)$ *in* $L^2_{[0,1]}$ *is linearly equivalent to the sequence of Form 3.*

Theorem 4. *If* G is *the kernel of operator* $(Af)(x) = \int\limits_0^x G(x-y)f(y)dy$ *satisfies the following conditions :*

$G(0) = G^{'}(0) = ... = G^{(n-2)}(0)$, $G^{(n-1)}(0) = \alpha \neq 0$ *and* $G^{(n+1)}(x) \in L^2_{[0,1]}$, *then ERS the* $Z_n(x) = A^n f_0(x)$ *in* $L^2_{[0,1]}$ *is linearly equivalent to the sequence of Form 4.*

The proofs of Theorem 2-4 are similar to that of Theorem 1, if the results of [10,11,12] on similarity of operator A to the operator $\hat{A}_2 f = i\int\limits_0^x f(y)dy$; A to the operator $\hat{A}_3 f = i\int\limits_0^x (x-y)f(y)dy$, i.e. to the square of operator $\hat{A}_2$ (to within the constant multiplier) and A to the operator $\hat{A}_4 f = \alpha\int\limits_0^x \frac{(x-y)^{n-1}}{(n-1)!}f(y)dy$, i.e. to the n$^{\text{th}}$ power of the operator $\hat{A}_2$ in Theorems 2,3,4 respectively.$\square$

In order that the corresponding of ERS be linearly equivalent to a sequence of the forms 1-4, it must be considered not only the sufficient conditions, but the necessary conditions too.

In [10] it has been obtained necessary and sufficient conditions for the linear equivalence of Volterra operators for some classes and operator $\hat{A}_2 f = i\int\limits_0^x f(y)dy$.
These conditions (Theorem 1-4) leads to the following criterion of reducing ERS to the Canonical form.

Theorem 5. *In order that the ERS* $Z_n(x) = A^n f_0(x)$ *in* $L^2_{[0,1]}$ *with the operator* $(Af)(x) = \int\limits_0^x G(x,y)f(y)dy$ *be linearly equivalent to the sequence of Form 2,it is necessary and sufficient that the kernel* $G(x,y)$ *of the operator* A *satisfies the conditions :*

1) $G(x,y) = 1 + G_1(x-y) + G_2(x-y)$,
2) $G_1(x)$ *is absolutely continuous function in [0,1] and* $G_1(0) = 0$,
3) $G_1^{'}(x) \geq 0$ *is non-increasing and absolutely continuous function in every closed interval* $[\varepsilon, 1], \forall\varepsilon > 0$,
4) $G_2(x) \geq 0$, $G_2(0) = 0$, *and* $G_2'(x)$ *is absolutely continuous function in* $[0,1]$.

Theorem 6. *In order that the ERS* $Z_n(x) = A^n f_0(x)$ *in* $L^2_{[0,1]}$ *with the operator* $(Af)(x) = \int\limits_0^x G(x,y)f(y)dy$ *be linearly equivalent to the sequence of Form 2, it is necessary and sufficient that the kernel* $G(x,y)$ *of the operator* A *satisfies the conditions :*

1) $G(x,y) = 1 + G_1(x-y) + G_2(x,y)$,
2) $G_1(x)$ *satisfies the conditions of the Theorem 5,*
3) $G_2(x,y) \geq 0$, $G_2(0,0) = 0$,
4) $G_2(x,y)$ *is absolutely continuous function of* x *for all* $y \in [0,1]$,
5) $\frac{\partial G_2(x,y)}{\partial x}$ *is absolutely continuous function of* y *for all* $x \in [0,1]$ *and* $\left|\frac{\partial G_2(x,y)}{\partial x}\right| < 0$,
6) $\frac{\partial^2 G(x,y)}{\partial x \partial y} \in L^1_{[0,1]}$.

Proofs of Theorems 5-6 can be obtained using the result of [10], where under the restrictions on the integral operator kernel there, the necessary and sufficient conditions for the similarity of operator A to the operator A_2 are given.The reasoning is similar to the proof of Theorem 1.

2. Let us analyze, in details, an example of a sequence of Form 2.
In order that the character of the sequence deviation from the sequence with the correlation function $K(n,m)$ (i.e. from the sequence of Form 1), we determine the correlation difference $W(n,m)$ of the form

$$W(n,m) = K(n+1,m) - K(n,m+1) = i\,\langle 2J_m A Z_{n,} Z_m \rangle\,.$$

For the sequence of Form 2, $2J_mA$ take the form

$(2J_mAf)(x) = \int\limits_0^1 (x,y)f(y)dy = \sum\limits_{\alpha,\beta=1}^{2} \langle f(x), g_\alpha \rangle\, J_{\alpha,\beta} g_{\beta}$, where the involutive matrix is :

$$J_{\alpha,\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the canal elements $g_1(x) = x$, $g_2(x) \equiv 1$.
Since the matrices J and I are distinct then the sequence in the example is not dissipative one .
And the correlation difference $W(n,m)$ has the form

$$W(n,m) = \sum_{\alpha,\beta=1}^{2} \Phi_\alpha(n) J_{\alpha,\beta} \overline{\Phi_\beta(m)},$$

where,

$$\Phi_\alpha(n) = \left\langle \hat{Z}_n(x), g_\alpha \right\rangle = \int\limits_0^1 \hat{Z}_n(x) \overline{g_\alpha} dx.$$

Since $\hat{Z}_n(x) = \hat{A}^n \hat{f}_0(x)$, then to find the scalar product it is possible to use the representation $\hat{A}^n$ through the resolvent of the operator A and so it is convenient to represent $\Phi_\alpha(n)$ in the form

$$\begin{aligned} \Phi_\alpha(n) &= \left\langle \hat{A}^n \hat{f}_0, g_\alpha \right\rangle = \left\langle \hat{f}_0, \hat{A}^{*^n} g_\alpha \right\rangle \\ &= \int\limits_0^1 \hat{f}_0(x) \overline{\Lambda_n(x)} dx, \end{aligned}$$

where,

$$\Lambda_n(x) = -\frac{1}{2\pi i} \oint\limits_\gamma \lambda^n (\hat{A}^* - \lambda I)^{-2} g_\alpha(x) dx,$$

and γ is the contour enclosing the spectrum of operator $\hat{A}$.

So, calculation of $\Phi_\alpha(n)$ reduces to finding the resolvent of operator A on the canal elements.
For example,

$$\Phi_2(n) = \int\limits_o^1 \frac{x^{2(n-1)}(-1)^{n+1}}{(2n+2)!} \hat{f}_0(x) dx,$$

then for $\Phi_1(n)$ one can find the similar expression, which is omitted in view of the inconvenience.

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Data availability : The data used to support the findings of this study are included within the article.

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