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# A NOVEL DISCUSSION ON POWER FUZZY GRAPHS AND THEIR APPLICATION IN DECISION MAKING

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Abstract. In this paper, Power fuzzy graphs is newly introduced by allotting fuzzy values on power graphs in such a way that the newly added edges, has the edge membership values between a closed interval which depends on vertex membership values and the length of the added edges. Power fuzzy subgraphs and total power fuzzy graphs are newly defined with properties and some special cases. It is observed that every power fuzzy graph is a fuzzy graph but the converse need not be true. Edges that are incident to vertices with the least vertex membership values are retained in the least power fuzzy subgraph. Further, the application of these concepts in real life time has been presented and discussed using power fuzzy graph model.

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*Key words and phrases* : Power graph, power fuzzy graph, total power fuzzy graph.

## 1. Introduction

A graph is a simple way of representing information as objects (vertices) and relations (edges). The vagueness found in the objects and relations are represented as fuzzy graphs. The interesting development on the $k$th power of graphs in the field of graph theory was the motivation to newly define power fuzzy graphs. The objective of the paper is to introduce a mathematical model based on the concept of power fuzzy graphs. Unlike in a power graph, the membership values for elements of a power fuzzy graph are allotted between [0,1]. Therefore, the novelty of the power fuzzy graph model is to minimize the interval for the allotment of membership values to edges, as the powers of the corresponding fuzzy graph increases. This results in a better mathematical methodology to find optimal solution to certain problems in uncertain scenarios. The problem

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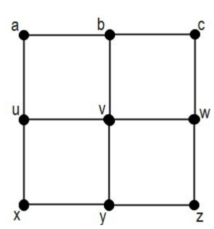
of taking decision regarding better places and routes in consideration of travel time, distance, safety, etc., can be presented and analysed using power fuzzy graph model.

## 2. Literature review

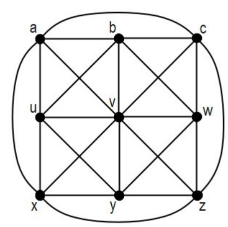
The concept of fuzzy sets and fuzzy relations were formulated by Lotfi Zadeh in 1965 [5]. The definition of fuzzy graphs based on Zadeh's concept was coined by Arnold Kaufmann in 1973 [2]. This was further developed by Azriel Rosenfeld in 1975 [6], and graphical terms such as path, connectedness, bridge, tree, cut node, and block were introduced by him. The degree, size, and order in fuzzy graphs were defined and studied by Nagoor Gani et al. [7]. Many other extensions of fuzzy graphs are found in [8,9,10,11]. Interesting findings on kth power of graphs such as the connectivity results of Hamiltonian power graphs [12], asteroidal number of power graphs [13], chromatic number of planar power graphs [15], co-comparability of power graphs [16], Weiner index of power graphs [17], reduced power graphs [18], spanning trees of power graphs [19], connected complement of power graphs [20] and power of soft graphs [21] have also contributed to this new concept of power fuzzy graphs and its related terms. Several techniques and models were developed for decision-making problems such as spherical fuzzy graph [23], picture fuzzy graph [24], q-rung picture fuzzy graph [25], picture fuzzy soft graph [26], and m-polar fuzzy graph [27]. The basic definitions required for this study are presented in section 3. Theoretical discussions on power fuzzy graphs are explained with results and examples in section 4. And properties of power fuzzy subgraphs are highlighted in section 5. Some special cases of power fuzzy graph and the application of power fuzzy graph is discussed in section 6 and section 7 respectively.

## 3. Preliminaries

**Definition 3.1.** [1] For any $k > 0$, the $k-th$ power $G^k$ of a graph $G = (V, E)$ has $V(G^k) = V(G)$, where two vertices $u$ and $v$ are adjacent in $G^k$ if and only if $d_G(u, v) \leq k$. $G^2$ and $G^3$ are called square and cube of $G$.




(A)




(B)

Figure 1. Power graphs $G^1$ and $G^2$

**Definition 3.2.** [3] A fuzzy Graph $G_f = (V, E, \sigma, \mu)$ corresponding to the crisp graph $G = (V, E)$ is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $u, v \in V$, $\mu(u, v) \leq min\{\sigma(u), \sigma(v)\}$, where $\sigma(u)$ and $\mu(u, v)$ represent the membership values of the vertex u and edge $(u, v)$ in $G$ respectively.

**Note 3.1.** The sequence of vertices $(v_1, v_2, \dots, v_n)$ in a fuzzy graph such that $\mu(v_i, v_{i+1}) > 0, i = 1, 2, \dots, (n-1)$ is called a path $P$. The number of edges in a path $P$ is called the length of $P$.

**Definition 3.3.** [7] Let $G_f = (V, E, \sigma, \mu)$ be a fuzzy graph. The size of $G_f$ is defined as $S(G_f) = \sum_{(v_1,v_2)\in V\times V} \mu(v_1, v_2)$. The order of $G_f$ is defined by $(O)G_f = \sum_{v\in V} \sigma(v)$. The degree of a vertex $v_1$ of $G_f$ is defined as $deg_{G_f}(v_1) = \sum_{v\neq v_1} \mu(v_1, v)$ for all v incident with $v_1$.

**Definition 3.4.** [3] Let $G_f = (V, E, \sigma, \mu)$ be a fuzzy graph. The strength of a path P $(v_1, v_2, \dots, v_n)$ is defined as $min\mu(v_i, v_{i+1}) : i = 1, 2, ..., n-1$. The maximum strengths of all paths between the vertices $v_1$ and $v_2$ in $G_f$ is called the strength of connectedness between the vertices $v_1$ and $v_2$ and it is denoted as $CONN_{G_f}(v_1, v_2)$ or $\mu_\infty(v_1, v_2)$.

## 4. Power fuzzy graphs

**Definition 4.1.** Let $G_f = (V, E, \sigma, \mu)$ be a fuzzy graph of a graph $G = (V, E)$. The power fuzzy graph of $G_f$ is defined as ${G_f}^k = (V, E_k, \sigma, \mu_k)$ where $E_k = E \cup E^*$, for any non-adjacent vertices $u, v \in V$ in $G_f$ there exists $uv \in E^*$ such that $l(u, v) \leq k$ where $l$ is the length from $u$ to $v$ and $k > 1$ provided $min(\sigma(u)^k, \sigma(v)^k) \leq \mu_k(u, v) \leq min(\sigma(u)^{k-1}, \sigma(v)^{k-1})$. ${G_f}^2$ is called the square of power fuzzy graph, and ${G_f}^3$ is called the cube of power fuzzy graph.

**Remark 4.1.** For $k = 1$, the power fuzzy graph ${G_f}^k$ is a crisp fuzzy graph.

**Example 4.2.** Consider the fuzzy graph ${G_f}^1$ in figure 2, which is a given fuzzy graph and satisfying the conditions of power fuzzy graph. ${G_f}^2$ in figure 3, is formed by adding edges $u_1u_3$ and $u_2u_4$ to the existing edges $u_1u_2, u_2u_3$ and $u_3u_4$ of ${G_f}^1$, since $l(u_1, u_3) = l(u_2, u_4) = 2$ and $l(u_1, u_2) = l(u_2, u_3) = l(u_3, u_4) = 1$.

| **Edge** | **Interval** | **Edge membership value** |
|---|---|---|
| $(u_1, u_2)$ | $[0.09, 0.3]$ | $\mu_2(u_1, u_2) = 0.2$ |
| $(u_2, u_3)$ | $[0.04, 0.2]$ | $\mu_2(u_2, u_3) = 0.04$ |
| $(u_3, u_4)$ | $[0.04, 0.2]$ | $\mu_2(u_3, u_4) = 0.08$ |
| $(u_1, u_3)$ | $[0.04, 0.2]$ | $\mu_2(u_1, u_3) = 0.1$ |
| $(u_2, u_4)$ | $[0.09, 0.3]$ | $\mu_2(u_2, u_4) = 0.2$ |

TABLE 1. Edge membership values of ${G_f}^2$

Similarly, ${G_f}^3$ in figure 4, is formed by adding a new edge $u_1u_4$ to the existing edges $u_1u_2, u_2u_3, u_3u_4, u_1u_3$ and $u_2u_4$ of ${G_f}^2$, since $l(u_i, u_j) \leq 3$, where $i, j = 1, 2, 3, 4$ and $i \neq j$.

| **Edge** | **Interval** | **Edge membership value** |
|---|---|---|
| $(u_1, u_2)$ | $[0.027, 0.09]$ | $\mu_3(u_1, u_2) = 0.05$ |
| $(u_2, u_3)$ | $[0.008, 0.04]$ | $\mu_3(u_2, u_3) = 0.02$ |
| $(u_3, u_4)$ | $[0.008, 0.04]$ | $\mu_3(u_3, u_4) = 0.01$ |
| $(u_1, u_3)$ | $[0.008, 0.04]$ | $\mu_3(u_1, u_3) = 0.09$ |
| $(u_2, u_4)$ | $[0.027, 0.09]$ | $\mu_3(u_2, u_4) = 0.03$ |
| $(u_1, u_4)$ | $[0.125, 0.25]$ | $\mu_3(u_1, u_4) = 0.15$ |

TABLE 2. Edge membership values of ${G_f}^3$

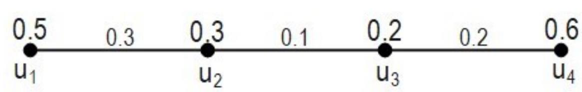



FIGURE 2. Power fuzzy graph ${G_f}^1$

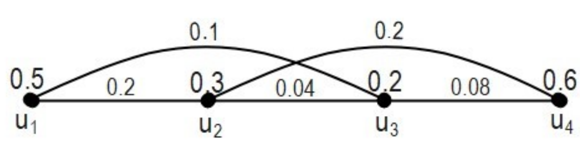



FIGURE 3. ${G_f}^2$ - Power fuzzy graph of ${G_f}^1$

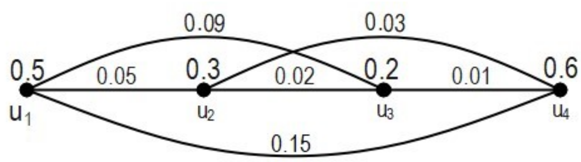



FIGURE 4. ${G_f}^3$ - Power fuzzy graph of ${G_f}^2$

**Definition 4.3.** The power fuzzy graph ${G_f}^k$ is called Total Power Fuzzy Graph ${T_f}^k$ where each vertex is adjacent to all other vertices. The number of edges incident to a vertex is denoted by $e({T_f}^k)$ and the maximum $k$ value referred as diameter is denoted by $k_m$.

**Example 4.4.** Consider ${G_f}^k = (V, E, \sigma, \mu_k)$ where $k = 1, 2, 3$ as in the Figure 4. Here no further edge can be added in ${G_f}^3$ since every vertex is adjacent to all other vertices. Therefore, ${G_f}^3$ is the total power fuzzy graph.

**Note 4.1.** For a trivial graph, $k_m = 1$

**Note 4.2.** For any integer n and decimal number $x, \lceil x \rceil + n = \lceil x + n \rceil$

**Remark 4.2.** For a power fuzzy graph ${G_f}^k = (V, E_k, \sigma, \mu_k)$, $\mu_k(u, v) \leq \mu_{(k-1)}(u, v)$ where $u, v \in V$ and $k > 2$.

**Proposition 4.5.** *For a path power fuzzy graph ${G_f}^k$ of $n$ vertices, $k_m = e({T_f}^k) = n - 1$, where $n > 1$.*

*Proof.* Proof by induction method. Let $k_m$ be denoted as $k_m(n)$ for $n$ vertices. When $n = 2$, $k_m(2) = 1$.

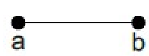
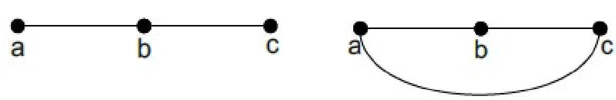



FIGURE 5. ${G_f}^1$ of 2-path graph

When $n = 3$, $k_m(3) = k_m(2) + 1 = 2$.




(A) (B)

FIGURE 6. ${G_f}^1$ and ${G_f}^2$ of 3-path graph

Assume that, the statement is true for n-1, (i.e.) $k_m(n-1) = k_m(n-2)+1 = n-2$. Hence, we have $k_m(n) = k_m(n-1)+k_m(1) = n-2+1 = n-1$. Similarly, $e({T_f}^k)(n) = n-1$.

□

**Proposition 4.6.** *For cycle power fuzzy graph ${G_f}^k$ of $n$ vertices, $k_m = \lceil \frac{(n-1)}{2} \rceil$, $e({T_f}^k) = n-1$, where $n > 1$.*

*Proof.* Proof by induction method. Let $k_m$ be denoted as $k_m(n)$ for $n$ vertices.
**Case 1:** $n$ is even, $k_m = \lceil \frac{(n-1)}{2} \rceil$
$n = 2$, $k_m(2) = \lceil \frac{1}{2} \rceil = 1$
$n = 4$, $k_m(4) = k_m(2) + 1 = \lceil \frac{3}{2} \rceil = 2$
Assume that, the statement is true for $n-2$, (i.e.) $k_m(n-2) = k_m(n-4)+1 = \lceil \frac{(n-3)}{2} \rceil$.
Hence, we have $k_m(n) = k_m(n-2) + k_m(2) = \lceil \frac{(n-3)}{2} \rceil + 1 = \lceil \frac{(n-1)}{2} \rceil$.
**Case 2:** $n$ is odd, $k_m = \frac{(n-1)}{2}$
$n = 3$, $k_m(3) = 1$
$n = 5$, $k_m(5) = \frac{4}{2} = 2$
$n = 7$, $k_m(7) = \frac{6}{2} = 3$
Assume that, the statement is true for $n-2$, (i.e.) $k_m(n-2) = \frac{(n-3)}{2}$.
Hence, we have $k_m(n) = k_m(n-2) - k_m(3) + k_m(5) = \frac{(n-1)}{2}$. Therefore, in general $k_m = \lceil \frac{(n-1)}{2} \rceil$ for $n$ vertices. Also, $e({T_f}^k) = n-1$ is true. □

**Definition 4.7.** Let ${G_f}^k = (V, E, \sigma, \mu_k)$ be a power fuzzy graph. The distance of any path $(x_1, x_2, \ldots, x_n)$ denoted as $P$ with distinct vertices and $n > 2$ of ${G_f}^k$ is defined as ${D_f}^k(x_1, x_2, ..., x_n) = \displaystyle\sum_{i=1}^{n-1} \mu_k(x_i, x_{i+1})$.

**Example 4.8.** Consider the path $(u_1, u_3, u_4)$ of ${G_f}^3 = (V, E_3, \sigma, \mu_3)$ in the Figure 4. Then the distance is ${D_f}^k(u_1, u_3, u_4) = 1$.

**Proposition 4.9.** *Let $P$ be any path in power fuzzy graph ${G_f}^k$ such that $P$ exists $\forall k$, then the distance of $P$ in ${G_f}^{k+1}$ is less than or equal to the distance of $P$ in ${G_f}^k$, for $k > 2$.*

*Proof.* Consider ${G_f}^k = (V, E_k, \sigma, \mu_k)$, a power fuzzy graph with $n > 2$ vertices $\{x_1, x_2, \ldots, x_n\}$. Let $\{P_1, P_2, \ldots P_m\}$ be set of all paths from $x_1$ to $x_n$ in ${G_f}^2$. Consider a path P from the set in ${G_f}^2$ then P exists $\forall {G_f}^k, k > 2$.Let the distance of $P$ in ${G_f}^{k+1}$ and the distance of $P$ in ${G_f}^k$ be denoted as ${D_p}^{k+1}$ and ${D_p}^k$ respectively. Let $(x_i, x_{i+1})$ be any edge in P. By the Remark 3.1.2, we get $\mu_k(x_i, x_{i+1}) \leq \mu_{k-1}(x_i, x_{i+1})$. Therefore, $\sum_{i=1}^{n-1} \mu_{k+1}(x_i, x_{i+1}) \leq \sum_{i=1}^{n-1} \mu_k(x_i, x_{i+1})$ and hence $D_p^{k+1} \leq D_p^k$. □

**Theorem 4.10.** *Every Power fuzzy graph is a fuzzy graph, but converse need not be true.*

*Proof.* Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be a power fuzzy graph, where $E_k$ is the set of edges in ${G_f}^k$ and $\mu_k$ is the set of corresponding membership values set. Also, note that in fuzzy graph any edge $(x, y)$, the membership value $\mu(x, y)$ lies between $[0, \min(\sigma(x), \sigma(y))]$. For any $k$, consider the edge $(x, y)$ in ${G_f}^k$ with the membership value which satisfies the condition $\min\left(\sigma(x)^k, \sigma(y)^k\right) \leq \mu_k(x, y) \leq \min\left(\sigma(x)^{k-1}, \sigma(y)^{k-1}\right)$.
**Case 1:** If $\sigma(x) = \sigma(y) = 0$ or if $\sigma(x) = 0$ and $\sigma(y) > 0$, then $\mu_k(x, y) = 0$. Since $\mu_k(x, y) \in [0, \min(\sigma(x), \sigma(y))], {G_f}^k$ is a fuzzy graph.
**Case 2:** If $\sigma(x) > 0, \sigma(y) > 0$ and $\sigma(x) > \sigma(y)$, then
$\min(\sigma(x)^{k-1}, \sigma(y)^{k-1}) = \sigma(y)^{k-1}$ and $\min(\sigma(x)^k, \sigma(y)^k) = \sigma(y)^k$. Hence $\mu_k(x, y) \in [\sigma(y)^k, \sigma(y)^{k-1}] \subset [0, \min(\sigma(x), \sigma(y))]$. Therefore, ${G_f}^k$ is a fuzzy graph.
Conversely, let $G_f = (V, E, \sigma, \mu)$ be the fuzzy graph and consider an edge $(x, y) \in E$. If $\mu(x, y)$ lies between $\left[\min\left(\sigma(x)^k, \sigma(y)^k\right), \min\left(\sigma(x)^{k-1}, \sigma(y)^{k-1}\right)\right]$ for $k > 1$, then $G_f$ is a power fuzzy graph. Suppose $\mu(x, y)$ lies between $\left(0, \min\left(\sigma(x)^k, \sigma(y)^k\right)\right)$ or $\left(\min\left(\sigma(x)^{k-1}, \sigma(y)^{k-1}\right), \min(\sigma(x), \sigma(y))\right)$ for $k > 1$, then $G_f$ is not a power fuzzy graph. Hence every $G_f$ need not be a power fuzzy graph. □

**Example 4.11.** The graph in Figure 4 satisfies the condition of fuzzy graph, (i.e.) $\mu(u, v) \leq min \sigma(u), \sigma(v)$, $\forall u,v \in V$. But does not satisfy the condition of power fuzzy graph.

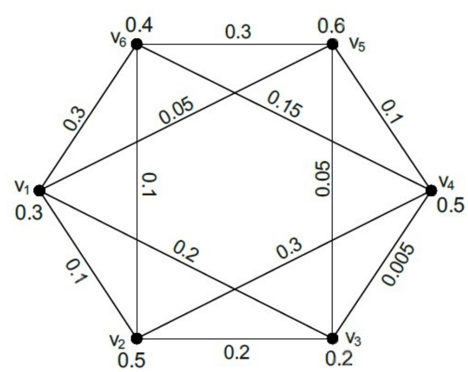



FIGURE 7. Fuzzy graph, not a ${G_f}^2$ Power fuzzy graph

**Proposition 4.12.** *For every connected power fuzzy graph ${G_f}^k$ with $k > 1$ and $n > 2$ vertices, there exist a cycle.*

*Proof.* Consider a power fuzzy graph, ${G_f}^k = (V, E, \sigma, \mu_k)$.
**Case 1:** Assume that ${G_f}^1$ contains a cycle, then for all power fuzzy graphs for $k > 1$ there exists a cycle.
**Case 2:** Let ${G_f}^1$ be a power fuzzy graph of $n > 2$ vertices without any cycle. Then there exists only one path between any two vertices in an acyclic graph.

Consider the two non-adjacent vertices in ${G_f}^1$ whose length $k \leq 2$, then the two vertices are joined by an edge in ${G_f}^2$. Since ${G_f}^2$ contains a cycle, there exist a cycle $\forall {G_f}^k$. □

**Theorem 4.13.** *Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be a power fuzzy graph. Then the strength for any path $P$ in ${G_f}^2$ is greater than or equal to strength of the path $P$ in ${G_f}^k$ where $k > 2$.*

*Proof.* Consider a path $P$ in ${G_f}^2$ then $P$ exists $\forall {G_f}^k$ and let $(x_1, x_2, \ldots, x_n)$ be the path $P$. Suppose $x_{n-1}\ x_n \in E_2$ is the weakest edge of $P$ in ${G_f}^2$, (i.e.) $\mu_k(x_{n-1}, x_n)$ is the least membership value, then let us denote strength of the path $P$ as $s_2 = \mu_k(x_{n-1}, x_n)$ in ${G_f}^2$. Since ${G_f}^2$ is a power fuzzy graph we have,

$$min(\sigma(x_{n-1})^2, \sigma(x_n)^2) \leq s_2 \leq min(\sigma(x_{n-1}), \sigma(x_n)) \tag{1}$$

Similarly, let $s_k$ be the strength of the path $P$ in ${G_f}^k$. Then

$$min(\sigma(x_{n-1})^k), \sigma(x_n)^k) \leq min(\sigma(x_{n-1})^{k-1}), \sigma(x_n)^{k-1}) \tag{2}$$

Therefore, by comparing the equations (1) and (2) we get, $s_2 \leq s_k$. Hence the theorem. □

**Corollary 4.14.** *Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be a power fuzzy graph. Then $CONN_{G_f}{}^2(a, b) \geq CONN_{G_f}{}^k(a, b)$ where $CONN_{G_f}{}^k(a, b)$ is the strength of connectedness in ${G_f}^k$ for $a, b \in V$.*

*Proof.* Consider ${G_f}^k = (V, E_k, \sigma, \mu_k)$, a power fuzzy graph with $n > 2$ vertices $\{x_1, x_2, \ldots, x_n\}$. Let $\{P_1, P_2, \ldots P_m\}$ be set of all paths from $x_1$ to $x_n$ in ${G_f}^2$, then ${G_f}^k$ contains more paths including this set. Let ${s_2}^1, {s_2}^2, \ldots, {s_2}^m$ be the strengths of $P_1, P_2, \ldots, P_m$ respectively in ${G_f}^2$. Similarly let ${s_k}^1, s_k^2, \ldots, {s_k}^m$ be the strengths of $P_1, P_2, \ldots, P_m$ respectively in ${G_f}^k$. By theorem 3.9 we have ${s_2}^1 \geq {s_k}^1, {s_2}^2 \geq {s_k}^2, \ldots, {s_2}^m \geq {s_k}^m$, hence we get $\max\left({s_2}^1, {s_2}^2, \ldots, {s_2}^m\right) \geq \max\left({s_k}^1, {s_k}^2, \ldots, {s_k}^m\right)$. This proves that $\mathrm{CON}\,N_{G_f}(a, b) \geq \mathrm{CONN}_{G_f}(a, b)$. □

**Corollary 4.15.** *Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be a power fuzzy graph and let $xy \in E_k$. Then $\mu_2(x, y) > \mu_k(x, y)\ \forall k > 1$ where $\mu_2(x, y) \in {G_f}^2$ and $\mu_k(x, y) \in {G_f}^k$.*

**Remark 4.3.** Order of ${G_f}^k$ is equal for every $k$ since the vertex membership function remains constant $\forall k$.

## 5. Subgraphs of power fuzzy graphs

**Definition 5.1.** A graph ${H_f}^k = (V_h, E_{hk}, \tau, v_k)$ is called Power fuzzy subgraph of ${G_f}^k = (V, E_k, \sigma, \mu_k)$ if $V_h \subseteq V$ and $E_{hk} \subseteq E_k$ where $E_k$ contains the edges which are incident with vertices in $V_h$ such that $\tau(u) = \sigma(u), \forall u \in V_h$ and $v_k(u, v) = \mu_k(u, v), \forall (u, v) \in E_k$.

**Definition 5.2.** A graph ${H_f}^k = (V, E_k, \tau, v_k)$ is called Partial power fuzzy subgraph of ${G_f}^k = (V, E_k, \sigma, \mu_k)$ if $\tau(u) \leq \sigma(u), \forall u \in V$ and $v_k(u, v) \leq \mu_k(u, v), \forall (u, v) \in E_k$.

**Definition 5.3.** A partial fuzzy subgraph ${H_f}^k = (V, E_k, \tau, v_k)$ is called Spanning power fuzzy subgraph of ${G_f}^k = (V, E_k, \sigma, \mu_k)$ if $\tau(u) = \sigma(u)\ \forall u \in V$.

**Definition 5.4.** A graph ${H_f}^k = (V, E_{hk}, \sigma, v_k)$ is called Least power fuzzy subgraph of ${G_f}^k$ if $E_{hk}$ has edges whose $\mu^k$ values lie between $\left[\sigma_l^k, \sigma_l^{k-1}\right]$ such that $E_{hk} \subset E_k$ where $\sigma_l$ is the least vertex membership value.

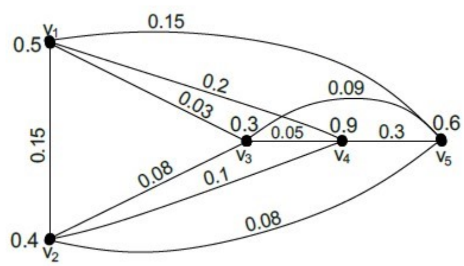



FIGURE 8. Total power fuzzy graph ${G_f}^3$ of Tadpole graph $T(3,2)$

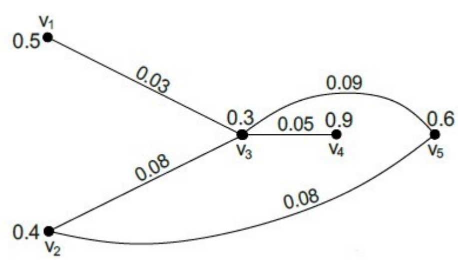



FIGURE 9. Least power fuzzy subgraph ${H_f}^3$ of Tadpole graph $T(3,2)$

**Lemma 5.5.** *The Least power fuzzy subgraph ${H_f}^k$ has all the edges incident to the vertices with $\sigma_l$ value of ${G_f}^k, \forall k > 0$.*

*Proof.* Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be a power fuzzy graph and let vertex $u \in V$ has the least membership value $\sigma_l$. Then the edges incident with vertex $u$ will have membership values between $\left[{\sigma_l}^k, \sigma_l^{k-1}\right]$. Since $H_f^k$ is the least power fuzzy graph, it has the edges with such $\mu_k$ values. □

**Corollary 5.6.** *Let number of edges incident to $\sigma_l$ of ${G_f}^k$ be denoted by $e_l$. Then the least power fuzzy subgraph ${H_f}^k$ has at least $e_l$ edges.*

**Example 5.7.** Consider the total power fuzzy graph ${G_f}^3$ in figure 8. Here, $\sigma_l = 0.3$ and $e_l = 4$. The total number of edges in the least power fuzzy subgraph ${H_f}^3$ in figure 9 is 5.

## 6. Special cases of power fuzzy graph

**Definition 6.1.** The power fuzzy graph ${G_f}^k = (V, E_k, \sigma, \mu_k)$ where $E_k = E \cup E^*$, for any non-adjacent vertices $u, v \in V$ in $G$ there exist $uv \in E^*$ such that $l(\mathrm{u,v}) \leq \mathrm{k}$ where $l$ is the length from $u$ to v and $k > 1$ provided $\mu_k(u,v) = \min\left(\sigma(u)^{k-1}, \sigma(v)^{k-1}\right)$ is called Maximal power fuzzy graph.

**Definition 6.2.** The power fuzzy graph ${G_f}^k = (V, E_k, \sigma, \mu_k)$ where $E_k = E \cup E^*$, for any non-adjacent vertices $u, v \in V$ in $G$ there exist $uv \in E^*$ such that $l(\mathrm{u,v}) \leq \mathrm{k}$ where $l$ is the length from $u$ to v and $k > 1$ provided $\mu_k(u,v) = \min\left(\sigma(u)^{k}, \sigma(v)^{k}\right)$ is called Minimal power fuzzy graph.

**Definition 6.3.** The power fuzzy graph ${G_f}^k = (V, E_k, \sigma, \mu_k)$ where $E_k = E \cup E^*$, for any non-adjacent vertices $u, v \in V$ in $G$ there exist $uv \in E^*$ such that $l(\mathrm{u,v}) \leq \mathrm{k}$ where $l$ is the length from $u$ to v and $k > 1$ provided $\mu_k(u,v) = \frac{\min\left(\sigma(u)^{k}, \sigma(v)^{k}\right) + \min\left(\sigma(u)^{k-1}, \sigma(v)^{k-1}\right)}{2}$ is called Median power fuzzy graph.

**Remark 6.1.** For a power fuzzy graph ${G_f}^k = (V, E_k, \sigma, \mu_k)$ which is either maximal or minimal or median then, $\mu_k(u,v) < \mu_{k-1}(u,v)$ where $u, v \in V$ and $k > 2$.

**Lemma 6.4.** *Let the values of $\sigma(u)$, $\forall u \in V$ in ${G_f}^1$ has only one decimal place.*

i *In a maximal power fuzzy graph ${G_f}^k$ with $k > 2$, the number of decimal places of $\mu_k$ values are $k-1$*

ii *In a minimal power fuzzy graph ${G_f}^k$ with $k > 2$, the number of decimal places of $\mu_k$ values are $k$.*

iii *In a median power fuzzy graph ${G_f}^k$ with $k > 2$ and when $\sigma(u) = 0.x$ where $x$ is even, the number of decimal places of $\mu_k$ values are $k$ similarly when $\sigma(u) = 0.y$ where $y$ is odd, the number of decimal places of $\mu_k$ values are $k+1$.*

**Lemma 6.5.** *Let ${G_f}^k = (V, E_k, \sigma, \mu_k)$ be the Maximal power fuzzy graph, then strength of any path in ${G_f}^1$ and ${G_f}^2$ are equal.*

*Proof.* Consider a path $P\{v_1, v_2, \ldots, v_n\}$ in ${G_f}^2$, then ${G_f}^1$ contains the same path. Let $s_1$ and $s_2$ be the strengths of path $P$ in ${G_f}^1$ and ${G_f}^2$ respectively. Since ${G_f}^1$, the Maximal power fuzzy graph is a crisp fuzzy graph the $\mu_1$ values for every edge takes the minimum membership value of the corresponding two vertices. For ${G_f}^2$ the edge membership values are defined as $\mu_2(u,v) = \min(\sigma(u), \sigma(v))$. Hence $\mu_1(u,v) = \mu_2(u,v) \forall (u,v) \in {G_f}^2$. This implies $s_1 = s_2$. Hence the proof. □

**Lemma 6.6.** *Let ${G_f}^k$ be either Maximal or Minimal power fuzzy graph, then for $k > 2$,* $\text{Size}\left({G_f}^2\right) > \text{Size}\left({G_f}^k\right)$.

*Proof.* Consider a ${G_f}^k = (V, E_k, \sigma, \mu_k)$ which is either maximal or minimal power fuzzy graphs $\forall k > 1$. By the Remark 5.4, we have $\mu_k(u,v) < \mu_{k-1}(u,v), \forall k >$

1 for $u, v \in V$ such that $u \neq$ v. Then $\mu_3(u,v) < \mu_2(u,v)$, $\mu_4(u,v) < \mu_3(u,v)$, and so, on. This implies $\mu_2(u,v) > \mu_3(u,v) > \mu_4(u,v) \ldots > \mu_k(u,v)$. Hence $\mu_2(u,v) > \mu_k(u,v)$ and $\sum\limits_{\forall (u,v)\in E_2} \mu_2(u,v) < \sum\limits_{\forall (u,v)\in E_k} \mu_k(u,v)$. Therefore Size$(G_f^2) >$ Size$({G_f}^k)$. $\square$

**Corollary 6.7.** *The Least power fuzzy subgraph* ${H_f}^k$ *of Maximal and Minimal power fuzzy graphs has only the edges incident with vertices having* $\sigma_l$ *value of* ${G_f}^k, \forall k > 1$.

*Proof.* Consider ${G_f}^k$ be Maximal power fuzzy graph and let there exist only one vertex, say $u$ with $\sigma_l$ value. Then $\mu_k(u,v) = {\sigma_l}^{k-1}$ where $(u,v) \in E_k, \forall v \in V - \{u\}$. Hence $\left[{\sigma_l}^k, {\sigma_l}^{k-1}\right] \not\subset \left[\sigma(v)^k, \sigma(v)^{k-1}\right]$. Since the least power fuzzy subgraph has the edges, whose values lie between $\left[{\sigma_l}^k, {\sigma_l}^{k-1}\right]$, the proof is complete. Similarly, when ${G_f}^k$ is a Minimal power fuzzy graph the statement is true. $\square$

## 7. Application

The concept of power fuzzy graph can be used in the road maps to find better destinations and the fastest routes to reach them. Considering a particular type of destinations in a location as vertices, the flaws in each destination regarding quality, popularity, facility, speciality, availability etc. can be viewed as vertex membership values. The membership values of edges(roads) are assigned according to the flaw values of vertices and distances, average traffic flow, road safety, etc. Hence the decision maker can find the way to reach the destination faster and safer accordingly by using the power fuzzy graph model.

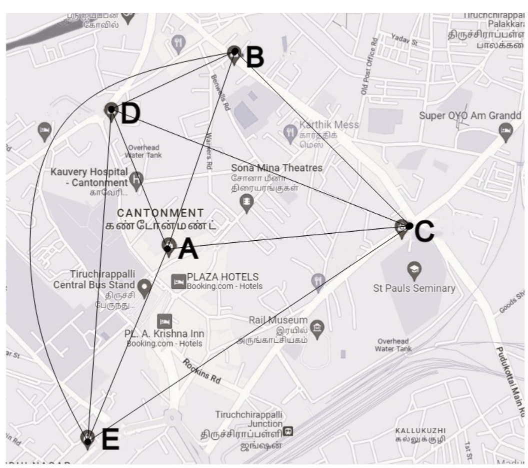



FIGURE 10. Shopping complexes in and around Trichy (Courtesy: Google Maps)

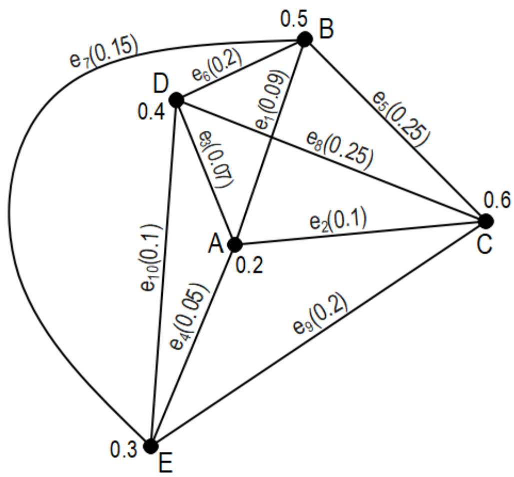



FIGURE 11. Total power fuzzy graph ${G_f}^2$ of five shopping complex

Let A, B, C, D and E be five shopping complexes connected by roadways considered as edges. By total power fuzzy graph model, we can find all the possible routes from each complex to the rest of the complexes.
According to the necessary and essential requirements of the buyer, brand, rate, quality, customer's reviews, the flaw values of vertices are calculated by the methods of performance analysis [28]. The edge membership values are calculated in Table 6.1 based on the interval computed from the vertex membership values, distance, route, road safety and travel time.

| **Edge** | **Interval** | **Edge membership value** |
|---|---|---|
| $e_1$ | $[0.04, 0.2]$ | $\mu_2(e_1) = 0.09$ |
| $e_2$ | $[0.04, 0.2]$ | $\mu_2(e_2) = 0.1$ |
| $e_3$ | $[0.04, 0.2]$ | $\mu_2(e_3) = 0.07$ |
| $e_4$ | $[0.04, 0.2]$ | $\mu_2(e_4) = 0.05$ |
| $e_5$ | $[0.25, 0.5]$ | $\mu_2(e_5) = 0.25$ |
| $e_6$ | $[0.16, 0.4]$ | $\mu_2(e_6) = 0.2$ |
| $e_7$ | $[0.09, 0.3]$ | $\mu_2(e_7) = 0.15$ |
| $e_8$ | $[0.16, 0.4]$ | $\mu_2(e_8) = 0.25$ |
| $e_9$ | $[0.09, 0.3]$ | $\mu_2(e_9) = 0.2$ |
| $e_{10}$ | $[0.09, 0.3]$ | $\mu_2(e_10) = 0.1$ |

TABLE 3. Edge membership values of ${G_f}^2$

| Destinations | Minimum path distance | Minimum path |
|---|---|---|
| A to E | 0.05 | $A, e_4, E$ |
| E to D | 0.12 | $E, e_4, A, e_3, D$ |
| D to B | 0.16 | $D, e_3, A, e_1, B$ |
| B to C | 0.19 | $B, e_1, A, e_2, C$ |

TABLE 4. Path traced from A to C using minimum path distance.

Therefore, by Figure11 the first complex to be visited is A, if the requirement is not in A, then visit E, and so on. Hence the order to visit the complex is A, E, D, B and C. The path from one destination to another is traced by calculating minimum distances which is given in the Table 4.

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**Data availability** : Not applicable

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