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LIMIT CYCLES FOR A CLASS OF FIFTH-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this research is to investigate sufficient conditions for the existence of limit cycles of the fifth-order differential equation

$$x^{(5)} + (p^2+q^2)\dddot{x} + p^2q^2\dot{x} = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}),$$

where p, q are rational numbers different from 0, $p \neq \pm q$, ε is a small real parameter, and F is a $2k\pi$-periodic function in the variable t. Also, we provide some applications.

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1. Introduction and statement of the main results

One of the main problems in differential equation theory is the study of their periodic orbits, their existence, their number, and their stability. A limit cycle of a differential equation is a periodic orbit isolated from the set of all periodic orbits of the differential equation.

In general, obtaining analytically periodic solutions to a differential system is difficult, usually impossible. Here, using the averaging theory, we reduce this difficult problem for the differential equation (4) to finding the zeros of a nonlinear system of fifth equations. It is known that, in general, the averaging theory for finding periodic solutions does not provide all the periodic solutions for the system. To explain this idea, there are two main reasons. The first way to look at the periodic solutions of a differential system is through the so-called displacement function. The zeros of this function give us periodic solutions. This displacement function, in general, is not global. Consequently, it cannot control all the periodic solutions, only those in its domain of definition are hyperbolic.

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The second part is that the displacement function is expanded in the power series of a small parameter ε. The averaging theory only controls the zeros of the displacement function's main term. When the dominant term is ε^k, we discuss the averaging theory of order k. For more details, see [3, 7, 11].

The averaging theory is a classical tool for studying the dynamics of nonlinear differential systems with periodic forcing, see [4, 8, 9, 12, 16, 17]. For a more modern exposition of the averaging theory, see section 2 of this paper and the book of Verhulst [19], Sanders and Verhulst [15].

In [5], the authors provide sufficient conditions for the existence of limit cycles of the fourth-order differential equation

$$\ddddot{u} + q\ddot{u} + pu = \varepsilon F(t, u, \dot{u}, \ddot{u}, \dddot{u}), \tag{1}$$

where $p,\ q$, ε are real parameters, and F is a nonlinear non-autonomous periodic function with respect to the variable t.

In [2], the authors studied the limit cycles of the following fourth-order differential equation

$$\ddddot{x} + (1 + p^2)\ddot{x} + p^2 x = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{2}$$

where p is a rational different from $-1, 0, 1$, ε is a small real parameter, and F is a non-autonomous periodic function in the variable t.

In [6], the authors studied the limit cycles of equation (2) in the case when $F = F(x, \dot{x}, \ddot{x}, \dddot{x})$, which is an autonomous function.

In [13], the authors studied the limit cycles of the following class of fifth-order differential equation

$$\overset{\cdot\cdot\cdot\cdot\cdot}{x} + \alpha\ddddot{x} + (\beta + \mu)\dddot{x} + \alpha(\beta + \mu)\ddot{x} + \beta\mu\dot{x} + \alpha\beta\mu x = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{3}$$

where $\alpha,\ \beta,\ \mu$ are rational numbers different from 0, such that $\alpha \neq \pm\beta,\ \alpha \neq \pm\mu$, and $\beta \neq \pm\mu$ with ε sufficiently small, and F is a non-autonomous periodic function in the variable t.

In this article, we investigate the existence of limit cycles of the equation (3) in the case when α is equal to 0, $\beta = p^2$, and $\mu = q^2$. The new differential equation becomes

$$x^{(5)} + (p^2 + q^2)\dddot{x} + q^2 p^2 \dot{x} = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{4}$$

where $p,\ q$ are rational numbers different from 0, and $p \neq \pm q$, ε is a small real parameter, and F is a $2k\pi$–periodic function with respect to the first variable t. This kind of differential equations appears frequently in many problems from physics, chemistry, economics, engineering,...

There are various applications for fifth-order differential systems, such as control theory and certain three-loop electric circuit problems (refer to Rosenvasser [14]). Additionally, numerous papers have been published on such systems and equations, including [10, 18, 20, 21].

Our main result concerning the limit cycles of differential equation (4) is presented in the following Theorem.

Theorem 1.1. *Assume that* p, q *are rational numbers different from* 0, *and* $p \neq \pm q$, *in the differential equation* (4). *We define*

$$\begin{aligned}
\mathcal{F}_1(X_0,Y_0,Z_0,U_0,V_0) &= \tfrac{1}{2k\pi}\textstyle\int_0^{2k\pi}\cos(pt)F(t,A(t),B(t),C(t),D(t),E(t))\,dt,\\
\mathcal{F}_2(X_0,Y_0,Z_0,U_0,V_0) &= -\tfrac{1}{2k\pi}\textstyle\int_0^{2k\pi}\sin(pt)F(t,A(t),B(t),C(t),D(t),E(t))\,dt,\\
\mathcal{F}_3(X_0,Y_0,Z_0,U_0,V_0) &= \tfrac{1}{2k\pi}\textstyle\int_0^{2k\pi}\cos(qt)F(t,A(t),B(t),C(t),D(t),E(t))\,dt,\\
\mathcal{F}_4(X_0,Y_0,Z_0,U_0,V_0) &= -\tfrac{1}{2k\pi}\textstyle\int_0^{2k\pi}\sin(qt)F(t,A(t),B(t),C(t),D(t),E(t))\,dt,\\
\mathcal{F}_5(X_0,Y_0,Z_0,U_0,V_0) &= \tfrac{1}{2k\pi}\textstyle\int_0^{2k\pi}F(t,A(t),B(t),C(t),D(t),E(t))\,dt,
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
A(t) &= \frac{X_0\cos(pt)-Y_0\sin(pt)}{p^2(p^2-q^2)}+\frac{U_0\sin(qt)-Z_0\cos(qt)}{q^2(p^2-q^2)}+\frac{V_0}{p^2q^2},\\
B(t) &= -\frac{Y_0\cos(pt)+X_0\sin(pt)}{p(p^2-q^2)}+\frac{Z_0\sin(qt)+U_0\cos(qt)}{q(p^2-q^2)},\\
C(t) &= -\frac{X_0\cos(pt)-Y_0\sin(pt)}{p^2-q^2}+\frac{U_0\sin(qt)-Z_0\cos(qt)}{p^2-q^2},\\
D(t) &= -\frac{Y_0\cos(pt)+X_0\sin(pt)}{p^2-q^2}-\frac{U_0\cos(qt)+Z_0\sin(qt)}{p^2-q^2},\\
E(t) &= \frac{p^2(X_0\cos(pt)-Y_0\sin(pt))}{p^2-q^2}+\frac{q^2(U_0\sin(qt)+Z_0\cos(qt))}{p^2-q^2},
\end{aligned} \tag{6}$$

and $p=\frac{p_1}{p_2}$, $q=\frac{q_1}{q_2}$, *where* p_1, p_2, q_1, q_2 *are integers numbers different form* 0, *and* k *be the least common multiple of* p_2 *and* q_2.

If F *is* $2k\pi$*-periodic, then for every simple zero* $(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)$ *of the system* $\mathcal{F}_k=0$, *with* $k=\overline{1,5}$, *satisfying*

$$\det\Big(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4,\mathcal{F}_5)}{\partial(X_0,Y_0,Z_0,U_0,V_0)}_{|(X_0,Y_0,Z_0,U_0,V_0)=(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)}\Big)\neq 0, \tag{7}$$

there exists a periodic solution $x(t,\varepsilon)$ *of equation* (4) *tending to the periodic solution given by*

$$x(t)=\frac{X_0^*\cos(pt)-Y_0^*\sin(pt)}{p^2(p^2-q^2)}+\frac{U_0^*\sin(qt)-Z_0^*\cos(qt)}{q^2(p^2-q^2)}+\frac{V_0^*}{p^2q^2}, \tag{8}$$

of the equation

$$x^{(5)}+(p^2+q^2)\dddot{x}+p^2q^2\dot{x}=0,$$

when $\varepsilon\to 0$. *Note that this solution is periodic of period* $2k\pi$.

Theorem 1.1 is proved in section 3. Its proof is based on the averaging theory of the first order. See section 2. The following corollaries provide an example of Theorem 1.1.

Corollary 1.2. *Suppose that* $f(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x})=(ax^2+bx+c)(\cos(t)+3)$, *then the differential equation* (4) *with* $p=\frac{3}{4}$, $q=1$, $a\neq 0$, *and* $b^2\geqslant 4ac$, *has* *four periodic solutions* $x_k(t,\varepsilon)$, *with* $k=\overline{1,4}$ *which tend to the periodic solutions* $x_k(t)$ *given by* (8) *for* $(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)$ *as equal to*

$$\begin{aligned}&(0,0,0,0,\frac{-9b+\sqrt{A_1}}{32a}),\\&(0,0,0,0,\frac{-9b-\sqrt{A_1}}{32a}),\\&(0,0,\frac{\sqrt{A_2}}{8a},0,\frac{-1071b-27\sqrt{A_2}}{3808a}),\\&(0,0,-\frac{\sqrt{A_2}}{8a},0,\frac{-1071b+27\sqrt{A_2}}{3808a}),\end{aligned}$$

with $A_1=-4ac+b^2,\ A_2=\dfrac{-56644ac+14161b^2}{2253}$, *of the equation*

$$x^{(5)}+\frac{25}{16}\dddot{x}+\frac{9}{16}\dot{x}=0, \tag{9}$$

when $\varepsilon\to 0$. *Note that these solutions are periodic of period* 8π.

Corollary 1.3. *Suppose that* $f(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x})=x-1+\dot{x}\sin(t)$, *then the differential system* (4) *with* $p=\frac{1}{3}$, $q=2$, *has two periodic solutions* $x_k(t,\varepsilon)$, *with* $k=1,2$ *which tend to the periodic solutions* $x_k(t)$ *given by* (8) *for* $(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)$ *as equal to*

$$\begin{aligned}&(0,0,0,0,\frac{4}{9}),\\&(0,0,0,0,-\frac{4}{9}),\end{aligned}$$

of the equation

$$x^{(5)}+\frac{37}{9}\dddot{x}+\frac{4}{9}\dot{x}=0, \tag{10}$$

when $\varepsilon\to 0$. *Note that these solutions are periodic of period* 6π.

2. Basic results on averaging theory

In this section, we present the basic result of the averaging theory that we will use to demonstrate the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\mathbf{x}'=F_0(t,\mathbf{x})+\varepsilon F_1(t,\mathbf{x})+\varepsilon^2F_2(t,\mathbf{x},\varepsilon), \tag{11}$$

with $\varepsilon=0$ to $\varepsilon\neq 0$ sufficiently small. Here the functions $F_0,F_1:\mathbb{R}\times\Omega\mapsto\mathbb{R}^n$ and $F_2:\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\mapsto\mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main assumption is that the unperturbed system

$$\mathbf{x}'=F_0(t,\mathbf{x}), \tag{12}$$

has a submanifold of periodic solutions.

We express by $\mathbf{x}(t,\mathbf{z})$ the solution of system (12) such that $\mathbf{x}(0,\mathbf{z})=\mathbf{z}$. The linearized system of the unperturbed system (12) along a periodic solution $\mathbf{x}(t,\mathbf{z})$ is

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z}))\mathbf{y}, \tag{13}$$

Let's $M_{\mathbf{z}}(t)$ the fundamental matrix of the linear differential system (13) and by $\xi:\mathbb{R}^k\times\mathbb{R}^{n-k}\to\mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates, i.e, $\xi(x_1,\dots,x_n)=(x_1,\dots,x_k)$.

Consider V as an open and bounded set with $Cl(V)\subset\Omega$, such that for each $\mathbf{z}\in Cl(V)$. In this context $\mathbf{x}(t,\mathbf{z})$ represents the periodic solution of the unperturbed system (12) with $\mathbf{x}(0,\mathbf{z})$. The set $Cl(V)$ is isochronous for the system (11); i.e., it is a set formed only by periodic orbits, all of them having the same period. The following result provides an answer to the problem of the bifurcation of T–periodic solutions from the periodic solutions $\mathbf{x}(t,\mathbf{z})$ that are contained in $Cl(V)$.

Theorem 2.1. *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V)\subset\Omega$ such that for each $\mathbf{z}\in\mathrm{Cl}(V)$, the solution $\mathbf{x}(t,\mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F}:\mathrm{Cl}(V)\to\mathbb{R}^n$ as*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{2\pi}\int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z})F_1(t,\mathbf{x}(t,\mathbf{z}))dt. \tag{14}$$

If there exists $a\in V$ with $\mathcal{F}(a)=0$, and $\det\left((d\mathcal{F}/d\mathbf{z})\,(a)\right)\neq 0$, then there exists a T–periodic solution $\mathbf{x}(t,\varepsilon)$ of system (11) such that $\mathbf{x}(0,\varepsilon)\to a$ as $\varepsilon\to 0$.

For a proof of Theorem 2.1, see Corollary 1 of [1].

3. PROOF OF THE RESULTS

Proof of Theorem 1.1. Introducing the variables $y=\dot{x},\ z=\ddot{x},\ u=\dddot{x},\ v=\ddddot{x}$, The fifth-order differential equation (4) can be written as the following differential system.

$$\begin{cases} \dot{x}=y,\\ \dot{y}=z,\\ \dot{z}=u,\\ \dot{u}=v,\\ \dot{v}=-(p^2+q^2)u-p^2q^2y+\varepsilon F(t,x,y,z,u,v). \end{cases} \tag{15}$$

As we can see, the unperturbed system of (15) has a single singular point at the origin with eigenvalues $\pm ip$, $\pm iq$, and 0 for $\varepsilon=0$. We'll write system (15) such that the linear part at the origin is in the real Jordan form. By using the change of variables

$$\begin{pmatrix} X\\ Y\\ Z\\ U\\ V \end{pmatrix} = \begin{pmatrix} 0 & 0 & q^2 & 0 & 1\\ 0 & pq^2 & 0 & p & 0\\ 0 & 0 & p^2 & 0 & 1\\ 0 & p^2q & 0 & q & 0\\ p^2q^2 & 0 & p^2+q^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y\\ z\\ u\\ v \end{pmatrix}, \tag{16}$$

we obtain the following system

$$\begin{cases} \dot{X} = -pY + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{Y} = pX, \\ \dot{Z} = -qU + \varepsilon G(t, X, Y, Z, U, V), \\ \dot{U} = qZ, \\ \dot{V} = \varepsilon G(t, X, Y, Z, U, V), \end{cases} \tag{17}$$

where $G(t, X, Y, Z, U, V) = F(t, A(t), D(t), C(t), D(t), E(t))$, and $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$ given in (6).

Note that the linear part of the differential system (17) at the origin is in its real normal jordan form and the change of variables (16) is defined when $p \neq \pm q$, because the determinant of the matrix of the change is $p^3 q^3 (p^2 - q^2)^2$. Now we'll apply Theorem 2.1 to the differential system (17), taking

$$\dot{\mathbf{x}} = \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix}, \ F_0(t, \mathbf{x}) = \begin{pmatrix} -pY \\ pX \\ -qU \\ qZ \\ 0 \end{pmatrix}, \ F_1(t, \mathbf{x}) = \begin{pmatrix} G(t, X, Y, Z, U, V) \\ 0 \\ G(t, X, Y, Z, U, V) \\ 0 \\ G(t, X, Y, Z, U, V) \end{pmatrix}.$$

We can see that the system (17) has an isochronous linear center at the origin with $\varepsilon = 0$. Using the notation of Theorem 2.1, the periodic solution $\mathbf{x}(t, \mathbf{z})$ of this center with $\mathbf{z} = (X_0, Y_0, Z_0, U_0, V_0)$ is

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} X_0 \cos(pt) - Y_0 \sin(pt) \\ Y_0 \cos(pt) + X_0 \sin(pt) \\ Z_0 \cos(qt) - U_0 \sin(qt) \\ U_0 \cos(qt) + Z_0 \cos(qt) \\ V_0 \end{pmatrix}. \tag{18}$$

This set of periodic orbits has the fifth dimension, and all have the same period $2k\pi$. We must compute the zeros $\alpha = (X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*)$ of the system $\mathcal{F}(\alpha) = 0$, where $\mathcal{F}(\alpha)$ is given by (14) as well as the fundamental matrix $M(t)$ of the differential system (17) with $\varepsilon = 0$, along any periodic solution is

$$M(t) = \begin{pmatrix} \cos(pt) & -\sin(pt) & 0 & 0 & 0 \\ \sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & \cos(qt) & -\sin(qt) & 0 \\ 0 & 0 & \sin(qt) & \cos(qt) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inverse matrix of $M(t)$ is

$$M^{-1}(t) = \begin{pmatrix} \cos(pt) & \sin(pt) & 0 & 0 & 0 \\ -\sin(pt) & \cos(pt) & 0 & 0 & 0 \\ 0 & 0 & \cos(qt) & \sin(qt) & 0 \\ 0 & 0 & -\sin(qt) & \cos(qt) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now computing the function $\mathcal{F}(\alpha)$ given in (14), we got that the system $\mathcal{F}(\alpha) = 0$, can be written as

$$\begin{cases} \mathcal{F}_1(X_0, Y_0, Z_0, U_0, V_0) = 0, \\ \mathcal{F}_2(X_0, Y_0, Z_0, U_0, V_0) = 0, \\ \mathcal{F}_3(X_0, Y_0, Z_0, U_0, V_0) = 0, \\ \mathcal{F}_4(X_0, Y_0, Z_0, U_0, V_0) = 0, \\ \mathcal{F}_5(X_0, Y_0, Z_0, U_0, V_0) = 0, \end{cases} \tag{19}$$

where $\mathcal{F}_k(X_0, Y_0, Z_0, U_0, V_0)$ for $k = \overline{1,5}$, are given in (5).

Then, the zeros $(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*)$ of the system (19) with respect to the variables X_0, Y_0, Z_0, U_0, and V_0 provide periodic orbits of the system (17) with $\varepsilon \neq 0$ sufficiently small if they are simple i.e if

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5)}{\partial(X_0, Y_0, Z_0, U_0, V_0)}_{|(X_0,Y_0,Z_0,U_0,V_0)=(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)}\right) \neq 0.$$

Going back through the change of variables, for every simple zero $(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*)$ of the system (19), we obtain $2k\pi$–periodic solutions $x(t, \varepsilon)$ of differential equation (4), for $\varepsilon \neq 0$ sufficiently small such that $x(t, \varepsilon)$ tends to the periodic solution

$$x(t) = \frac{X_0^* \cos(pt) - Y_0^* \sin(pt)}{p^2(p^2 - q^2)} + \frac{U_0^* \sin(qt) - Z_0^* \cos(qt)}{q^2(p^2 - q^2)} + \frac{V_0^*}{p^2 q^2},$$

of $x^{(5)} + (p^2 + q^2)\dddot{x} + p^2 q^2 \dot{x} = 0$ when $\varepsilon \to 0$. Note that this solution is periodic of period $2k\pi$. This completes the proof of the Theorem 1.1. $\square$

Proof of corollary 1.2. We have the equation

$$x^{(5)} + \frac{25}{16}\dddot{x} + \frac{9}{16}\dot{x} = \varepsilon((ax^2 + bx + c)(\cos(t) + 3)), \tag{20}$$

which corresponds to the case $p = \dfrac{3}{4}$, $q = 1$, and $F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = (ax^2 + bx + c)(\cos(t) + 3)$.

The functions $\mathcal{F}_k(X_0,Y_0,Z_0,U_0,V_0)$ for $k=\overline{1,5}$ of Theorem 1.1 are

$$\begin{aligned}
\mathcal{F}_1(X_0,Y_0,Z_0,U_0,V_0) &= -\frac{128}{1323}X_0(224V_0+48Z_0)a-\frac{128}{21}X_0b,\\
\mathcal{F}_2(X_0,Y_0,Z_0,U_0,V_0) &= -\frac{128}{1323}Y_0(224V_0+48Z_0)a-\frac{128}{21}Y_0b,\\
\mathcal{F}_3(X_0,Y_0,Z_0,U_0,V_0) &= (\frac{16384}{3969}X_0^2+\frac{16384}{3969}Y_0^2+\frac{96}{49}Z_0^2+\frac{32}{49}U_0^2+\frac{128}{81}V_0^2\\
&\quad+\frac{256}{21}Z_0V_0)a+(\frac{24}{7}Z_0+\frac{8}{9}V_0)b+\frac{1}{2}c,\\
\mathcal{F}_4(X_0,Y_0,Z_0,U_0,V_0) &= \frac{8}{147}U_0(224V_0+48Z_0)a+\frac{24}{7}U_0b,\\
\mathcal{F}_5(X_0,Y_0,Z_0,U_0,V_0) &= (\frac{32768}{1323}X_0^2+\frac{32768}{1323}Y_0^2+\frac{384}{49}Z_0^2+\frac{384}{49}U_0^2+\frac{256}{27}V_0^2\\
&\quad+\frac{256}{63}Z_0V_0)a+(\frac{8}{7}Z_0+\frac{16}{3}V_0)b+3c.
\end{aligned}$$

The system $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=\mathcal{F}_4=\mathcal{F}_5=0$, has four real solutions given by

$$\begin{aligned}
&(0,0,0,0,\frac{-9b+9\sqrt{A_1}}{32a}), && (0,0,0,0,\frac{-9b-9\sqrt{A_1}}{32a}),\\
&(0,0,\frac{\sqrt{A_2}}{8a},0,\frac{-1071b-27\sqrt{A_2}}{3808a}), && (0,0,-\frac{\sqrt{A_2}}{8a},0,\frac{-1071b+27\sqrt{A_2}}{3808a}).
\end{aligned}$$

Since the Jacobians (7) for these solutions $(X_0^*,Y_0^*,Z_0^*,U_0^*,V_0^*)$ are

$$\pm\frac{142606336}{64827}(-4ac+b^2)^{5/2},\ \mp\frac{713031680000}{8884685756961}\sqrt{-9012ac+2253b^2}(4ac-b^2)^2.$$

By Theorem 1.1, equation (20) has four periodic solutions, tending to the periodic solutions of equation (9) given in the statement of Corollary 1.2. □

Proof of corollary 1.3. We have the equation

$$x^{(5)}+\frac{37}{9}x^{(3)}+\frac{4}{9}\dot{x}=\varepsilon(x^2-1+\ddddot{x}\sin(t)),\tag{21}$$

which corresponds to the case $p=\frac{1}{3}$, $q=2$, and $F(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x})=x^2-1+\ddddot{x}\sin(t)$. The functions $\mathcal{F}_k(X_0,Y_0,Z_0,U_0,V_0)$ for $k=\overline{1,5}$ of Theorem 1.1 are

$$\begin{aligned}
\mathcal{F}_1(X_0,Y_0,Z_0,U_0,V_0) &= -\frac{729}{140}X_0V_0,\\
\mathcal{F}_2(X_0,Y_0,Z_0,U_0,V_0) &= -\frac{729}{140}Y_0V_0,\\
\mathcal{F}_3(X_0,Y_0,Z_0,U_0,V_0) &= \frac{81}{560}Z_0V_0,\\
\mathcal{F}_4(X_0,Y_0,Z_0,U_0,V_0) &= \frac{81}{560}U_0V_0,\\
\mathcal{F}_5(X_0,Y_0,Z_0,U_0,V_0) &= \frac{6561}{2450}X_0^2+\frac{6561}{2450}Y_0^2+\frac{81}{39200}Z_0^2+\frac{81}{39200}U_0^2+\frac{81}{16}V_0^2-1.
\end{aligned}$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = 0$, has two real solutions given by $(0, 0, 0, 0, \frac{4}{9}),\ (0, 0, 0, 0, -\frac{4}{9})$.

Since the Jacobians (7) for these solutions $(X_0^*, Y_0^*, Z_0^*, U_0^*, V_0^*)$ are $\pm\frac{4782969}{48020000}$.

By Theorem 1.1, equation (21) has two periodic solutions, tending to the periodic solutions of equation (10) given in the statement of Corollary 1.3.

□

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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