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ON DYNAMICS OF A SIXTH-ORDER MULTIPLE-ROOT FINDER FOR NONLINEAR EQUATIONS†

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Abstract. A family of sixth-order multiple-root solver have been developed and the special case of weight function is investigated. The dynamical analysis of selected iterative schemes with uniparametric polynomial weight function are studied using Möbius conjugacy map applied to the form $((z-A)(z-B))^m$ and the stability surfaces of the strange fixed points for the conjugacy map are displayed. The numerical results are shown through various parameter spaces.

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1. Introduction

The problem of finding roots occurs in diverse fields are modeled into nonlinear equations [1, 2]. Researchers are interested in developing efficient iteration schemes [3, 5, 7, 8, 9, 10] and investigating the dynamics [12, 13, 16, 17, 18] of higher order method to find the zeros of nonlinear equations. A root α of $f(x) = 0$ is called a multiple zero with multiplicity m if $f^{(i)}(\alpha) = 0,\ i = 0, 1, 2, \cdots, m-1$ and $g^{(m)}(\alpha) \neq 0$ [15].

We study the dynamics of a class of sixth-order multiple-zero solvers developed by Geum-Kim-Neta[11] below:

$$\begin{cases} y_n = x_n - m \cdot h(x_n),\ h(x_n) = \frac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot A_f(s) \cdot h(x_n),\ s = (\frac{f(y_n)}{f(x_n)})^{\frac{1}{m}}, \\ x_{n+1} = x_n - m \cdot B_f(s,v) \cdot h(x_n),\ v = (\frac{f(w_n)}{f(x_n)})^{\frac{1}{m}}, \end{cases} \tag{1}$$

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where $A_f : C \to C$ is a analytic function in a small neighborhood of O and $B_f : C^2 \to C$ is holomorphic in a small neighborhood of $(0,0)$.

The biparametric family of (1) is selected by

$$x_{n+1} = I_f(x_n, \lambda, \eta), n = 0, 1, 2, \cdots,$$

with the iteration function [4, 13, 14, 19]

$$I_f(x_n, \lambda, \eta) = x_n - m \cdot B_f(s, v) \cdot h(x_n),$$

and

$$A_f(s) = 1 + s + 2s^2,$$
$$B_f(s, v) = 1 + s + 2s^2 + \lambda s^5 + (1 + 2s + \eta s^2)v, \ for\ \lambda,\ \eta \in C.$$

In this work, we consider one complex parameter η choosing $\lambda = 0$.

Definition 1.1. Let $l_1 : X \to X$ and $l_2 : Y \to Y$ be the dynamical systems. If there is a function $k_1 : X \to Y$ such that $k_1 \circ l_1 = l_2 \circ k_1$, l_1 and l_2 are conjugate. Then the map k_1 is called a conjugacy [6].

The primary aim of this paper is to study the complex dynamical analysis on the Riemann sphere by investigating the parameter spaces related with the free critical points for the uniparametric family of sixth-order multiple-root finders. Such research from a viewpoint of complex dynamics could restrict us from treating the real dynamics for real nonlinear equations. However, the main motivation for investigating the relevant complex dynamics lies in finding the dynamical behavior of the iterative method via Möbius conjugacy map by presenting η-parameter spaces.

The outline of the next sections is : the conjugacy maps of selected method and the stability surfaces are described in Section 2. The complex dynamics with parameter spaces and conclusion are shown in Section 3.

2. Dynamical analysis

Using conjugacy map $M(z) = \frac{z-A}{z-B}$ considered by Blanchard [6], when applied to $f(z) = ((z - A)(z - B))^m$, the iterative method I_f is conjugate to $J(z, \eta)$ as follows

$$J(z, \eta) = -\frac{z^6(5 + 4z + z^2)\Psi(z)}{(1 + 4z + 5z^2)p(z)}, \qquad (2)$$

where $\Psi(z) = (6 + 218z^3 + 205z^4 + 120z^5 + 45z^6 + 10z^7 + z^8 - 5z^2(-27 + \eta) - 4z(-11 + \eta) - \eta)$ and $p(z) = (-1 - 10z - 45z^2 - 120z^3 - 205z^4 - 218z^5 + 5z^6(-27 + \eta) + 4z^7(-11 + \eta) + z^8(-6 + \eta))$.

We find that $J(z, \eta)$ is dependent on a parameter η, independently of m, A and B. To figure out the dynamics behind iterative map, we describe the fixed points of $J(z,\eta)$ and their stability [12]. Since $M(z)$ is a fixed point of $J(z,\eta)$ for a fixed point z of I_f with $M^{-1}(z) = \frac{zB-A}{z-1}$, we have

$$\phi(z,\eta) = z - J(z,\eta) = \frac{z(z-1)T(z)}{t(z)}, \tag{3}$$

with

$T(z) = (1 + 15z + 105z^2 + 455z^3 + 1365z^4 + 1365z^4 + 1365z^{10} + 455z^{11} + 105z^{12} + 15z^{13} + z^{14} + +z^5(2973+5\eta) + z^9(2973+5\eta) + 3z^6(1587+8\eta_3 z^8(1587+8\eta) + z^7(5578+42\eta))$,

$t(z) = (1+4z+5+z^2)(1+10z+45z^2+120z^3+205z^4+218z^5-5z^6(-27+\eta)-4z^7(-11+\eta)-z^8(-6+\eta))$.

Theorem 2.1. *(1) If $\eta = \frac{392}{5}$, then we get a factor $(z-1)$ for $t(z)$ and*

$$\phi\,(z,\ \eta) = \frac{z\ T_1(z)}{(1+4z+5z^2)\ t_1(z)},$$

where $T_1(z) = 5z^{14} + 5 + 75z + 525z^2 + 2275z^3 + 16825z^5 + 6825z^4 + 33213z^6 + 44354z^7 + 33213z^8 + 6825z^{10} + 16825z^9 + 2275z^{11} + 525z^{12} + 75z^{13}$ and $t_1(z) = 5 + 55z + 280z^2 + 880z^3 + 1905z^4 + 2995z^5 + 1710z^6 + 362z^7$.

(2) If $\eta = -\frac{6232}{25}$, then we have

$$\phi(z,\ \eta) = -\frac{(-1+z)^3 z\ T_2(z)}{(1+4z+5z^2)\ t_2(z)},$$

where $T_2(z) = 6382z^8 25 + 425z + 3450z^2 + 17850z^3 + 66375z^4 + 158065z^5 + 219212z^6 + 158065z^7 + 66375z^8 + 3450^{10} + 17850z^9 + 425z^{11} + 25z^{12}$ and $t_2(z) = 25 + 250z + 1125z^2 + 3000z^3 + 5125z^4 + 5450z^5 + 3435z^6 + 26028z^7$.

(3) If $z \neq 0$ is a η- dependent fixed point of J for $\eta \notin \{\frac{392}{5}, -\frac{6232}{25}\}$, then $\frac{1}{z}$ is a fixed point. For a given η, the strange fixed points are found from the fourteen numerical roots of $T(z) = 0$.

Proof. (1) We find $T(1) = 20(-392+5\eta) = 0$ for $\eta = \frac{392}{5}$ to have a factor $(z-1)$

(2) We have $T(1) = 4(6232 + 25\eta) = 0$, $T'(1) = 28(6232 + 25\eta) = 0$ for $\eta = -\frac{6232}{25}$ to have a factor $(z-1)^2$.

(3) Since $T(\frac{1}{z}) = T(z)/z^{14}$ for $z \in C - \{0\}$, $\frac{1}{z}$ is a fixed point of $J(z,\eta)$. We get the strange fixed points z with $T(z) = 0$ for a values of $\eta \notin \{\frac{392}{5}, -\frac{6232}{25}\}$. If we find out seven roots $v_i, i \in \{1,2,\cdots 7\}$ of $T(z) = 0$, we factor $T(z) = \Pi^7{}_{i=1}(z-v_i)(z-\frac{1}{v_i})$. □

Let $\eta \notin \{\frac{392}{5}, -\frac{6232}{25}\}$ and $v_1, v_2, v_3, v_4, v_5, v_6, v_7 \in C - \{0\}$ be seven roots of $T(z)$. Then $T(z)$ can be factored with seven second-degree polynomials oven the real field in case of

$$T(z) = \prod_{i=1}^{7}(1 + s_i z + z^2) = \prod_{i=1}^{7}(z - v_i)(z - \frac{1}{v_i})$$

where $s_i = -(v_i + \frac{1}{v_i})$, for $i \in \{1, 2, \cdots 7\}$ in terms of η.

Computing $J'(z, \eta)$, the derivative of J, we have

$$J'(z,\eta) = -\frac{2z^5(1+z)^{14}Q(z)}{q(z)^2}, \tag{4}$$

where

$$\begin{aligned} Q(z) =& 15(-6+\eta) + 15z^6(-6+\eta) + 5z^2(-374+5\eta) + 5z^4(-374+5\eta) \\ &+ 7z(-92+\eta) + 7z^5(-92+7\eta) - 2z^3(1316+39\eta), \\ q(z) =& (1+4z+5+z^2)(1+10z+45z^2+120z^3+205z^4+218z^5 \\ &- 5z^6(-27+\eta) - 4z^7(-11+\eta) - z^8(-6+\eta)). \end{aligned}$$

Lemma 2.2. $Q(\frac{1}{z}) = \frac{Q(z)}{z^6}$ *holds for* $z \in C - \{0\}$.

Theorem 2.3. *(1) If* $\eta = \frac{392}{5}$*, we have*

$$J'(z,\eta) = -\frac{20z^5(1+z)^{14}\ Q_1(z)}{(1+4z+5z^2)^2 q_1(z)^2},$$

where $Q_1(z) = 2715 + 13424z + 24358z^2 + 13424z^3 + 2715z^4$ *and* $q_1(z) = 5 + 55z + 280z^2 + 880z^3 + 1905z^4 + 2995z^5 + 1710z^6 + 362z^7$.

(2) If $\eta = -\frac{6232}{25}$*, then*

$$J'(z,\eta) = \frac{100z^5(1+z)^{14}\ Q_2(z)}{(1+4z+5z^2)^2\ q_2(z)^2},$$

where $Q_2(z) = 47865 + 160734z + 101275z^2 - 210148z^3 + 101275z^4 + 160734z^5 + 47865z^6)$ *and* $q_2(z) = 25 + 250z + 1125z^2 + 3000z^3 + 5125z^4 + 5450z^5 + 34535z^6 + 26028z^7 + 6382z^8$.

(3) If $\eta = 6$*, we have*

$$J'(z,\eta) = \frac{20z^5(1+z)^{14}\ Q_3(z)}{(1+4z+5z^2)^2\ q_3(z)^2},$$

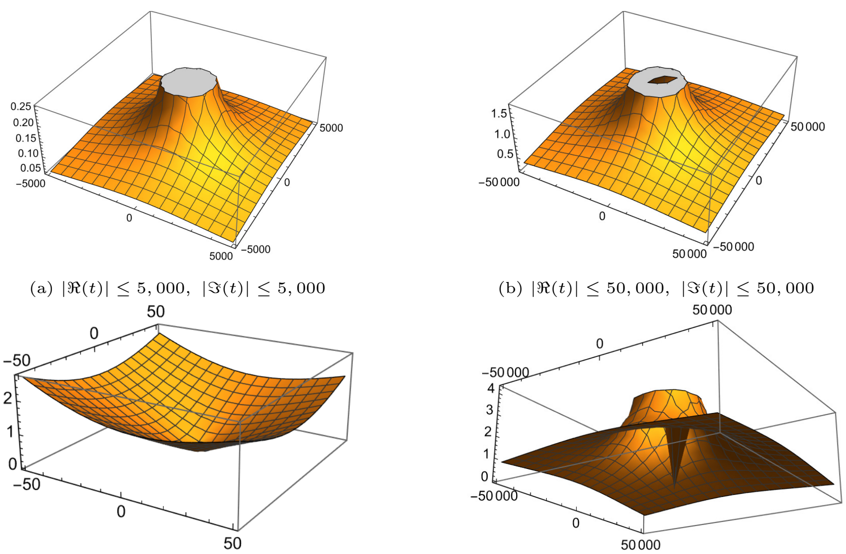
where $Q_3(z) = 35 + 172z + 310z^2 + 172z^3 + 35z^4$ *and* $q_3(z) = 1 + 10z + 45z^2 + 120z^3 + 205z^4 + 218z^5 + 105z^6 + 20z^7$.

Proof. (1) Assume that for z , $q(z) = 0$ and $Q(z) = 0$. Eliminating η from $q(z) = 0$ and $Q(z) = 0$, we have the relations $(-1+z)(1+z)(1+4z+5z^2)(15 + 124z + 450z^2 + 935z^3 + 1185z^4 + 890z^5 + 392z^6 + 95z^7 + 10z^8)$. Then $(z-1)$ is considered to be a common division of $Q(z)$ and $q(z)$. After computing $q(1) = 7840 - 100\eta = 0$ and $Q(1) = 20(-392 + 5\eta) = 0$, we have $\eta = \frac{392}{5}$ for a common factor $(z-1)$.

(2) The fixed point for $\eta = -\frac{6232}{25}$ is considered in Theorem 1.

(3) We have $Q(z)$ as a factor of z for $Q(0) = 15(-6+\eta) = 0$ fwith $\eta = 6$, . ∎

In Figure 1, the typical stability of fixed points are shown by illustrative conical surfaces . The critical point of the iteration map are given by $J'(z, \eta) = 0$, and we know that the points $z = 0$ and $z = \infty$ in domain of f are critical points associated with the zeros A and B of the form $((z - A)(z - B))^m$. The critical values that are not the zeros of the polynomial $((z-A)(z-B))^m$ are free critical points. We find that one critical point $z = 1$ is a free critical point.



(a) $|\Re(t)| \leq 5,000, \;\; |\Im(t)| \leq 5,000$

(b) $|\Re(t)| \leq 50,000, \;\; |\Im(t)| \leq 50,000$

(c) $|\Re(t)| \leq -50, \;\; |\Im(t)| \leq 50$

(d) $|\Re(t)| \leq 50,000, \;\; |\Im(t)| \leq 50,000$

FIGURE 1. Stability surfaces.

3. Conclusion

Let $\mathcal{P} = \{\eta \in C : \text{ a critical orbit of } z \text{ converges to a number } w_p \in \overline{\mathbb{C}}\}$. It is called the parameter space. There are finite periods in the orbit if the number w_p is a finite constant. Otherwise, the orbit is not periodic however it is bounded or goes to infinity.

We utilize a systematic method for coloring a point $t \in \mathcal{P}$ according to the orbital period of z under $J(z, \eta)$. Then the point t is painted in corresponding color C_k if t induces a k-periodic orbit with $k \in N \cup \{0\}$ under $J(z, \eta)$. We use a tolerance of 10^{-6} after up to 1000-2000 iterations [20] to allow for desired k periodic convergence of an orbit related to $\mathcal{P}$. We use the color C_q according to the color palette in Table 1.

Lemma 3.1. *Let* $g : C^2 \to C,\; g(\eta, z) = p_0(z) + p_1(z) \cdot \eta + p_2(z) \cdot \eta^2$ *be complex polynomials with real coefficients with* $p_i(z),\; i \in \{1, 2\}$. $g(\eta, z) = 0$. *Let* $\bar{z}$ *be a complex conjugate of* z. *Then the following holds.*

(1) $\eta(\bar{z}) = \overline{\eta(z)}$.
(2) If $z(\eta)$ *is a root of* g*, then so is* $\bar{z}(\bar{\eta})$.

Proof. (1) Solving $g(\eta, z) = 0$ for η, we have

$$\eta(z) = \frac{-p_1(z) \pm \sqrt{p_1(z)^2 - 4p_1(z)p_2(z)}}{2p_1(z)}$$

Since $p_i(z)$ are complex polynomial with real coefficients, we get $p_i(\bar{z}) = \overline{p_i(z)}$.

$$\eta(\bar{z}) = \frac{-p_1(\bar{z}) \pm \sqrt{p_1(\bar{z})^2 - 4p_1(\bar{z})p_2(\bar{z})}}{2p_1(\bar{z})} = \frac{-\overline{p_1(z)} \pm \sqrt{\overline{p_1(z)}^2 - 4\overline{p_1(z)p_2(z)}}}{2\overline{p_1(z)}}.$$

Then $\eta(\bar{z}) = \overline{\eta(z)}$.

(2) Let $z(\eta)$ be a zero of $g(\eta, z)$. Then $g(\eta, z) = \overline{g(\eta, z)} = \overline{p_0(z)} + \overline{p_1(z)} \cdot \overline{\eta} + \overline{p_2(z)} \cdot \overline{\eta}^2 = p_0(\overline{z}) + p_1(\overline{z}) \cdot \overline{\eta} + p_2(\overline{z}) \cdot \overline{\eta}^2 = g(\overline{\eta}, \overline{z})$ □

Theorem 3.2. *Let* $z(\eta)$ *be a free critical point of iteration map* $J(z, \eta)$*. Then the parameter space is symmetrical about horizontal axis.*

Proof. If $z(\eta)$ is a zero of $Q(z)$, then $\bar{z}(\bar{\eta})$ is a zero of $Q(z)$. From conjagted map $J(z, \eta)$, we have

$$|J(z, \eta)| = |J(z(\eta), \eta)| = |\overline{J(z(\eta), \eta)}| = |J(\overline{z(\eta)}, \bar{\eta})| = |J(\bar{z}(\bar{\eta}), \bar{\eta})|,$$

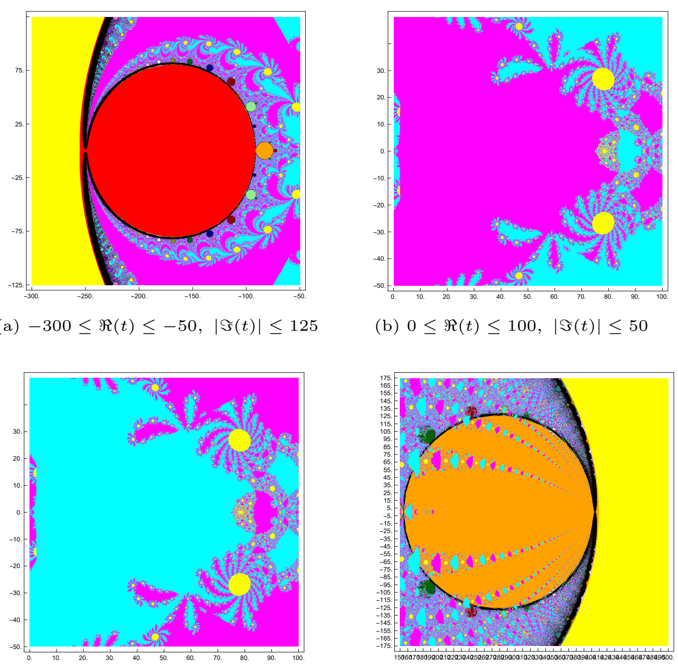
which implies that the parameter spaces related with map $J(z, \eta)$ is symmetric with respect to its horizontal axis. □

In Figures 2, we display the parameter spaces $\mathcal{P}$. A point $\epsilon \in \mathcal{P}$ is painted using the coloring scheme shown in Table 1. In terms of numerical phenomena, every point of the parameter space $\mathcal{P}$ whose color is none of magenta(root $z = B$), cyan(root $z = A$), red or yellow is not a better choice of t. We find the complicated but fancy pattern and for $n \in N - \{1\}$, n-periodic orbit is budding at period-1 component and 6-periodic component is budding at period-3 component.

TABLE 1. Color palette for a i-periodic orbit with $i \in \mathbb{N} \cup \{0\}$

i	C_i
$i = 1$	$C_1 = \begin{cases} \text{magenta}, \infty, \\ \text{cyan}, 0, \\ \text{yellow}, 1, \\ \text{red, for other strange fixed point}, \end{cases}$
$2 \le i \le 68$	C_2 = orange, C_3 = light green, C_4 = dark red, C_5 = dark blue, C_6 = dark green, C_7 = dark yellow, C_8 = floral white, C_9 = light pink, C_{10} = khaki, C_{11} = dark orange, C_{12} = turquoise, C_{13} = lavender, C_{14} = thistle, C_{15} = plum, C_{16} = orchid, C_{17} = medium orchid, C_{18} = blue violet, C_{19} = dark orchid, C_{20} = purple, C_{21} = power blue, C_{22} = sky blue, C_{23} = deep sky blue, C_{24} = dodger blue, C_{25} = royal blue, C_{26} = medium spring green, C_{27} = spring green, C_{28} = medium sea green, C_{29} = sea green, C_{30} = forest green, C_{31} = olive drab, C_{32} = bisque, C_{33} = moccasin, C_{34} = light salmon, C_{35} = salmon, C_{36} = light coral, C_{37} = Indian red, C_{38} = brown, C_{39} = fire brick, C_{40} = peach puff, C_{41} = wheat, C_{42} = sandy brown, C_{43} = tomato, C_{44} = orange red, C_{45} = chocolate, C_{46} = pink, C_{47} = pale violet red, C_{48} = deep pink, C_{49} = violet red, C_{50} = gainsboro, C_{51} = light gray, C_{52} = dark gray, C_{53} = gray, C_{54} = charteruse, C_{55} = electric indigo, C_{56} = electric lime, C_{57} = lime, C_{58} = silver, C_{59} = teal, C_{60} = pale turquoise, C_{61} = sandy brown, C_{62} = honeydew, C_{63} = misty rose, C_{64} = lemon chiffon, C_{65} = lavender blush, C_{66} = gold, C_{67} = crimson, C_{68} = tan.
$i = 0^*$ or $i > 69$	C_i = black.

$*$: $i = 0$: the orbit is non-periodic but bounded.



(a) $-300 \le \Re(t) \le -50,\ |\Im(t)| \le 125$ (b) $0 \le \Re(t) \le 100,\ |\Im(t)| \le 50$

(c) $0 \le \Re(t) \le 100,\ |\Im(t)| \le 50$ (d) $150 \le \Re(t) \le 450,\ |\Im(t)| \le 180$

FIGURE 2. Parameter spaces.

Möbius conjugacy maps along with the property of dynamical analysis for the uniparametric family of sixth-order multiple-root finders are studied and the complex dynamical analysis on the Riemann sphere is analyzed by investigating the parameter spaces related with the free critical points. The stability surfaces of the strange fixed points for the conjugacy map are displayed. As the future work, we deal with the visualization of different types of numerical methods by improving the current work. In addition, we investigate the convergent region and the basins of attraction of the developed multiple-root finder in detail.

Conflicts of interest : The author declare no conflict of interest.

Data availability : Not applicable

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