J. Appl. Math. & Informatics Vol. 42(2024), No. 2, pp. 223 - 235
https://doi.org/10.14317/jami.2024.223

EXPLORING NOVEL APPROACHES FOR ESTIMATING FRACTIONAL STOCHASTIC PROCESSES THROUGH PRACTICAL APPLICATIONS

NABIL LAICHE, LAID GASMI, RAMAN VINOTH, AND HALIM ZEGHDOUDI*

Abstract. In this paper, our primary focus revolves around the examination of a set of fractional stochastic models. Through our investigation, we can establish the presence of a solution and its distinctiveness. Additionally, we employ a moment-based algorithm to estimate the coefficients within these models and provide evidence that these estimations maintain their asymptotic characteristics. To support this claim, we conduct experimental studies using simulations and numerical examples.

AMS Mathematics Subject Classification : 34K50, 60G22, 60G20.
Key words and phrases : Fractional stochastic models, Caputo fractional derivatives, stochastic process.

1. Introduction

Stochastic models have garnered increasing attention due to their capacity to represent complex systems. This research area has become essential not only for mathematicians and statisticians but also for professionals in fields such as chemistry, biology, economics, and physics see [9]. In economics, a substantial portion of economic phenomena necessitates stochastic modeling, and similarly, in physics, several phenomena demand the application of stochastic theories see [17] and [18] . Looking back historically, credit is owed to the biologist R. Brown in 1827 for his observation of the highly irregular motion exhibited by a pollen particle submerged in a fluid. The applications of stochastic modeling extends beyond biology, and it wasn't until approximately 80 years later Einstein and Smoluchowski provided a coherent interpretation of this phenomenon. By a broad definition, we can assert that Brownian motion stands as the quintessential problem in stochastic process theory [19], [20]. Moreover, it has assumed

---

Received September 16, 2022. Revised November 24, 2023. Accepted February 16, 2024.
*Corresponding author.

significant proportions. With our current knowledge, we would approach modeling this system as follows.

Second section: This involves expanding upon the theoretical investigation of a set of stochastic processes featuring fractional derivatives and presenting the generalization of certain theorems in the realm of stochastic analysis.

Third section: Enhancing the study with explicit fractional elements, the model coefficients are estimated using the moments-based approach.

Fourth section: This section relies on the cited algorithm to estimate coefficients in fractional models, considering it as a broader and more encompassing scenario.

Fifth section: Numerical illustration to prove the value of theoretical study.

We commence with a vibrant and instructive example to demonstrate the significance of this model type. The Brownian particle experiences a counteracting viscous drag force due to the fluid's viscosity (assuming the fluid is stationary)

$$F_{drag} = -\theta m v$$

Here, θ represents the kinematic viscosity of the medium, and m denotes the mass of the Brownian particle. In the field of physics, it is observed that the smaller fluid particles move significantly faster than the Brownian particle we aim to describe. To account for this, we will simplify the interactions between the Brownian particle and the fluid particles. Consequently, the random changes in velocity experienced by the Brownian particle (v) will be modeled using a Wiener process characterized by a strength parameter, g. We can also establish the presence of external force acting on the Brownian particles, denoted as h. This problem can be formulated in the following manner

$$d^q x(t) = g(t, x(t))dt + h(t, x(t))d^r w(t), \ 0 < q, r \leq 1 \tag{1}$$

Where d^q denotes the fractional derivative which we will define. First we will expose the preliminary bases for the maternal model

$$\begin{cases} dx(t) = g(t, x(t)dt + h(t, x(t))dw(t) \\ x(t_0) = c_0, \ t_0 \leq t \leq M < \infty \end{cases} \tag{2}$$

2. Preliminaries

This section is dedicated to elucidating and expanding upon the essential concepts related to stochastic equations, as well as the renowned theorems in the field of stochastic analysis, and their relevance to our research. Additionally, we delve into the fundamental principles of fractional calculus. Following this, we create the data set that forms the focal point of our investigation, one that incorporates both stochastic and fractional derivative characteristics. We commence with the following theorem, as referenced in see [19].

Theorem 1. *Let a stochastic process $(x(t))_{t\in\mathbb{R}}$ defined in a probability space (Ω, A, P) with expression (2)* *where $w(t)$ is Brownian motion, g and h tow measurable functions on the interval $[t_0, M]$, and these functions are checked on the following conditions. There exists a constant $C > 0$ such that*

$\mathbf{A}: |g(t, x(t) - g(t, y(t)| + |h(t, x(t) - h(t, y(t)| \leq C\,|x(t) - y(t)|$

(This property is known by the Liptischizian condition)

$\mathbf{B}: |g(t, x(t))|^2 + |h(t, x(t))|^2 \leq C^2(1 + x^2(t))$.

(This condition is called restriction on growth).

so the process accepts the unique solution

Proof

This theorem must be proven because of its great importance in the following remarks, first we shall prove the uniqueness of solution we would like to show that

$$E\,|x(t) - y(t)|^2 = 0 \text{ for all } t \in [t_0, M]$$

And we suppose that

$$E\,|x(t) - y(t)| = 0$$

Such as $y(t)$ and $x(t)$ are two continuous solutions, in this case must be define a function φ

$$\varphi_k(t) = \begin{cases} 1, \text{ if } |x(t)| \leq k \text{ and } |y(t)| \leq k \\ 0, \text{ otherwise} \end{cases}$$

Since

$$\varphi_k(t) = \varphi_k(t)\varphi_k(s),\ s \leq t$$

We have

$$\begin{aligned} \varphi_k(t)(x(t) - y(t)) &= \varphi_k(t)\int_{t_0}^{M} \varphi_k(s)\{g(s, x(s)) - g(s, y(s))\}ds \\ &\quad + \int_{t_0}^{M} \varphi_k(s)\{h(s, x(s)) - h(s, y(s))\}dw(s) \end{aligned}$$

we apply Lipschitiz condition to bound the integral

$$\Phi(s) = |g(s, x(s)) - g(s, y(s)| + |h(s, x(s)) - h(s, y(s)|$$

$$\varphi_k(s)\Phi(s) \leq k\varphi_k(s)\,|x(s) - y(s)| \leq 2k^2$$

and by the Schwartz inequality we find

$$\begin{aligned} E\left[\varphi_k(t)\,|x(t) - y(t)|^2\right] &\leq 2E\left|\int_{t_0}^{M} \varphi_k(s)E\{g(s, x(s)) - g(s, y(s))\}ds\right|^2 \\ &\quad +2E\left|\int_{t_0}^{M} \varphi_k(s)\{h(s, x(s)) - h(s, y(s))\}dw(s)\right|^2 \\ &\leq 2(M - t_0)\int_{t_0}^{t} E\varphi_k(s)\,|g(s, x(s)) - g(s, y(s))|^2 \\ &\quad + |h(s, x(s)) - h(s, y(s))|^2 \end{aligned}$$

$$\leq \quad C_0 \int_{t_0}^{t} E\varphi_k(s) \left|x(s) - y(s)\right|^2 ds = 0$$

Which shows that

$$E \left|x(t) - y(t)\right|^2 = 0$$

Remark 2. *If $x(t)$ and $y(t)$ are two solutions for equation (2), then*

$$P\left(\sup_{t_0 \leq t \leq M} |x(t) - y(t)| = 0)\right) = 1$$

Proof : See reference [1].

Before studying the process (1), we recall the fractional derivatives.

$[a, b]$ $(-\infty < a < b < +\infty)$ be a finite interval of $\mathbb{R}$, the left and right Riemann -Liouville fractional integrals $I^q_{(a,t)}$ and $I^q_{(t,b)}$ of order r are defined as follows

$$I^q_{(a,t)} x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-u)^{q-1} x(u) du, \ t > a, \ q > 0$$

and

$$I^q_{(t,b)} x(t) = \frac{1}{\Gamma(q)} \int_t^b (u-t)^{q-1} x(u) du, \ t < b, \ q > 0$$

where the gamma function is defined

$$\Gamma(q) = \int_0^{+\infty} u^{q-1} \exp(-u) du, \ q > 0$$

Definition 3. *The left and right Riemann- Liouville fractional derivatives $d^q_{(a,t)}$ and $d^q_{(t,b)}$ of order $q > 0$, are defined*

$$\begin{aligned} d^q_{(a,t)} x(t) &= \frac{d^n}{dt^n} \left\{ I^{n-\beta}_{(a,t)}(x(t)) \right\} \\ &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-u)^{n-q-1} x(u) du, \ t > 0 \end{aligned}$$

and

$$\begin{aligned} d^q_{(t,b)} &= (-1)^n \frac{d^n}{dt^n} \left\{ I^{n-q}_{(t,b)}(x(t)) \right\} \\ &= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (u-t)^{n-q-1} x(u) du, \ t > 0 \end{aligned}$$

Definition 4. *The left and right Caputo fractional derivative $d^{(C,r)}_{(a,t)}$ and $d^{(C,r)}_{(t,b)}$ of order $\beta > 0$ are defined by*

$$d^{(C,q)}_{(a,t)} f(t) \quad = \quad d^q_{(a,t)} \left\{ x(t) - \sum_{j=0}^{n-1} \frac{x^{(j)}(a)}{j!} (t-a)^j \right\}$$

$$\begin{aligned} &= d^q_{(a,t)}x(t) - d^q_{(a,t)} \sum_{j=0}^{n-1} \frac{x^{(j)}(a)}{j!}(t-a)^j \\ &= d^q_{(a,t)}x(t) - \sum_{j=0}^{n-1} \frac{x^{(j)}(a)}{\Gamma(j-q+1)}(t-a)^{j-q} \end{aligned}$$

and the same for $d^{(C,q)}_{(a,t)}x(t)$

$$d^{(C,q)}_{(t,b)}x(t) = d^{\beta}_{(t,b)}x(t) - \sum_{j=0}^{n-1} \frac{x^{(j)}(b)}{\Gamma(j-q+1)}(b-t)^{j-q}$$

where $n = [q]+1$, in the case where $0 < q < 1$ we find

$$d^{(C,q)}_{(a,t)}x(t) = d^q_{(a,t)}x(t) - \frac{x(a)}{\Gamma(1-q)}(t-a)^{-q}$$

and

$$d^{(C,q)}_{(t,b)}x(t) = d^q_{(t,b)}x(t) - \frac{x(b)}{\Gamma(1-q)}(b-t)^{-q}$$

$$d^s x(t) = \{\beta_0(t)x(t) + \beta_1(t)\}\, dt + \{\gamma_0(t)x(t) + \gamma_1(t)\}\, dw(t) \tag{3}$$

We can write this stochastic model as a generalized function

$$\left\{ \begin{array}{c} d^{(C,q)}_{(t_0,t)}x(t) = F(t, x(t), w(t)) \\ x(t_0) = \omega(t_0),\ 0 < r < 1 \\ F(t, x(t), w(t)) = g(t, x(t)dt + h(t, x(t))dw(t) \end{array} \right. \tag{4}$$

The case where $s = 1$ has been studied by A. Bibi and F. Mrahi in reference **[25]**. And the existence and uniqueness of Itô solution process is ensured by the general results, this solution according to Le Breton and Musiela is given by

$$x(t) = \varphi(t) \left\{ x(t_0) + \int_{t_0}^{t} \varphi^{-1}(s)(\beta_1(t) - \gamma_0(s)\gamma_1(s))ds + \int_{t_0}^{t} \varphi^{-1}(s)\gamma_1(t)dw(s) \right\}$$

Where

$$\varphi(t) = \exp\left\{ \int_{t_0}^{t} \beta_0(s) - \frac{1}{2}\gamma_0(t)ds + \int_{t_0}^{t} \gamma_0(t)dw(t) \right\}, t \geq 0.$$

Now we will look for the existence and uniqueness of the model in the case (4). First, the question that arises is that this type of model follows the conditions of the first theorem, in this case we need to state the theorem

Theorem 5. *The stochastic process (4) accepts a unique solution written in the form*

$$x(t) = \omega(t_0) + \frac{1}{\Gamma(q)} \left\{ \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))dt + \int_{t_0}^{t} (t-s)^{q-1} h(s, x(s))dw(s) \right\}$$

Proof

This theorem is a particular case of theorem under above in it suffices to prove the following theorem.

Theorem 6. *Let process $(x(t))_{t\in\mathbb{R}}$ defined with its fractional stochastic expression*

$$\begin{cases} d^{(C,q)}_{(t_0,t)}\{x(t)-\Phi(t,x(t))\} = F(t,x(t),w(t)) \\ x(t_0)=\omega(t_0) \end{cases} \tag{5}$$

If the model satisfies the following conditions

H_1) $F(t,x(t),w(t)) = \Psi_1(t,x(t))dt + \Psi_1(t,x(t))dw(t)$

H_2) $F(t,x(t),w(t))$ *is measurable with respect to t on I.*

H_3) $F(t,x(t),w(t))$ *is continuous on $C(I,\mathbb{R}^n)$.*

H_4) *There exist $\delta>0$, a real valued function $v(t)\in L^{\frac{1}{\delta}}$ such that for any $x(t)$*

$$F(t,x(t),w(t)) \leq v(t)$$

H_5) *For any $x_0(t), x_1(t)$*

$$|\Phi(t,x_0(t))-\Phi(t,x_1(t))| \leq \theta\,|x_0(t)-x_1(t)|$$

Then, the model (5) accepts a unique solution defined by its following expression

$$\begin{aligned} x(t) &= \omega(t_0)-\Phi(t_0,\omega(t_0))+\Phi(t,x(t)) \qquad (6) \\ &+\frac{1}{\Gamma(q)}\left\{\int_{t_0}^{t}(t-s)^{q-1}\Psi_1(s,x(s))dt+\int_{t_0}^{t}(t-s)^{q-1}\Psi_1(s,x(s))dw(s)\right\} \end{aligned}$$

Proof

We will make an extension for the proof. First it is easy to obtain that F is lebesgue measurable, Based on **H_2** and **H_3** we can notice that

$$(t-s)^{q-1}\in L^{\frac{1}{1-q}}([t_0,t],\mathbb{R})$$

logically the function $(t-s)^{q-1}F(t,x(s),w(s))$ will be Lebesgue integrable with respect to $[t_0,t]$ for all $t\in\mathbf{I}_0$, and with Holder's inequality we bound our function

$$\int_{t_0}^{t}\left|(t-s)^{q-1}F(s,x(s),w(s))\right|ds \leq \|v(t)\|\left\|(t-s)^{q-1}\right\|_{\frac{1}{1-q}[t_0,t]}$$

now just substitute the solution in the generalized equation (5), then

$$\begin{aligned} & d^{(C,q)}_{(t_0,t)}\{x(t)-\Phi(t,x(t))\} \\ = \; & d^{(C,q)}_{(t_0,t)}\left\{\begin{array}{c} \omega(t_0)-\Phi(t_0,\omega(t_0))-\Phi(t,\omega(t)) \\ +\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\Psi_1(s,x(s))dt \\ +\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\Psi_1(s,x(s))dw(s) \end{array}\right\} \\ = \; & d^{(C,q)}_{(t_0,t)}\left(i^q_{(t_0,t)}F(t,x(t),w(t))\right)-I^q_{(t_0t)}F(t,x(t),dw(t))_{t=t_0}\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \end{aligned}$$

through $d_{(t_0,t)}^{(C,q)}\left(i_{(t_0,t)}^{q}F\right) = F$ (it is a fundamental property in Caputo's Fractional Calculus see [**15**]), and when $i_{(t_0,t)}^{q}F(t,x(t),w(t))_{t=t_0} = 0$, so we will get to that.

$$d_{(t_0,t)}^{(C,q)}\{x(t)-\Phi(t,x(t))\} = F(t,x(t),w(t))$$

2.1. General theorem construction. Are the conditions of the first theorem on a stochastic process valid if the process is stochastic and fractional. Acronym. we will call these models fractional stochastic process. Now, we construct the following theorem.

Theorem 7. *Let $(x(t))_{t\in\mathbb{R}}$ fractional stochastic process defined in a probability space (Ω, A, P) with the following expression*

$$d_{(t_0,t)}^{(C,q)}x(t) = g(t,x(t)dt + h(t,x(t))dw(t) \tag{7}$$

g and h are defined in the previous theorem and are checked on the following conditions. Let a constant $C > 0$ such that
1 : $|g(t,x(t)-g(t,y(t)| + |h(t,x(t)-h(t,y(t)| \leq C\,|x(t)-y(t)|$
2 : $|g(t,x(t))|^2 + |h(t,x(t))|^2 \leq C^2(1+x^2(t))$.
then, the process (7) accepts the solution.

Proof
The proof is very simple if we write process (7) under the following form

$$I_{(t_0,t)}^{q}d_{(t_0,t)}^{(C,q)}(x(t)) = I_{(t_0,t)}^{q}g(t,x(t)dt + I_{(t_0,t)}^{q}h(t,x(t))dw(t)$$

In addition to being

$${}_{(t_0,t)}^{(C,q)}(x(t)) = d_{(t_0,t)}^{q}(x(t)-x(t_0)),\ 0<q<1$$

and

$$\begin{aligned} I_{(t_0,t)}^{q}d_{(t_0,t)}^{(C,q)}(x(t)) &= I_{(t_0,t)}^{q}d_{(t_0,t)}^{q}(x(t)-x(t_0)),\ 0<q<1 \\ &= x(t)-x(t_0) \end{aligned}$$

Also, the two functions $I_{(t_0,t)}^{q}g$ and $I_{(t_0,t)}^{q}h$ are conducive the tow propreties **1** and **2**. Thus, we will reach the following mathematical writing

$$x(t) = I_{(t_0,t)}^{q}g(t,x(t)dt + I_{(t_0,t)}^{q}h(t,x(t))dw(t) + x(t_0)$$

which shows that the sample of fractional stochastic models is a generalized case for stochastic models.

3. Estimation

There are many approaches to estimation, and the samples of different stochastic models are innumerable and very complicated, so the nature of the approach is not always valid for estimation. In the stochastic literature, there are several methods for estimating nonlinear models in the continuous case, but the best known is the method of moments. For our sample, the COGARCH models were compiled by the statisticians Haug, Kluperllberg, Linder and Zapp in 2005 see [8]. Moreover, this type of model is a special case of continuous stochastic models. And besides, the moment approach gives better estimation results. And to apply this method, it is necessary to specify a private category of fractional stochastic models. This model has acquired quite some attention in the physics literature relatively to its probabilistic properties and asymptotic behavior of its statistical inference

$$\begin{cases} d_{(0,t)}^{(C,q)}x(t) = \{\gamma_0(t)x(t) + \gamma_1(t)\}\,dt + \gamma_2(t)d^r w(t). \\ x(0) = x_0,\ 0 < q, r < 1. \end{cases} \tag{8}$$

$(x(t))_{t>0}$ defined on some probability space (Ω, A, P) denoted by a fractional derivative in the Caputo sense, where $\{(\gamma_i(t))_{t\in\mathbb{R}}, i = 0, 1 \text{ } or \text{ } 2\}$ the coefficients part of the fractional stochastic model, such as $\gamma_0(t)\gamma_1(t)\gamma_2(t) \neq 0$, $w(t)$ represents brownian motion process. In the simplest process where we assume that $q = r$ and $\gamma_2(t) = 0$. Then, in this case we will find the classic solution method

$$d^q x(t) - d^q w(t) = \{\gamma_0(t)x(t) + \gamma_1(t)\}\,dt$$

Then

$$d^q\,(x(t) - w(t)) = \{\gamma_0(t)x(t) + \gamma_1(t)\}\,dt$$

Which gives the solution directly according to the definition of general fractional derivative

$$I^q d^q\,(x(t) - w(t)) = I^q\,\{\gamma_0(t)x(t) + \gamma_1(t)\}\,dt.$$

Assumption 01 : see [3]. Under the following conditions, for any $T > 0$

$$\textbf{a)} \int_0^T |\gamma_i(t)|\,dt < \infty,\ i = 0, 1.$$

and

$$2\gamma_0(t) < 0$$

Theorem 8. *Under* ***Assumption 01*** *we have the mean defined* $E(x(t)) = m(t)$, *and the variance* $v(t)$ *and*

$$C(t,s) = E\,\{(x(t) - m(t))(x(s) - m(t))\}\,, t > s$$

functions of process (8) generated by its fractional stochastic expression are written respectively by

$$\begin{aligned} m(t) &= \varphi(t)m(0) \\ v(t) &= \varphi(t)\left\{v(0) + \int_0^t \varphi^{-1}(s)\gamma_1^2(s)dt\right\} \\ C(t,s) &= \varphi(t)\varphi^{-1}(s)v(s),\ t \geq s \geq 0 \end{aligned}$$

where

$$\varphi(t) = \exp\left\{\int_0^t 2\gamma_0(z)dz\right\}$$

Proof

To demonstrate this theorem it is necessary to write the model (8) in the form of a product between a classical derivative and a function. Then, we will apply the theorem 2.1 of reference [3].

$$\begin{aligned} d_{(0,t)}^{(C,q)}x(t) &= \theta(t)dx(t) \\ &= \gamma_0(t)x(t)dt + \gamma_1(t)dw(t) \end{aligned}$$

so, the model will be written in a pure stochastic way

$$dx(t) = \theta^{-1}(t)\left\{\gamma_0(t)x(t)dt + \gamma_1(t)dw(t)\right\}$$

The coefficients which we will estimate and we will designate by the following vector

$$\delta = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$$

Corollary 9. *Under assumption 01 we have the following results*

$$\begin{cases} m\,(0) = -\frac{\gamma_1(t)}{\gamma_0(t)} \\ v(0) = \frac{\gamma_2^2(t)}{\left|2\gamma_0^2(t)\right|} \\ v(h) = v(0)\exp\left\{\gamma_0(t)\,|h|\right\} \end{cases}$$

Corollary 10. *It is assumed here that* $\hat{\delta} = (\hat{\gamma}_{0,N}(t), \hat{\gamma}_{1,N}(t), \hat{\gamma}_{2,N}(t))$*, where* N *represents the sample size. Then we have*

$$\begin{cases} \hat{m}\,(0) = -\frac{\hat{\gamma}_{1,N}(t)}{\hat{\gamma}_{0,N}(t)} \\ \hat{v}(0) = \frac{(\hat{\gamma}_{2,N}(t))^2}{2|\hat{\gamma}_{0,N}(t))|^2} \\ \hat{v}(h) = v(0)\exp\left\{\hat{\gamma}_{0,N}(t))\,|h|\right\} \end{cases}$$

We seek the estimators through sub-conditions $\hat{v}(0) \to 0$ *and* $\hat{v}(h) \to 0$.

4. Simulation

This section is devoted to the simulation of the model (8) with real coefficients, the case where $q = 1$ represents the simulation of a classic model, which shows that fractional derivation is a generalization of normal derivation. We keep the same values for coefficients γ_0, γ_1 and γ_2 in order to make a constructive comparison between the two situations. we use here some convergence criteria noted as follows, γ_i, $i = 0, 1$ and 2 true values, but compared to the estimated values noted $\hat{\gamma}_{i,N}$, $i = 0, ..., 2$, where N sample size, NS (number of simulations) , $RMSE$ noted the root mean square error, here we consider the true values to be $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0, 1$.

To make a simulation it is necessary to take an example of Brownian movement see reference of [20]

$$w(t) = \sqrt{2}\sum_{i=1}^{\infty}\frac{\sin\left\{(i-0.5)\pi t\right\}}{(i-0.5)\pi}\varepsilon_i, t \in [0,1]$$

Where the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ is mutually independent standard Gaussian random variables. and by the polygonal approximation we have

$$w^n = w_{t_i} + (w_{t_i+1} - w_{t_i})\frac{t-t_i}{t_i - t_{i+1}}\ t \in [t_i, t_{i+1}]$$

Table 1: Estimation for fractional stochastic (8) model with true values $\gamma_0 = 0.015, \gamma_1 = 0.05, \gamma_2 = 0,1$ and $r = 1$.

	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
$q = 0.90$	1000	250	0.0245	0.0396	0.1456
	1500	250	0.0185	0.0511	0.0986
	3000	250	0.0162	0.0587	0.1234

Table 2: Estimation for fractional stochastic (8) model with true values $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0,1$

	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{2,N}$
$q = 0.5$	1000	500	0.0164	0.0548	1.4857
	1500	500	0.0245	0.0785	1.0142
	3000	500	0.0137	0.0864	0.9765
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$\beta = 0.2$	1000	500	0.1009	0.0458	1.2227
	1500	500	0.0354	0.2012	1.0165
	3000	500	0.0199	0.1025	0.1253
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$\beta = 0.1$	1000	500	0.0175	0.0589	1.1245
	1500	500	0.0402	0.0412	1.7856
	3000	500	0.0114	0.0415	0.9896

Table 3: Estimation for fractional stochastic (8) model with true values $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0,1$

	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{2,N}$
$q, r = 0.5$	1000	500	0.0180	0.0555	1.0446
	1500	500	0.0235	0.0396	1.1034
	3000	500	0.0767	0.0454	0.9003
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$q, r = 0.2$	1000	500	0.1156	0.0325	1.3453
	1500	500	0.0354	0.2012	1.0345
	3000	500	0.0234	0.3563	0.2351
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$q, r = 0.1$	1000	500	0.1004	0.0456	1.1345
	1500	500	0.0105	0.0412	1.8765
	3000	500	0.0322	0.0418	0.9956

Table 4: RMSE for all situations simulation

			$RMSE$		
β	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
0.7	1000	1000	0.0125	0.0102	0.0245
0.8	2000	2000	0.3125	0.0012	0.0290
0.9	5000	5000	0.0123	0.0223	0.0103

4.1. Conclusion and Futur works. We can observe that the classical COGARCH(1,1) case offers a more accurate approximation between the estimated values and the actual values. This observation highlights the robust asymptotic behavior of the estimators, as demonstrated in **Table 1**. It's noteworthy that as we increase both the sample size (N) and the number of simulations (NS), the estimators gradually converge towards values that closely resemble the actual ones, particularly when $N = 3000$ and $NS = 500$.

In the case of fractional stochastic models, we observe a similar asymptotic behavior in the estimated values as we increase NS and N, as illustrated in **Table 2**. The numerical illustration in **Table 2** of our model simulations reveal that increasing the fractional derivative value (q) enhances the convergence between the model coefficients and their estimators for each q value ($q = \{0.2, 0.5$ or $0.9\}$).

Comparing **Tables 1**, **2**, it becomes evident that the moment-based approach is highly effective when dealing with models featuring fractional components. The estimated values for the $q = 0.9$ case are especially close to those in **Table 1**, and this closeness further improves with the increase in both sample size (N) and the number of simulations (NS).

Among most physicists, the Caputo derivative is considered the most accurate approximation. We notice that as q approaches 1, the approximation becomes increasingly accurate, indicating that the fractional stochastic model is the best fit in our simulations. It's worth noting that the $RMSE$ criterion approaches zero as q approaches 1, able it with some minor fluctuations. Based on findings from previous studies and estimations in this field, it can be inferred that fractional stochastic models represent a broader class of models that encompasses stochastic models. We also deduce when we simulate the model in **Table 3** with the two equal fractional derivatives we found a small disturbance in our simulation which shows that the fractional derivative on the Brownian motion of the model had an impact on our results. In the next, we will study with numerical methods but with two different fractional derivatives r and q, that is to say, we will generalize the study of models, this type of model plays a fundamental role in modeling physical phenomena.

Conflicts of interest: The authors declares no conflict of interest.

Data availability: Not applicable

Acknowledgments: The authors are grateful for the comments and suggestions by the referee and the editor. Their comments and suggestions greatly improved the article.

References

1. T. Andersen, T. Bollerslev, & A. Hadi, *ARCH and GARCH models*, John Wiley & Sons, New York, 2014.
2. A. Behme, C. Klüppelberg, & K. Mayr, *Asymmetric COGARCH processes*, Journal of Applied Probability **51(A)** (2014), 161-173.
3. Bibi, Abdelouahab, and Fateh Merahi, *Yule-Walker type estimator of first-order time-varying periodic bilinear differential model for stochastic processes*, Communications in Statistics-Theory and Methods **49** (2020), 4046-4072.
4. Bishwal, P.N. Jaya, *Parameter estimation in stochastic differential equations*, Springer, 2007.
5. T. Bollerslev, *Generalized autoregressive conditional heteroskedasticity*, Journal of econometrics **31** (1986), 307-327.
6. Doob, L. Joseph, and Joseph L. Doob, *Stochastic processes*, Vol. 7, Wiley, New York, 1953.
7. C. Francq, & J.M. Zakoian, *GARCH models: structure, statistical inference and financial applications*, John Wiley & Sons, 2019.
8. S. Haug, C. Klüppelberg, A. Lindner, & M. Zapp, *Method of moment estimation in the COGARCH (1, 1) model*, The Econometrics Journal **10** (2007), 320-341.
9. R. Hilfer, *Applications of fractional calculus in physics*, World scientific, 2000.
10. C. Klüppelberg, A. Lindner, & R. Maller, *A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour*, Journal of Applied Probability **41** (2004), 601-622.
11. J. Kallsen, & B. Vesenmayer, *COGARCH as a continuous-time limit of GARCH (1, 1)*, Stochastic Processes and their Applications **119** (2009), 74-98.
12. C. Klüppelberg, A. Lindner, & R. Maller, *Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models*, In From stochastic calculus to mathematical finance (pp. 393-419), Springer, Berlin, Heidelberg, 2006.
13. C. Klüppelberg, R. Maller, & A. Szimayer, *The COGARCH: a review, with news on option pricing*, Surveys in stochastic processes, **4** (2011), 1–30.
14. F.Z. Meghraoui, & A. Bibi, *Inférence statistique dans les modéles non linéaires à temps continu*, Doctoral dissertation, Université Mentouri Constantine, 2012.
15. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.
16. T.S. Rao, & M.M. Gabr, *An introduction to bispectral analysis and bilinear time series models*, Vol. 24, Springer Science & Business Media, 2012.
17. Scalas, Enrico, Rudolf Gorenflo, and Francesco Mainardi. *Fractional calculus and continuous-time finance*, Physica A: Statistical Mechanics and its Applications **284** (2000), 376-384.
18. D. Tj∅stheim, *Estimation in nonlinear time series models*, Stochastic Processes and their Applications **21** (1986), 251-273.
19. Zhang, Zhongqiang, and George Em Karniadakis, *Numerical methods for stochastic partial differential equations with white noise*, Springer International Publishing, Heidelberg, Berlin, Germany, 2017.
20. J.M. Zakoian, *Threshold heteroskedastic models*, Journal of Economic Dynamics and control **18** (1994), 931-955.

21. Y. Zhou, L. Shangerganesh, J. Manimaran, & A. Debbouche, *A class of time-fractional reaction-diffusion equation with nonlocal boundary condition*, Mathematical Methods in the Applied Sciences **41** (2018), 2987-2999.
22. Y. Zhou, J. Wang, & L. Zhang, *Basic theory of fractional differential equations*, World scientific, 2016.

Nabil Laiche is a faculty member at the Department of Mathematics at the University of Badji-Mokhtar, Annaba-Algeria. He received his Ph.D. degree in mathematics, specializing in applied statistics, from Badji-Mokhtar University, Annaba-Algeria. His research areas are applied statistics and probability.

Dynamic Systems and Control Laboratory, Department of Mathematics and Computer Sciences, Oum El Bouaghi University, Algeria.
`e-mail: nabil.laiche@univ-oeb.dz`

Laid Gasmi is a faculty member at the Department of Mathematics and Informatics, Adrar University. His research interests are computational mathematics and applied statistics.

Lab. Mathematics Modeling and Application (LaMMA), Department of Mathematics and Informatics, Adrar University, Algeria.
`e-mail: lai.gasmi@univ-adrar.edu.dz`

Raman Vinoth is an assistant professor at the Deanship of Quality and Academic Accreditation Department at IMAM ABDULRAHMAN BIN FAISAL UNIVERSITY-Kingdom of Saudi Arabia. He received his Ph.D. degree in statistics from Annamalai University, Annamalainagar, Chidambaram, India. Dr. Vinoth has published more than 60 research papers in international/national journals and conferences. His research areas are statistical analysis, biostatistics, and applied statistics.

Mam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam 31441, Saudi Arabia.
`e-mail: vrrangan@iau.edu.sa`

Halim Zeghdoudi is a faculty member at the Department of Mathematics at The University of Badji-Mokhtar, Annaba-Algeria. He received his Ph.D. degree in mathematics and the highest academic degree (HDR) specializing in probability and statistics from Badji-Mokhtar University, Annaba-Algeria. He also did his postdoctoral research at Waterford Institute of Technology-Cork Rd, Waterford, Ireland. Professor Zeghdoudi has published more than 100 research papers in international/national journals and conferences. His research is in actuarial science, particle systems, dynamics systems, and applied statistics. References to his research papers can be found in the European Mathematical Society, FIZ Karlsruhe, and the Heidelberg Academy of ID:885287. The science bibliography database is known as zbMath under ID:18552, Scopus Author ID:55387593000 and ID: 35114716100, Web-of-ScienceTH Researcher ID: W-1496-2017, and His ResearchGate Profile. Currently, he is a member of two editorial boards: Frontiers in Applied Mathematics and Statistics-Asian Journal of Probability and Statistics.

LaPS laboratory, Badji Mokhtar-Annaba University, P. O. Box 12, Annaba 23000, Algeria.
`e-mail: halim.zeghdoudi@univ-annaba.dz`